Palestine Journal of Mathematics

Vol. 12(1)(2023) , 361–380

ON THE COEFFICIENT ESTIMATES OF NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS

Krishnat D. Masalkar, Uday H.Naik and Channayya Hiremath

Communicated by Ayman Badawi

MSC 2010 Classifications:Primary 30C45;Secondary 30C50.

Keywords and phrases: Bi-univalent functions, Subordination principle, Schwartz function, Starlike functions.

Abstract In this paper, we introduce two subclasses of the class of bi-univalent fucntions on the open unit disk using subordination principle.We estimate sharp upper bounds on moduli the first five coefficients in Maclaurin's series of the functions in these subclasses. Also, we have obtained Fekete-Szegö inequality and similar type of inequalities for these subclasses.

1 Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} \colon |z| < 1\}$ be a open unit disk in the complex plane $\mathbb{C}$. The class

$$\mathcal{S} = \{f : \mathbb{U} \longrightarrow \mathbb{C} \colon f \text{ is analytic, univalent and } f(0) = 0 = f'(0) - 1\}$$

has very nice properties. The class $\mathcal{S}$ is invarient under rotation, dilation, complex conjugation but not invarient under addition of functions. Also the class $\mathcal{S}$ is invarient under disk automorphisms, range transformations, omitted value transformations, square root transformations.

If $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{S}$ then $Area(f(\mathbb{U})) = \iint_{\mathbb{U}} |f'(z)|^2 dxdy = \pi \sum_{n=1}^{\infty} n|a_n|^2$.

Also if $f \in \mathcal{S}$ then

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \text{ for all } z \in \mathbb{U},$$

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2} \text{ for all } z \in \mathbb{U}$$

and

$$\frac{1-|z|}{1+|z|} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|} \text{ for all } z \in \mathbb{U}.$$

A function $f \in \mathcal{S}$ has the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and it is starlike if $f(z) \in f(\mathbb{U})$ whenever $rf(z) \in f(\mathbb{U}),$ for all $0 < r < 1$. The function f is starlike if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \text{ for all } |z| < 1.$$

(see [5]). Caratheodory ([4],[5]) proved in 1907 that for an analytic function $g(z) = 1 + \sum_{n=1} b_n z^n$, if $Re\,(g(z)) > 0$, for all $|z| < 1$ then $|b_n| \leq 2$. Using this, one can easily prove that if a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is starlike then $|a_n| \leq n$. If a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent in $\mathbb{U}$ and f^{-1} is also univalent in $\mathbb{U}$ then f is called as bi-univalent in $\mathbb{U}$. The class of all biunivalent functions on unit disk $\mathbb{U}$ is denoted by Σ. In 1967,

Lewin ([11]) introduced the class of bi-univalent functions and shown that $|a_2| \leq 1.51$. Brannan and Clunie ([2]) conjectured that , if the function f is bi-univalent on $\mathbb{U}$ then $|a_2| \leq \sqrt{2}$. Netanyahu ([13]) proved that, if the function is bi-univalent then $|a_2| \leq \frac{4}{3}$. Still the coefficient bounds for $|a_3|, |a_4|, |a_5|$.... is an open problem. The study of subclasses of bi-univalent functions was continued by Brannan and Taha (see [3] ,[21]) by introducing certain subclasses of bi-univalent functions, $S^*(\alpha)$ and $K(\alpha)$ $(0 \leq \alpha < 1)$ of starlike and convex functions of oder α respectively. Srivastava et al.([16]) also contributed by introducing certain subclasses of bi-univalent functions and found some initial coeffiecients bounds. Ma and Minda ([12]) introduced the classes $S^*(\phi) = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \colon f \text{ is univalent on } \mathbb{U}, \ \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}$ and $K^*(\phi) = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \colon f \text{ is univalent on } \mathbb{U}, \ 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}$, where ϕ is an analytic function with positive real part on the unit disk $\mathbb{U}$, $\phi(0) = 1, \phi'(0) > 1$ and maps $\mathbb{U}$ into a region which is starlike with respect to 1 and symmetric with respect to the real axis. These classes include several well known subclasses of starlike and convex functions respectively as special cases. In 1925, J.E. Littlewood proved the subordination theorem in operator theory and complex analysis. It states that if h is analytic univalent function from the unit disk $\mathbb{U}$ onto itself such that $h(0) = 0$ then the composition operator $C_h(f) = foh$ defines linear operator with operator norm less than 1 on various function spaces of analytic functions on the unit disk $\mathbb{U}$. These spaces include the Hardy spaces, Bergman spaces and Dirichlet spaces.
Various spaces of orthogonal polynomials are connected with bi-univalent functions (see [6], [17],[23],[24].) The recurrence relation for the Horadam polynomials $h_n(x)$ was studied by Horzum Kocer ([9]) and is given by

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \text{ for all } n = 3, , 4...$$

with

$$h_1(x) = a, \quad h_2(x) = bx, \quad \text{where } \ a, b, p, q \quad \text{are real constants.}$$

The spaces of Fibbonacci polynomials, Lucas polynomials, Pell-Lucas polynomials, Chebyshev polynomials are special cases of the space of Horadam polynomials.

2 Preliminaries

The following lemma is well known and gives bounds on absolute values of coefficients of Schwarz function.

Lemma 2.1. *([5],[10],[15]) If $u(z) = \sum_{n=0}^{\infty} u_n z^n$ is an analytic function from open unit disk $\mathbb{U} = \{z \in \mathbb{C} \colon |z| < 1\}$ to itself then*

$$|u_0| \leq 1, \quad |u_n| \leq 1 - |u_0|^2 \ \textit{for all n=1,2,3,...}.$$

Further, if $u_0 = 0$ then $\left|u_2 - \mu u_1^2\right| \leq \max\{1, |\mu|\}$ for any real number μ. The result is sharp for functions $u(z) = z$ and z^2.
If $f(z) = z + \sum_{n=1}^{\infty} a_n z^n$ is an analytic function on unit disk $\mathbb{U}$ with positive real part then $|a_n| \leq 2$ for all n.

If $p(z) = \sum_{i=0}^{n} a_i z^i$ is a polynomial of degree n then Bernstein([1]) proved that

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|$$

and equality holds if all zeros of p are at origin. Further for a polynomial p which has all zeros in $|z| \leq 1$ then Turan ([21]) proved that $\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|$. This inequality is sharp and

equality holds for all polynomials that has all zeros on $|z| = 1$. Also Gardener et al.([8]) proved that, if p has all zeros in $|z| \leq k \leq 1$ then $\frac{|a_{n-1}|}{|a_n|} \leq nk$. This motivates us to study the maximum modulus of polynomial function of several complex variables.
Let $A, B_1, ..., B_k$ be real numbers and $u_1, u_2, \ldots, u_n$ be complex numbers with $|u_k| \leq 1$. From triangle inequality it is clear that

$$\max_{|u_k|\leq 1} \left| A + \sum_{k=1}^{n} B_k u_k \right| \leq |A| + \sum_{k=1}^{n} |B_k| .$$

In following lemma we assert that equality holds in the above inequality. Now We prove

Lemma 2.2. *Let* $\overline{\mathbb{U}}^n = \{(u_1, u_2, ..., u_n) \in \mathbb{C}^n : |u_k| \leq 1\}$ *and* $\partial\overline{\mathbb{U}}^n$ *be its booundary.*
If $A, B_1, B_2, ..., B_n \in \mathbb{R}$*, and* C *is a constant such that* $|C| \leq \left| A + \sum_{k=1}^{n} B_k u_k \right|$ *on* $\partial\overline{\mathbb{U}}^n$ *then*

$$|C| \leq \sqrt{A^2 + \sum_{k=1}^{n} B_k^2} \leq |A| + \sum_{k=1}^{n} |B_k|.$$

Proof Define a function f on $\overline{\mathbb{U}}^n$ as

$$f(u_1, ..., u_n) = A + \sum_{k=1}^{n} B_k u_k.$$

Let g and h be functions on $[0, 2\pi]^n$, defined as below:

$$g(\theta_1, \theta_2, ..., \theta_n) = A^2 + \sum_{k=1}^{n} B_k^2 + \sum_{k=1}^{n} AB_k \cos(\theta_k) + \sum_{j\neq k} B_j B_k \cos(\theta_k - \theta_j)$$

and

$$h(\theta_1, \theta_2, ..., \theta_n) = A^2 + \sum_{k=1}^{n} B_k^2 + \sum_{k=1}^{n} |AB_k| \cos(\theta_k) + \sum_{j\neq k} |B_j B_k| \cos(\theta_k - \theta_j).$$

Observe that

$$g(\theta_1, \theta_2, ..., \theta_n) = \left(A + \sum_{k=1}^{n} B_k e^{i\theta_k} \right) \left(A + \sum_{k=1}^{n} B_k e^{-i\theta_k} \right) = |f(e^{i\theta_1}, e^{i\theta_2}, ...e^{i\theta_n})|^2$$

Consider the case $A > 0$. Similar proof will work for $A < 0$. Define ϕ_k as below:

$$\phi_k = \begin{cases} \theta_k - \pi & \text{if } B_k < 0 \\ \theta_k & \text{if } B_k > 0 \end{cases}.$$

Therefore

$$B_k \cos(\theta_k) = \begin{cases} |B_k| \cos\theta_k = |B_k| \cos\phi_k & \text{if } \quad B_k > 0. \\ -|B_k| \cos(\theta_k) = |B_k| \cos(\theta_k - \pi) = |B_k| \cos(\phi_k) & \text{if } \quad B_k < 0. \end{cases}.$$

If $B_j B_k < 0$ then either $B_j < 0,\ B_k > 0$ or $B_j > 0,\ B_k < 0$. If $B_j < 0,\ B_k > 0$ then

$$B_j B_k \cos(\theta_j - \theta_k) = -|B_j B_k| \cos(\pi + \phi_j - \phi_k) = |B_j B_k| \cos(\phi_j - \phi_k).$$

Similarly if $B_j > 0,\ B_k < 0$ then

$$B_j B_k \cos(\theta_j - \theta_k) = -|B_j B_k| \cos(\phi_j - \phi_k - \pi) = |B_j B_k| \cos(\phi_j - \phi_k).$$

Also if $B_jB_k > 0$ then either $B_j > 0,\ B_k > 0$ or $B_j < 0,\ B_k < 0$. If $B_j > 0,\ B_k > 0$ then $B_jB_k\cos(\theta_j-\theta_k) = |B_jB_k|\cos(\phi_j-\phi_k)$. If $B_j < 0,\ B_k < 0$ then

$$B_jB_k\cos(\theta_j-\theta_k) = |B_jB_k|\cos(\pi+\phi_j-\pi-\phi_k) = |B_jB_k|\cos(\phi_j-\phi_k).$$

Therefore

$$g(\theta_1,...,\theta_n) = A^2+\sum_{k=1}^{n}B_k^2+2\sum_{k=1}^{n}AB_k\cos(\theta_k)+2\sum_{j<k}B_jB_k\cos(\theta_k-\theta_j)$$
$$= A^2+2\sum_{k=1}^{n}B_k^2+2\sum_{k=1}^{n}|AB_k|\cos(\phi_k)+2\sum_{j<k}|B_jB_k|\cos(\phi_j-\phi_k) = h(\phi_1,...,\phi_n).$$

Hence

$$\begin{aligned}
&\left(|A|+\sum_{k=1}^{n}|B_k|\right)^2\\
&\geq \max_{0\leq\theta_1,\theta_2,\ldots,\theta_n<2\pi} g(\theta_1,...,\theta_n)\\
&= \max_{0\leq\phi_1,\phi_2,\ldots,\phi_n<2\pi} h(\phi_1,...,\phi_n)\\
&\geq h(\phi_1,...,\phi_n)|_{\phi_1=0,\ldots,\phi_n=0}\\
&= A^2+\sum_{k=1}^{n}B_k^2+2\sum_{k=1}^{n}|AB_k|+2\sum_{j<k}|B_jB_k|\\
&= \left(|A|+\sum_{k=1}^{n}|B_k|\right)^2.
\end{aligned}$$

By maximum modulus principle for functions of several complex variables,

$$\begin{aligned}
&\max_{(u_1,u_2,\ldots,u_n)\in\overline{\mathbb{U}}^n}|f(u_1,...,u_n)|\\
&= \max_{(u_1,u_2,\ldots,u_n)\in\partial\overline{\mathbb{U}}^n}|f(u_1,...,u_n)| = \max_{0\leq\theta_1,\theta_2,\ldots,\theta_n\leq2\pi}|f(e^{i\theta_1},e^{i\theta_2},...,e^{i\theta_n})|\\
&= \sqrt{\max_{0\leq\theta_1,\theta_2,\ldots,\theta_n\leq2\pi} g} = \max_{0\leq\theta_1,\theta_2,\ldots,\theta_n\leq2\pi}\sqrt{g}\\
&= A+\sum_{k=1}^{n}|B_k|.
\end{aligned}$$

Also

$$\int_{[0,2\pi]^n} g = 2\pi^n\left(A^2+\sum_{k=1}^{n}B_k^2\right)+2\sum_{k=1}^{n}AB_k\int_{[0,2\pi]^n}\cos(\theta_k)+2\sum_{j<k}B_jB_k\int_{[0,2\pi]^n}\cos(\theta_k-\theta_j)$$
$$= (2\pi)^n\left(A^2+\sum_{k=1}^{n}B_k^2\right).$$

Therefore

$$(2\pi)^n\left(A^2+\sum_{k=1}^{n}B_k^2\right) = \int_{[0,2\pi]^n} g \leq \int_{[0,2\pi]^n}\max_{0\leq\theta_1,\theta_2,\ldots,\theta_n\leq2\pi} g = (2\pi)^n\max_{0\leq\theta_1,\theta_2,\ldots,\theta_n\leq2\pi} g.$$

This gives

$$\sqrt{A^2+\sum_{k=1}^{n}B_k^2} \leq \sqrt{\max_{0\leq\theta_1,\theta_2,\ldots,\theta_n\leq2\pi} g} = \max_{(u_1,u_2,\ldots,u_n)\in\overline{\mathbb{U}}^n}|f(u_1,...,u_n)|.$$

Similarly, if C is constant such that $|C| \le |f(u_1, ..., u_n)|$ on $\partial\overline{\mathbb{U}}^n$ then $|C|^2 \le g$ on $[0, 2\pi]^n$. Hence $\int_{[0,2\pi]^n} |C|^2 \le \int_{[0,2\pi]^n} g$. This gives $|C| \le \sqrt{A^2 + \sum_{k=1}^{n} B_k^2}$. □

Remark 2.3. Let $a_k \in \mathbb{C}$ for $k = 1, 2, ..., n$. Let $F(z_1, z_2, .., z_n) = \sum_{k=1}^{n} a_k z_k$ be a function defined on $\overline{\mathbb{U}}^n$. If $a_k = |a_k|e^{it_k}$ and $z_k = |z_k|e^{i\theta_k}$ for all $k = 1, 2, ..., n$. then

$$|f(z_1, z_2, ..., z_n)|^2 = \left(\sum_{j=1}^{n} |a_j||z_j|e^{i(t_j+\theta_j)}\right)\left(\sum_{k=1}^{n} |a_k||z_k|e^{-i(t_k+\theta_k)}\right)$$

That is

$$|f(z_1, z_2, ..., z_n)|^2 = \sum_{k=1}^{n} |a_k|^2|z_k|^2 + 2\sum_{1\le j<k\le n} |a_j||a_k||z_j||z_k| \cos(t_j - t_k + \theta_j - \theta_k).$$

If we choose $\theta_j = t_j$, $\theta_k = t_k$, and $|z_j| = |z_k| = 1$ for all $j, k = 1, 2, ...n$ then

$$|f(z_1, z_2, ..., z_n)|^2 = \sum_{k=1}^{n} |a_k|^2 + 2\sum_{1\le j<k\le n} |a_j||a_k|.$$

If C be a constant such that $|C| \le f$ on $\partial\overline{\mathbb{U}}^n$ then

$$\int_{\partial\overline{\mathbb{U}}^n} |C|^2 \le (2\pi)^n \left(\sum_{k=1}^{n} |a_k|^2\right) + 2\sum_{1\le j<k\le n} |a_j||a_k| \int_{[0,2\pi]^n} \cos(t_j - t_k + \theta_j - \theta_k).$$

But $\int_{[0,2\pi]^n} \cos(t_j - t_k + \theta_j - \theta_k) = 0$. Therefore $|C| \le \sqrt{\sum_{k=1}^{n} |a_k|^2}$.

Similar to the lemma (2.2) the following lemma holds for a polynomial function of one complex variable.

Lemma 2.4. *If $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial function then*

$$\max_{|z|\le 1} |p(z)| = \max_{|z|=1} |p(z)| \le \sum_{k=0}^{n} |a_k|.$$

Equality holds in above inequality if all a_k are have same argument. Also

$$\int_{|z|=1} |p(z)|^2 dz = 2\pi \sum_{k=1}^{n} a_k^2.$$

Moreover, C is constant such that $|C| \le |p(z)|$ on $|z| \le 1$ then

$$|C| \le \sqrt{\sum_{k=1}^{n} a_k^2} \le \sup_{|z|\le 1} |p(z)|.$$

Proof Assume that $a_k = |a_k|e^{it_k}$ for all $k = 0, 1, ...n$. Then

$$\begin{aligned} &|p(e^{i\theta})|^2 \\ &= \left(\sum_{j=0}^{n} |a_j|e^{i(t_j+j\theta)}\right)\left(\sum_{k=0}^{n} |a_k|e^{-i(t_k+k\theta)}\right) \\ &= \sum_{k=0}^{n} a_k^2 + \sum_{0\le i<k\le n} \left(\left(a_j a_k (e^{i(t_j-t_k+j-k)\theta} + e^{-i(t_j+t_k+j-k)\theta}\right)\right). \end{aligned}$$

That is,

$$|p(e^{i\theta})|^2 = \sum_{k=0}^{n} a_k^2 + 2 \sum_{0\le i<k\le n} \left(a_j a_k \cos((t_j - t_k + j - k)\theta)\right).$$

That is,

$$|p(e^{i\theta})|^2 \le \sum_{k=0}^{n} a_k^2 + 2 \sum_{0\le j<k\le n} |a_j a_k| = \left(\sum_{k=0}^{n} |a_k|\right)^2.$$

By Maximum modulus principle,

$$\max_{|z|\le 1} |p(z)| = \max_{0\le\theta<2\pi} |p(e^{i\theta})| \le \sum_{k=0}^{n} |a_k|$$

Moreover if all $a_1, a_2, ..., a_n$ have same argument then $t_j - t_k = 0$ for all $j, k = 0, 1, 2, ..., n$ and $|p(0)| = \sum_{k=0}^{n} |a_k|$.
If C is a constant such that $|C| \le |p(z)|$ on $|z| = 1$ then

$$\int_{|z|=1} |C|^2 \le \int_{|z|=1} |p(z)|^2 = 2\pi \sum_{k=1}^{n} a_k^2 + \sum_{0\le j<k\le n} a_j a_k \int_0^{2\pi} \cos((t_j - t_k + j - k)\theta) d\theta.$$

Since $\int_0^{2\pi} \cos((t_j - t_k + j - k)\theta) d\theta = 0, \ \ |C| \le \sqrt{\sum_{k=0}^{n} a_k^2}$. □

These above lemmas are useful to find bounds on $|a_n|$ for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in a subclass of the class of bi-univalent functions Σ. Once we obtain these bounds in general, then it is possible that to find an extremal functions of that subclass of Σ.

Definition 2.5. Principle of subordination: Let f and g be two functions from $\mathbb{U}$ to $\mathbb{C}$, f is said to be subordinated by g, denoted by $f \prec g$ if

$$f(\mathbb{U}) \subseteq g(\mathbb{U})$$

Subordination principle says that, there exist Schwarz function u such that $u(0) = 0$ and

$$f(z) = g(u(z)) \quad \text{for all} \ \ z \in \mathbb{U}.$$

3 Two classes: $\Sigma^G_{\alpha,\beta,\gamma}$ and $\Sigma^G_{\lambda,\mu,\nu,\delta}$

Let $\alpha, \beta, \gamma, \mu, \nu, \delta$ be real numbers. Denote

$$\mathcal{A} = \{f : \mathbb{U} \longrightarrow \mathbb{C} \colon f \text{ is analytic function on } \mathbb{U} \text{ satisfying } f(0) = 0,\ f'(0) = 1\}.$$

Let $\mathcal{S}$ denote the subclass of functions of $\mathcal{A}$ which are univalent(ie. one-one). As each $f \in \mathcal{S}$ is one-one, f^{-1} exist but it may not be defined on $\mathbb{U}$, but defined at least on $\left\{z \in \mathbb{C} \colon |z| < \frac{1}{4}\right\}$.
(By Kobe-quarter theorem, for each $f \in \mathcal{S}$, $\left\{z \in \mathbb{C} \colon |z| < \frac{1}{4}\right\} \subseteq f(\mathbb{U})$.)

Let $\Sigma = \left\{f \in \mathcal{S} \colon f^{-1} \in \mathcal{S}\right\}$. This is called as class of bi-univalent functions on $\mathbb{U}$.

For any function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the class Σ, define two functions F_1 and F_2 as below:

$$F_1(f)(z) = \frac{(1-\alpha)f(z) + \alpha z f'(z) + \beta z^2 f''(z)}{(1-\gamma)f(z) + \gamma z f'(z)},$$

and

$$F_2(f)(z) = \lambda \left(\frac{f(z)}{z}\right)^{\mu} \left(\frac{zf'(z)}{f(z)}\right)^{\nu} + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\delta}.$$

Let $(p_n = p_n(x))_{n=1}^{\infty}$ be sequence of real valued polynomial functions of a real variable x and

$$G(x,z) = \sum_{n=1}^{\infty} p_n(x) z^{n-1}$$

be the generating function. Now we define two subclasses of Σ as below:

Definition 3.1. The class $\Sigma^G_{\alpha,\beta,\gamma}$ of bi-univalent functions defined as

$$\Sigma^G_{\alpha,\beta,\gamma} = \left\{ f \in \Sigma \colon F_1(f)(z) \prec G(x,z) + 1 - a \text{ and } F_1(f^{-1})(z) \prec G(x,z) + 1 - a \right\}$$

Definition 3.2. The class $\Sigma^G_{\lambda,\mu,\nu,\delta}$ of bi-univalent functions defined as

$$\Sigma^G_{\lambda,\mu,\nu,\delta} = \left\{ f \in \Sigma \colon F_2(f)(z) \prec G(x,z) + 1 - a \text{ and } F_2(f^{-1})(z) \prec G(x,z) + 1 - a \right\}.$$

In this paper, we find bounds on $|a_n|'s$ for $n = 1,2,3,4,5.$ for the functions in following classes $\Sigma^G_{\alpha,\beta,\gamma}$ and $\Sigma^G_{\lambda,\mu,\nu,\delta}$.

Remark 3.3. Note that if we put values $0,1$ to parameters $\alpha,\beta,\gamma,\mu,\nu,\delta$, we get various perticular cases of the classes in deinitions (5) and (6). These paticular cases are as:

(i) $\Sigma^G_{\alpha,\beta,0} = \left\{ f \colon (1-\alpha) + \alpha z \dfrac{f'(z)}{f(z)} + \beta z^2 \dfrac{f''(z)}{f(z)} \prec G(x,z) + 1 - a \right\}.$

(ii) $\Sigma^G_{\alpha,\beta,1} = \left\{ f \colon (1-\alpha) \dfrac{f(z)}{zf'(z)} + \alpha + \beta z \dfrac{f''(z)}{f'(z)} \prec G(x,z) + 1 - a \right\}.$

(iii) $\Sigma^G_{1,0,0} = \left\{ f \colon z \dfrac{f'(z)}{f(z)} \prec G(x,z) + 1 - a \right\}.$

(iv) $\Sigma^G_{1,1,1} = \left\{ f \colon 1 + z \dfrac{f''(z)}{f'(z)} \prec G(x,z) + 1 - a \right\}.$

(v) $\Sigma^G_{1,\beta,\gamma} = \left\{ f \colon \dfrac{zf'(z) + \beta z^2 f''(z)}{(1-\gamma)f(z) + \gamma z f'(z)} \prec G(x,z) + 1 - a \right\}.$

(vi) $\Sigma^G_{0,\beta,\gamma} = \left\{ f \colon \dfrac{f(z) + \beta z^2 f''(z)}{(1-\gamma)f(z) + \gamma z f'(z)} \prec G(x,z) + 1 - a \right\}.$

(vii) $\Sigma^G_{1,\mu,\nu,\delta} = \left\{ f \colon \left(\dfrac{f(z)}{z}\right)^{\mu} \left(\dfrac{zf'(z)}{f(z)}\right)^{\nu} \prec G(x,z) + 1 - a \right\}.$

Let μ be any real number. Many authers give bounds on $|a_2|$, $|a_3 - \mu a_2^2|$ for a function f in these and similar type of classes.(see [7],[14],[17], [19],[20]). We found bounds on $|a_1|, |a_2|, |a_3|, |a_4|, |a_5|$, which are given in results of next sections.

4 Bounds on coefficients of functions in the class $\Sigma^G_{\alpha,\beta,\gamma}$

We begin with

Theorem 4.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \Sigma^G_{\alpha,\beta,\gamma}$, *then*

(i) $|a_2| \leq \min \left\{ \left| \dfrac{A_2 p_2^3}{\left(4A_2A_3^2 - 2(\gamma+1)A_3\right) p_2^2 + 2A_2A_3p_3} \right|, \dfrac{|p_2|}{|A_2|} \right\},$

(ii) $|a_3 - \mu a_2^2| \leq \left| B_1 - \mu \dfrac{p_2^2}{A_2^2} \right| + |B_2|,$

(iii) $\left|a_4-5\mu(a_2a_3-a_2^3)\right| \leq \sqrt{2}|\mu+1|\sqrt{C_2^2+C_3^2}+\sqrt{2C_2^2+(\mu^2+1)C_1^2}+|\mu-2||C_1,$

(iv) $\left|a_5-\mu\left(3a_3^2+6a_2a_4-21a_2^2a_3+14a_2^4\right)\right|$
$\leq(|\mu|+|\mu-1|)\left(\sqrt{2}\,|D_5|+\sqrt{2}\,|D_4|+\sqrt{6}\,|D_3|+\sqrt{2}\,|D_2|\right)+\sqrt{(1-2\mu)^2D_1^2+D_6^2}$

where $B_1, B_2, C_1, C_2, C_3, C_4, D_1, D_2, D_3, D_4, D_5, D_6$ *are given by (4.2,4.3, 4.4.)*

Proof Let

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\Sigma_{\alpha,\beta,\gamma}^{G}.$$

Consider f^{-1} into its Taylor's series,

$$f^{-1}(w)$$
$$=w+\sum_{n=2}^{\infty}c_nw^n$$
$$=w-a_2w^2+(2a_2^2-a_3)w^3+(-a_4+5a_2a_3-5a_2^3)w^4+(-a_5+3a_3^2+6a_2a_4-21a_2^2a_3+14a_2^4)w^5+o(w^6$$

By definition of class $\Sigma_{\alpha,\beta,\gamma}^{G}$ and subordination principle, there exist Schwarz functions $u(z)=\sum_{n=1}^{\infty}u_nz^n$ and $v(w)=\sum_{n=1}^{\infty}v_nw^n$ such that

$$F_1(f)(z)=G(x,u(z))\ \text{ and }\ F_1(f^{-1})(w)=G(x,v(w))\quad\text{for all }\ z,w\in\mathbb{U}.$$

On substituting series expressions of f and f^{-1} in above equations, we get

$$\frac{(1-\alpha)\left(z+\sum_{n=2}^{\infty}a_nz^n\right)+\alpha\left(z+\sum_{n=2}^{\infty}a_nnz^n\right)+\beta\sum_{n=2}^{\infty}a_nn\,(n-1)\,z^n}{(1-\gamma)\left(z+\sum_{n=2}^{\infty}a_nz^n\right)+\gamma\left(z+\sum_{n=2}^{n}a_nnz^n\right)},$$
$$=\sum_{n=1}^{\infty}\left(p_n(x)\left(\sum_{n=1}^{\infty}u_nz^n\right)^{n-1}\right)+1-a,$$

and

$$\frac{(1-\alpha)\left(z+\sum_{n=2}^{\infty}c_nz^n\right)+\alpha\left(z+\sum_{n=2}^{\infty}c_nnz^n\right)+\beta\sum_{n=2}^{\infty}c_nn\,(n-1)\,z^n}{(1-\gamma)\left(z+\sum_{n=2}^{\infty}c_nz^n\right)+\gamma\left(z+\sum_{n=2}^{n}c_nnz^n\right)},$$
$$=\sum_{n=1}^{\infty}\left(p_n(x)\left(\sum_{n=1}^{\infty}v_nz^n\right)^{n-1}\right)+1-a.$$

Let

$$A_n=\alpha+n\beta-\gamma\quad\text{for all}\quad n=1,2,3,4...$$

If we expand both sides of above equations into Taylor's series axpansion and equate like powers of z, on both sides we get that

$$p_1(x) = a.$$

$$a_2 = \frac{p_2}{A_2}u_1, \ \ c_2 = -a_2 = \frac{p_2}{A_2}v_1, \tag{4.1}$$

$$a_3 = B_1u_1^2 + B_2u_2, \ \ c_3 = 2a_2^2 - a_3 = B_1v_1^2 + B_2v_2, \tag{4.2}$$

$$\text{Where} \ \ B_1 = \frac{(\gamma+1)p_2^2 + A_2p_3}{2A_2A_3}, \ \ B_2 = \frac{p_2}{2A_3}$$

$$a_4 = C_1u_1^3 + C_2u_1u_3 + C_3u_3, \ \ \ c_4 = -a_4 + 5a_2a_3 - 5a_2^3 = C_1v_1^3 + C_2v_1v_3 + C_3v_3, \tag{4.3}$$

$$\text{Where} \ \ C_1 = \frac{(2\gamma^2+3\gamma+1)p_2^3 + (7A_2 + 2(\gamma+1)\beta))p_3p_2 + 2A_2A_3p_4}{6A_2A_3A_4},$$

$$C_2 = \frac{2(2A_2+\beta)\gamma + 3A_2}{6A_2A_3A_4}, \ \ C_3 = \frac{1}{3A_4},$$

$$a_5 = D_1u_1^4 + D_2u_1^2u_2 + D_3u_2^2 + D_4u_1u_3 + D_5u_4 + D_6u_1$$

$$c_5 = -a_5 + 3a_3^2 + 6a_2a_4 - 21a_2^2a_3 + 14a_2^4$$

$$= D_1v_1^4 + D_2v_1^2v_2 + D_3v_2^2 + D_4v_1v_3 + D_5v_4 + D_6v_1. \tag{4.4}$$

$$\text{Where } D_1 = \frac{B}{8A_2^2A_3A_5}, D_2 = \frac{C}{8A_2^2A_3A_5}, D_3 = \frac{D}{8A_2^2A_3A_5},$$

$$D_4 = \frac{E}{8A_2^2A_3A_5}, D_5 = \frac{F}{8A_2^2A_3A_5}, D_6 = \frac{G}{8A_2^2A_3A_5}.$$

Also the real parameters B, C, D, G are given as below

$$B = \left(-4(A_2+\gamma)(A_3+\gamma) + 10\gamma^3 + 6(\beta+3A_2)\gamma^2 + 4A_2(\beta+3A_2)\gamma\right)p_5$$
$$- \left(\gamma^3 + 5\gamma^2 + 4\gamma + 1\right)p_2^4 - 2A_3\left(\gamma^2+2\gamma+1\right)p_2^2p_3 + (2\gamma+1)A_2^2p_3^2$$

$$C = \left(-2(\gamma+1)^2A_3 + 8\gamma^2\right)p_2^3 + \left(8\gamma^3 + (16\beta + 8\alpha + 8A_2)\gamma^2 + 4\gamma A_2^2 + 2A_2^2\right)p_3p_2$$
$$+ \left(-4\gamma^3 + (-6\beta - 14A_2 + 2A_3)\gamma^2 - 12\left(\beta + \frac{4}{3}A_2 - 1/3A_3\right)A_2\gamma + 2A_2^2A_3\right)p_4.$$

$$D = 6{A_2}^2\left(\left(\frac{1}{3}{p_2}^2 - \frac{1}{3}p_3\right)\gamma + \frac{1}{6}{p_2}^2 + \frac{1}{3}p_3(A_3+\gamma)\right), E = 8A_2^2A_3p_3,$$

$$F = 6{A_2}^2\left(-\frac{1}{3}p_2\gamma + \frac{1}{3}(A_3+\gamma)p_2\right)$$

$$G = 4\left(-3\gamma^2 + 3\left(A_3 + \gamma - \frac{2}{3}\right)\gamma + (7\beta + 2\alpha)\right)A_2A_3p_2.$$

From equations (4.1) we get

$$u_1 = -v_1, \ a_2^2 = \frac{p_2^2}{2A_2^2}(u_1^2+v_1^2). \ \ \text{Therefore} \ \ |a_2| \le \frac{|p_2(x)|}{|A_2|}.$$

Adding equations in (4.2), we get

$$a_2^2 = \frac{B_1}{2}(u_1^2+v_1^2) + \frac{B_2}{2}(u_2+v_2) = B_1u_1^2 + \frac{B_2}{2}(u_2+v_2). \ \text{Hence} \ \ a_2^2 = B_1\frac{a_2^2}{p_2^2} + \frac{B_2}{2}(u_2+v_2).$$

If we solve the above equation for a_2, we get

$$a_2^2 = \frac{B_2p_2^2(u_2+v_2)}{2(p_2^2 - B_1)} = \frac{p_2^3}{4A_3\left(p_2^2 - \dfrac{(\gamma+1)p_2^2 + A_2p_3}{2A_2A_3}\right)} = \frac{A_2p_2^3}{\left(4A_2A_3^2 - 2(\gamma+1)A_3\right)p_2^2 + 2A_2A_3p_3}.$$

$$\text{Hence} \ \ |a_2| \le \min\left\{\left|\frac{A_2p_2^3}{\left(4A_2A_3^2 - 2(\gamma+1)A_3\right)p_2^2 + 2A_2A_3p_3}\right|, \frac{|p_2|}{|A_2|}\right\}.$$

Let μ be any real number. Using equations (4.2), we get

$$a_3 - \mu a_2^2 = B_1 u_1^2 + B_2 u_2 - \mu a_2^2$$

Now by lemma (2.1), $|u_2| \leq 1 - |u_1^2|,\ |v_2| \leq 1 - |v_1^2|$. So we can find real numbers x_2, y_2 such that $u_2 = (1-u_1^2)x_2,\ \ v_2 = (1-u_1^2)y_2$.
Therefore $a_3 - \mu a_2^2 = B_1 u_1^2 + B_2 x_2 (1-u_1^2) - \mu \frac{p_2^2}{A_2^2} u_1^2$.
That is, $a_3 - \mu a_2^2 = \left(B_1 - \mu \frac{p_2^2}{A_2^2}\right) u_1^2 + B_2 x_2 (1-u_1^2)$.
By triangle inequality, we get $\left|a_3 - \mu a_2^2\right| \leq \left|B_1 - \mu \dfrac{p_2^2}{A_2^2}\right| + |B_2|$.
Now subtracting equations in (4.3), we get

$$5a_2 a_3 - 5a_2^3 = 2C_1 u_1^3 + C_2 u_1 (v_3 - u_3) + C_3 (v_3 - u_3).$$

Also for any real number μ, again from (4.3), we get

$$a_4 - 5\mu(a_2 a_3 - a_2^3) = C_1(2-\mu)u_1^3 + C_2 u_1 (u_3 - \mu(v_3 - u_3)) + C_3 (u_3 - \mu(v_3 - u_3)).$$

Using lemma (2.1), we can write $u_3 = (1-u_1^2)x_3,\ v_3 = (1-u_1^2)y_3$ with $|x_3| \leq 1, |y_3| \leq 1$.
Therefore,

$$\begin{aligned} & a_4 - 5\mu(a_2 a_3 - a_2^3) \\ &= \left(-C_2 u_1^3 - C_3 u_1^2 + C_2 u_1 + C_3\right)(\mu+1)\, x_3 + \left(C_2 u_1^3 + C_3 \mu u_1^2 - C_2 u_1 - C_3 \mu\right) y_3 \\ &\quad + (-\mu+2)\, C_1 u_1^3. \end{aligned}$$

By using triangle inequality, we get

$$\begin{aligned} & \left|a_4 - 5\mu(a_2 a_3 - a_2^3)\right| \\ & \leq \left|\left(-C_2 u_1^3 - C_3 u_1^2 + C_2 u_1 + C_3\right)(\mu+1)\right| + \left|\left(C_2 u_1^3 + C_3 \mu u_1^2 - C_2 u_1 - C_3 \mu\right)\right| \\ & + \left|(-\mu+2)\, B_1\right|. \end{aligned}$$

Further using lemma (2.2), we have

$$\left|a_4 - 5\mu(a_2 a_3 - a_2^3)\right| \leq \sqrt{2}|\mu+1|\sqrt{C_2^2 + C_3^2} + \sqrt{2C_2^2 + (\mu^2+1)C_1^2} + |2-\mu||C_1|.$$

Now from equation (4.4)

$$\begin{aligned} & 3a_3^2 + 6a_2 a_4 - 21 a_2^2 a_3 + 14 a_2^4 \\ & = D_1 v_1^4 + D_1 u_1^4 + D_2 v_1^2 v_2 + D_2 u_1^2 u_2 + D_3 v_2^2 + D_3 u_2^2 + D_4 v_1 v_3 + D_4 u_1 u_3 + D_5 v_4 + D_5 u_4. \end{aligned}$$

Using Lemma (2.1) we can find numbers such that $|x_i| \leq 1,\ \ |y_i| \leq 1$ and $u_i = (1-u_1^2)x_i$, $v_i = (1-u_1^2)y_i$ for all $i = 2, 3, 4.$ and from the last equation we get

$$\begin{aligned} & 3a_3^2 + 6a_2 a_4 - 21 a_2^2 a_3 + 14 a_2^4 \\ & = 2D_1 u_1^4 - D_2 u_1^4 y_2 + D_2 u_1^2 y_2 + D_3 u_1^4 y_2^2 - 2D_3 u_1^2 y_2^2 + D_3 y_2^2 + D_4 u_1^3 y_3 - D_4 u_1 y_3 - D_5 u_1^2 y_4 \\ & + D_5 y_4 + D_2 u_1^2 \left(-{u_1}^2 + 1\right) x_2 + D_3 \left(-u_1^2 + 1\right)^2 x_2^2 + D_4 u_1 \left(-u_1^2 + 1\right) x_3 + D_5 \left(-u_1^2 + 1\right) x_4. \end{aligned}$$

Hence we have

$$\begin{aligned} a_5 - \mu b = & \left(\mu D_5 u_1^2 - \mu D_5\right) y_4 + \left(-\mu D_4 u_1^3 + \mu D_4 u_1\right) y_3 \\ & + \left(-\mu D_3 u_1^4 + 2\mu D_3 u_1^2 - \mu D_3\right) y_2^2 + \left(\mu D_2 u_1^4 - \mu D_2 u_1^2\right) y_2 + \left(\mu D_5 u_1^2 - D_5 u_1^2 - \mu D_5 + D_5\right) x_4 \\ & + \left(\mu D_4 u_1^3 - D_4 u_1^3 - \mu D_4 u_1 + D_4 u_1\right) x_3 + \left(-\mu D_3 u_1^4 + D_3 u_1^4 + 2\mu D_3 u_1^2 - 2D_3 u_1^2 - \mu D_3 + D_3\right) x_2^2 \\ & + \left(\mu D_2 u_1^4 - D_2 u_1^4 - \mu D_2 u_1^2 + D_2 u_1^2\right) x_2 - 2\mu D_1 u_1^4 + D_6 u_1 + D_1 u_1^4. \end{aligned}$$

By triangle inequality we have,

$$\begin{aligned}|a_5-\mu b| &\le \left|\mu D_5u_1^2-\mu D_5\right|+\left|-\mu D_4u_1^3+\mu D_4u_1\right| \\ &+\left|-\mu D_3u_1^4+2\mu D_3u_1^2-\mu D_3\right|+\left|\mu D_2u_1^4-\mu D_2u_1^2\right|+\left|\mu D_5u_1^2-D_5u_1^2-\mu D_5+D_5\right| \\ &+\left|\mu D_4u_1^3-D_4u_1^3-\mu D_4u_1+D_4u_1\right|+\left|-\mu D_3u_1^4+D_3u_1^4+2\mu D_3u_1^2-2D_3u_1^2-\mu D_3+D_3\right| \\ &+\left|\mu D_2u_1^4-D_2u_1^4-\mu D_2u_1^2+D_2u_1^2\right|+\left|(1-2\mu)D_1u_1^4+D_6u_1\right|.\end{aligned}$$

Therefore, by lemma (2.4) we get,

$$\begin{aligned}|a_5-\mu b| &\le \sqrt{2}\,|\mu D_5|+\sqrt{2}\,|\mu D_4|+\sqrt{6}\,|\mu D_3|+\sqrt{2}\,|\mu D_2|+\sqrt{2}\,|(\mu-1)D_5| \\ &+\sqrt{2}\,|(\mu-1)D_4|+\sqrt{6}\,|(\mu-1)D_3|+\sqrt{2}\,|(\mu-1)D_2|+\sqrt{(1-2\mu)^2D_1^2+D_6^2}.\end{aligned}$$

□

Following corollaries give bounds on absolute values of coefficients of functions in various subclasses of the class $\Sigma^G_{\alpha,\beta,\gamma}$. These subclasses are obtained by just taking values 0, 1 to parameters α,β,γ.

Corollary 4.2. *If $f\in\Sigma^G_{\alpha,\beta,0}$ then*

(i) $|a_2|\le \min\left\{\dfrac{p_2}{\alpha+2\beta},\dfrac{1}{2}\dfrac{(\alpha+2\beta)\,p_2^3}{(\alpha+3\beta)\left(2\alpha^2+10\alpha\beta+p_3\alpha+12\beta^2+2p_3\beta-p_2^2\right)}\right\},$

(ii) $\left|a_3-\mu a_2^2\right|\le\left|\dfrac{1}{2}\dfrac{p_2^2+(\alpha+2\beta)\,p_3}{(\alpha+2\beta)\,(\alpha+3\beta)}-\dfrac{\mu p_2^2}{(\alpha+2\beta)^2}\right|+\dfrac{1}{2}\left|\dfrac{p_2}{\alpha+3\beta}\right|,$

(iii)
$$\begin{aligned}\left|a_4-5\mu(a_2a_3-a_2^3)\right| &\le \frac{1}{3\sqrt{2}}\,|\mu+1|\,\sqrt{\frac{(3\alpha+6\beta)^2}{(\alpha+2\beta)^2(\alpha+3\beta)^2(\alpha+4\beta)^2}+\frac{4}{(\alpha+4\beta)^2}} \\ &+\frac{1}{6}\sqrt{2\frac{(3\alpha+6\beta)^2}{(\alpha+2\beta)^2(\alpha+3\beta)^2(\alpha+4\beta)^2}\hat{}+\frac{(\mu^2+1)\left(p_2^3+(7\alpha+16\beta)\,p_2p_3+(2\alpha+4\beta)\,(\alpha+3\beta)\,p_4\right)^2}{(\alpha+2\beta)^2(\alpha+3\beta)^2(\alpha+4\beta)^2}} \\ &+\frac{1}{6}\,|\mu-2|\left|\frac{p_2^3+(7\alpha+16\beta)\,p_2p_3+(2\alpha+4\beta)\,(\alpha+3\beta)\,p_4}{(\alpha+2\beta)\,(\alpha+3\beta)\,(\alpha+4\beta)}\right|,\end{aligned}$$

(iv)
$$\begin{aligned}&\left|a_5-\mu\left(3a_3^2+6a_2a_4-21a_2^2a_3+14a_2^4\right)\right| \\ &\le(|\mu|+|\mu+1|)\left(\frac{\sqrt{3}}{\sqrt{2}}\sqrt{\frac{(\alpha/3+\beta)\,p_2}{(\alpha+3\beta)\,(\alpha+5\beta)}}+\sqrt{\frac{p_3}{\alpha+5\,\beta}}+\frac{3}{\sqrt{2}}\sqrt{\frac{1/6p_2{}^2+(\alpha/3+\beta)\,p_3}{(\alpha+3\beta)\,(\alpha+5\beta)}}\right) \\ &+(|\mu|+|\mu+1|)\left(\frac{1}{2}\sqrt{\frac{(-2\,\alpha-6\,\beta)\,p_2^3+2\,(\alpha+2\beta)^2\,p_3p_2+2\,(\alpha+2\beta)^2\,(\alpha+3\beta)\,p_4}{(\alpha+2\beta)^2\,(\alpha+3\beta)\,(\alpha+5\beta)}}\right)\end{aligned}$$

Corollary 4.3. *If $f\in\Sigma^G_{\alpha,\beta,1}$ then*

(i) $|a_2|\le$
$$\min\left\{\frac{1/2\,(\alpha+2\,\beta-1)\,p_2{}^3}{(\alpha+3\,\beta-1)\,(p_3(\alpha+2\beta-1)-2\,p_2{}^2+2\alpha^2+12\beta^2+10\alpha\beta-4\alpha-10\beta+2)},\frac{p_2}{\alpha+2\,\beta-1}\right\},$$

(ii) $\left|a_3-\mu a_2^2\right|$
$$\le\left|\frac{2\,p_2{}^2+(\alpha+2\,\beta-1)\,p_3}{2\,(\alpha+2\,\beta-1)\,(\alpha+3\,\beta-1)}-\frac{\mu\,p_2{}^2}{(\alpha+2\,\beta-1)^2}\right|+\left|\frac{p_2}{2(\alpha+3\beta-1)}\right|,$$

(iii) $\left|a_4-5\mu(a_2a_3-a_2^3)\right|$
$$\le\frac{1}{3\sqrt{2}}\,|\mu+1|$$
$$\sqrt{\frac{(7\alpha+16\,\beta-7)^2}{(\alpha+2\,\beta-1)^2(\alpha+3\,\beta-1)^2(\alpha+4\,\beta-1)^2}+\frac{4}{(\alpha+4\,\beta-1)^2}}+$$

$$\sqrt{\frac{2\left(7\,\alpha+16\,\beta-7\right)^{2}+\left(\mu^{2}+1\right)\left(6\,{p_{2}}^{3}+\left(7\,\alpha+18\,\beta-7\right)p_{2}\,p_{3}+\left(2\,\alpha+4\,\beta-2\right)\left(\alpha+3\,\beta-1\right)p_{4}\right)^{2}}{36\left(\alpha+2\,\beta-1\right)^{2}\left(\alpha+3\,\beta-1\right)^{2}\left(\alpha+4\,\beta-1\right)^{2}}}$$
$$+\frac{1}{6}\left|\mu-2\right|\left|\frac{6\,{p_{2}}^{3}+\left(7\,\alpha+18\,\beta-7\right)p_{2}\,p_{3}+\left(2\,\alpha+4\,\beta-2\right)\left(\alpha+3\,\beta-1\right)p_{4}}{\left(\alpha+2\,\beta-1\right)\left(\alpha+3\,\beta-1\right)\left(\alpha+4\,\beta-1\right)}\right|,$$

(iv) $\left|a_5-\mu\left(3a_3^2+6a_2a_4-21a_2^2a_3+14a_2^4\right)\right|$
$$\leq 1/2\,\left(|\mu|+|\mu+1|\right)\sqrt{6}\sqrt{\frac{p_2}{3\,\alpha+15\,\beta-3}}$$
$$+1/2\left(|\mu|+|\mu+1|\right)\sqrt{\frac{T}{\left(\alpha+2\,\beta-1\right)^{2}\left(\alpha+3\,\beta-1\right)\left(\alpha+5\,\beta-1\right)}}$$
$$+\left(|\mu|+|\mu+1|\right)\left(\sqrt{2}\sqrt{\frac{p_3}{\alpha+5\,\beta-1}}+1/2\,\sqrt{2}\sqrt{\frac{3\,\left(6\,\beta+2\,\alpha-2\right)p_3+9\,{p_2}^{2}}{\left(\alpha+3\,\beta-1\right)\left(\alpha+5\,\beta-1\right)}}\right),$$

Where
$$T=\left(6\,\alpha^{2}+24\,\beta\,\alpha+24\,\beta^{2}+4\,\alpha+8\,\beta+6\right)p_2\,p_3$$
$$+\left(2\alpha^{3}+14\alpha^{2}\beta+32\,\alpha\,\beta^{2}+24\beta^{3}-18\alpha^{2}-84\,\beta\,\alpha-96\beta^{2}+18\alpha+42\beta-6\right)p_4$$
$$+\left(-8\,\alpha-24\,\beta+16\right){p_2}^{3}.$$

Corollary 4.4. *If* $f\in\Sigma^{G}_{1,\beta,1}$ *then*

(i) $|a_2|\leq\min\left\{\frac{\beta\,{p_2}^{3}}{36\,\beta^{3}+6\,\beta^{2}p_3-6\,\beta\,{p_2}^{2}},\frac{p_2}{2\beta}\right\},$

(ii) $\left|a_3-\mu a_2^2\right|\leq\left|\frac{\beta\,p_3+{p_2}^{2}}{6\beta^{2}}-\frac{\mu\,{p_2}^{2}}{4\beta^{2}}\right|+\left|\frac{p_2}{6\beta}\right|,$

(iii) $\left|a_4-5\mu(a_2a_3-a_2^3)\right|\leq\frac{1}{18\sqrt{2}}\left|\mu+1\right|\sqrt{\frac{16}{\beta^{4}}+\frac{9}{\beta^{2}}}$
$$+\frac{1}{144}\sqrt{\frac{512}{\beta^{4}}+\frac{\left(\mu^{2}+1\right)\left(12\,\beta^{2}p_4+18\,\beta\,p_2\,p_3+6\,{p_2}^{3}\right)^{2}}{\beta^{6}}}+\frac{|\mu-2|}{144}\left|\frac{12\,\beta^{2}p_4+18\,\beta\,p_2\,p_3+6\,{p_2}^{3}}{\beta^{3}}\right|,$$

(iv) $\left|a_5-\mu\left(3a_3^2+6a_2a_4-21a_2^2a_3+14a_2^4\right)\right|$
$$\leq\frac{\left(|\mu|+|\mu+1|\right)\left(3\,\sqrt{6}\sqrt{2\,p_3\,\beta+{p_2}^{2}}\beta+\sqrt{6}\sqrt{W}+12\,\beta^{3/2}\sqrt{p_3}+6\,\beta^{3/2}\sqrt{p_2}\right)\sqrt{10}}{60\beta^{2}}$$
Where $W=6\,\beta^{3}p_4+6\,\beta^{2}p_2\,p_3-6\,{p_2}^{3}\beta-16\,\beta^{2}p_4+8\,\beta\,p_2\,p_3+2\,{p_2}^{3}-7\,\beta\,p_4+4\,p_3\,p_2-p_4.$

Corollary 4.5. *If* $f\in\Sigma^{G}_{1,\beta,0}$ *then*

(i) $|a_2|\leq\min\left\{\frac{2\left(1+2\beta\right){p_2}^{3}}{\left(1+3\beta\right)\left(12\beta^{2}+2\beta p_3-{p_2}^{2}+10\beta+p_3+2\right)}\frac{p_2}{1+2\beta}\right\},$

(ii) $\left|a_3-\mu a_2^2\right|\leq\left|\frac{{p_2}^{2}+\left(1+2\beta\right)p_3}{2\left(1+2\beta\right)\left(1+3\beta\right)}-\frac{\mu{p_2}^{2}}{\left(1+2\beta\right)^{2}}\right|+\left|\frac{p_2}{2+6\beta}\right|,$

(iii) $\left|a_4-5\mu(a_2a_3-a_2^3)\right|\leq\frac{|\mu+1|}{\sqrt{3}}\sqrt{\frac{\left(3+6\,\beta\right)^{2}}{\left(1+2\beta\right)^{2}\left(1+3\beta\right)^{2}\left(1+4\beta\right)^{2}}+\frac{4}{\left(1+4\beta\right)^{2}}}$
$$+\frac{1}{6}\sqrt{\frac{2\left(3+6\,\beta\right)^{2}}{\left(1+2\beta\right)^{2}\left(1+3\beta\right)^{2}\left(1+4\,\beta\right)^{2}}+\frac{\left(\mu^{2}+1\right)\left({p_2}^{3}+\left(7+16\beta\right)p_2p_3+\left(2+4\beta\right)\left(1+3\beta\right)p_4\right)^{2}}{\left(1+2\beta\right)^{2}\left(1+3\beta\right)^{2}\left(1+4\beta\right)^{2}}}$$
$$+\frac{1}{6}\left|\mu-2\right|\left|\frac{{p_2}^{3}+\left(7+16\beta\right)p_2p_3+\left(2+4\beta\right)\left(1+3\beta\right)p_4}{\left(1+2\beta\right)\left(1+3\beta\right)\left(1+4\beta\right)}\right|,$$

(iv) $\left|a_5-\mu\left(3a_3^2+6a_2a_4-21a_2^2a_3+14a_2^4\right)\right|$
$$\leq\left(|\mu|+|\mu+1|\right)\sqrt{2}\left(1/2\,\sqrt{3}\sqrt{\frac{\left(1/3+\beta\right)p_2}{\left(1+3\,\beta\right)\left(1+5\,\beta\right)}}+\sqrt{\frac{p_3}{1+5\,\beta}}+3/2\,\sqrt{\frac{1/6\,{p_2}^{2}+\left(1/3+\beta\right)p_3}{\left(1+3\,\beta\right)\left(1+5\,\beta\right)}}\right)$$

$$+(|\mu|+|\mu+1|)\sqrt{2}\left(1/4\sqrt{2}\sqrt{\frac{(-2-6\beta)p_2{}^3+2(1+2\beta)^2p_3p_2+2(1+2\beta)^2(1+3\beta)p_4}{(1+2\beta)^2(1+3\beta)(1+5\beta)}}\right).$$

Corollary 4.6. *If* $f\in\Sigma_{1,1,1}^G$ *then*

(i) $|a_2|\le\min\left\{\frac{p_2{}^3}{-6p_2{}^2+6p_3+36},\frac{p_2}{2}\right\}$,

(ii) $|a_3-\mu a_2^2|\le\left|-\frac{1}{6}p_2{}^2-\frac{p_3}{6}+\frac{1}{4}\mu p_2{}^2\right|+\frac{1}{6}|p_2|$,

(iii) $|a_4-5\mu(a_2a_3-a_2^3)|\le,$

(iv) $|a_5-\mu(3a_3^2+6a_2a_4-21a_2^2a_3+14a_2^4)|$

$$\le\frac{5\sqrt{2}|\mu+1|}{36}+\frac{\sqrt{(p_2{}^3+3p_2p_3+2p_4)^2\mu^2+p_2{}^6+6p_2{}^4p_3+4p_2{}^3p_4+9p_2{}^2p_3{}^2+12p_2p_3p_4+4p_4{}^2+128/9}}{24}$$
$$+\frac{1}{6}|\mu-2|\left|\frac{1}{4}p_2{}^3+\frac{3}{4}p_2p_3+\frac{p_4}{2}\right|.$$

Corollary 4.7. *If* $f\in\Sigma_{1,0,0}^G$ *then*

(i) $|a_2|\le\min\left\{\frac{p_2{}^3}{-2p_2{}^2+2p_3+4},p_2\right\}$,

(ii) $|a_3-\mu a_2^2|\le\left|-\frac{1}{2}p_2{}^2-\frac{p_3}{2}+\mu p_2^2\right|+\frac{1}{2}|p_2|$,

(iii) $|a_4-5\mu(a_2a_3-a_2^3)|$

$$\le\frac{\sqrt{13}}{\sqrt{3}}|\mu+1|+\frac{1}{6}\sqrt{18+(\mu^2+1)(p_2{}^3+7p_2p_3+2p_4)^2}+\frac{1}{6}|\mu-2|\left|p_2{}^3+7p_2p_3+2p_4\right|,$$

(iv) $|a_5-\mu(3a_3^2+6a_2a_4-21a_2^2a_3+14a_2^4)|$

$$\le(|\mu|+|\mu+1|)\left(\sqrt{p_2/2}+\sqrt{2p_3}+\frac{\sqrt{3}}{2}\sqrt{p_2{}^2+2p_3}+\frac{1}{\sqrt{2}}\sqrt{-p_2{}^3+p_2p_3+p_4}\right).$$

5 Bounds on coefficients of functions in the class $\Sigma_{\lambda,\mu,\nu,\delta}^G$

Let μ,ν,δ,λ be real parameters .For any analytic function f let us recall the definition of operator F_2 as,

$$F_2(f)(z)=\lambda\left(\frac{f(z)}{z}\right)^\mu\cdot\left(\frac{zf'(z)}{f(z)}\right)^\nu+(1-\lambda)\left(1+\frac{zf''(z)}{f'(z)}\right)^\delta. \quad (5.1)$$

Also recall the definition.

Definition 5.1. The class $\Sigma_{\lambda,\mu,\nu,\delta}^G$ of bi-univalent functions defined as

$$\Sigma_{\lambda,\mu,\nu,\delta}^G=\left\{f\in\Sigma\colon F_2(f)(z)\prec G(x,z)+1-a \text{ and } F_2(f^{-1})(z)\prec G(x,z)+1-a\right\}.$$

Let $f\in\Sigma_{\lambda,\mu,\nu,\delta}^G$. Hence there exist Schwartz functions $u(z)=\sum_{n=1}^{\infty}u_nz^n$ and $v(w):\sum_{n=1}^{\infty}v_nw^n$ form $\mathbb{U}$ to $\mathbb{U}$ such that $u(0)=0,v(0)=0$ and

$$F_2(f(z))=G(x,u(z))+1-a,\quad F_2(f^{-1}(w))=G(x,v(w))+1-a\quad\text{for all } z,w\in\mathbb{U} \quad (5.2)$$

On substituting series expansions of f and f^{-1},

$$f(z) = 1 + \sum_{n=2}^{\infty} a_n z^n, \quad f^{-1}(w) = 1 + \sum_{n=1}^{\infty} c_n w^n$$

in left hand side of equation (2) we get that

$$F(z) = \lambda \left(1 + \sum_{i=2}^{\infty} a_i z^{i-1}\right)^{\mu} \left(\frac{z + \sum_{i=2}^{\infty} i a_i z^i}{z + \sum_{i=2}^{\infty} a_i z^i}\right)^{\nu} + (1-\lambda)\left(1 + \frac{\sum_{i=1}^{\infty} (i+1)\, i a_{i+1} z^i}{1 + \sum_{i=1}^{\infty} (i+1)\, a_{i+1} z^i}\right)^{\delta}$$

and

$$F(w) = \lambda \left(1 + \sum_{i=2}^{\infty} c_i z^{i-1}\right)^{\mu} \left(\frac{z + \sum_{i=2}^{\infty} i c_i z^i}{z + \sum_{i=2}^{\infty} c_i z^i}\right)^{\nu} + (1-\lambda)\left(1 + \frac{\sum_{i=1}^{\infty} (i+1)\, i c_{i+1} z^i}{1 + \sum_{i=1}^{\infty} (i+1)\, c_{i+1} z^i}\right)^{\delta}.$$

Further expansion of above expression into power series in z, we have

$$\begin{aligned} F(z) =& 1 + A_{11} a_2 z + \left(A_{21}\, {a_2}^2 + A_{22}\, a_3\right) z^2 + \left(A_{31}\, {a_2}^3 + A_{32}\, a_2\, a_3 + A_{33}\, a_4\right) z^3 \\ &+ \left(A_{41}\, {a_2}^4 + A_{42}\, {a_2}^2 a_3 + A_{43}\, {a_3}^2 + A_{44}\, a_2\, a_4 + A_{45}\, a_5\right) z^4 \\ &+ \left(A_{51}\, {a_2}^5 + A_{52}\, {a_2}^3 a_3 + A_{53}\, a_2\, {a_3}^2 + A_{54}\, {a_2}^2 a_4 + A_{55}\, a_3\, a_4 + A_{56}\, a_2\, a_5 + A_{57}\, a_6\right) z^5 + O(z^6). \end{aligned}$$

$$\begin{aligned} F(z) =& 1 + A_{11} c_2 w + \left(A_{21}\, {a_2}^2 + A_{22}\, c_3\right) w^2 + \left(A_{31}\, {c_2}^3 + A_{32}\, c_2\, c_3 + A_{33}\, c_4\right) w^3 \\ &+ \left(A_{41}\, {c_2}^4 + A_{42}\, {c_2}^2 c_3 + A_{43}\, {c_3}^2 + A_{44}\, c_2\, c_4 + A_{45}\, c_5\right) w^4 \\ &+ \left(A_{51}\, {c_2}^5 + A_{52}\, {c_2}^3 c_3 + A_{53}\, c_2\, {a_3}^2 + A_{54}\, {c_2}^2 c_4 + A_{55}\, c_3\, c_4 + A_{56}\, c_2\, c_5 + A_{57}\, c_6\right) w^5 + O(w^6). \end{aligned}$$

Also, we can write the power series expession for $G(x, u(z))$ and $G(x, v(w))$ as below:

$$\begin{aligned} G(x, u(z)) =& \sum_{n=1}^{\infty} \left(p_n \left(\sum_{n=1}^{\infty} u_n z^n\right)^{n-1}\right) \\ =& p_1 + 1 - a + p_2 u_1 z + \left(p_3 {u_1}^2 + p_2 u_2\right) z^2 + \left(p_4 {u_1}^3 + 2\, p_3 u_1 u_2 + p_2 u_3\right) z^3 \\ &+ \left({u_1}^4 p_5 + 3\, {u_1}^2 p_4 u_2 + 2\, u_1 p_3 u_3 + p_3 {u_2}^2 + p_2 u_4\right) z^4 \\ &+ \left({u_1}^5 p_6 + p_2 u_5 + 2\, u_1 p_3 u_4 + 3\, {u_1}^2 p_4 u_3 + 3\, u_1 p_4 {u_2}^2 + 4\, {u_1}^3 p_5 u_2 + 2\, p_3 u_2 u_3\right) z^5 + O(z^6) \end{aligned}$$

and

$$\begin{aligned} G(x, v(w)) =& \sum_{n=1}^{\infty} \left(p_n \left(\sum_{n=1}^{\infty} u_n z^n\right)^{n}\right) \\ =& (p_1 + 1 - a + p_2 v_1 w + \left(p_3 {v_1}^2 + p_2 v_2\right) w^2 + \left(p_4 {v_1}^3 + 2\, p_3 v_1 v_2 + p_2 v_3\right) w^3 \\ &+ \left({v_1}^4 p_5 + 3\, {v_1}^2 p_4 v_2 + 2\, v_1 p_3 v_3 + p_3 {v_2}^2 + p_2 v_4\right) w^4 \\ &+ \left(p_2 v_5 + 2\, v_1 p_3 v_4 + 3\, {v_1}^2 p_4 v_3 + 3\, v_1 p_4 {v_2}^2 + 4\, {v_1}^3 p_5 v_2 + {v_1}^5 p_6 + 2\, p_3 v_2 v_3\right) w^5 + O(w^6). \end{aligned}$$

Equating coefficients of like powers of z, of power series expansion of left hand side and right hand side of equation (2) ,we get the equations for coefficients $a_n's$

$$1 = p_1 + 1 - a, \quad A_{11}a_2 = p_2u_1, \quad A_{21}a_2^2 + A_{22}a_3 = p_3u_1^2 + p_2u_2, \tag{5.3}$$

$$A_{31}a_2^3 + A_{32}a_2a_3 + A_{33}a_4 = p_4u_1^3 + 2p_3u_1u_2 + p_2u_3, \tag{5.4}$$

$$A_{41}a_2^4 + A_{42}a_2^2a_3 + A_{43}a_3^2 + A_{44}a_2a_4 + A_{45}a_5$$
$$= u_1^4p_5 + 3u_1^2p_4u_2 + 2u_1p_3u_3 + p_3u_2^2 + p_2u_4, \tag{5.5}$$

$$A_{51}a_2^5 + A_{52}a_2^3a_3 + A_{53}a_2a_3^2 + A_{54}a_2^2a_4 + A_{55}a_3a_4 + A_{56}a_2a_5 + A_{57}a_6,$$
$$= u_1^5p_6 + p_2u_5 + 2u_1p_3u_4 + 3u_1^2p_4u_3 + 3u_1p_4u_2^2 + 4u_1^3p_5u_2 + 2p_3u_2u_3. \tag{5.6}$$

Solving these equations for a_2, a_3, a_4, a_5 we get

$$a_2 = \frac{p_2}{A_{11}}u_1 = Bu_1. \tag{5.7}$$

$$a_3 = \frac{p_2u_2}{A_{22}} + \frac{p_3u_1^2}{A_{22}} - \frac{A_{21}p_2^2}{A_{11}^2A_{22}} = Cu_2 + Du_1^2 - E. \tag{5.8}$$

$$a_4 = \frac{p_4u_1^3}{A_{33}} - \frac{A_{32}\,p_2p_3u_1^2}{A_{11}A_{22}\,A_{33}} + \frac{p_2u_3}{A_{33}} - \frac{A_{32}p_2^2u_2}{A_{11}A_{22}A_{33}} + \frac{A_{21}\,A_{32}p_2^3 - A_{22}A_{31}p_2^3}{A_{11}^3A_{22}A_{33}}.$$

$$ie., a_4 = Fu_1^3 - Gu_1^2 + Hu_3 - Iu_2 + J. \tag{5.9}$$

Where

$$B = \frac{p_2}{A_{11}}, C = \frac{p_2}{A_{22}}, D = \frac{p_3}{A_{22}}, E = \frac{A_{21}p_2^2}{A_{11}^2A_{22}}, F = \frac{p_4}{A_{33}}, G = \frac{A_{32}\,p_2p_3}{A_{11}A_{22}\,A_{33}},$$
$$H = \frac{p_2}{A_{33}}, I = \frac{A_{32}p_2^2}{A_{11}A_{22}A_{33}}, J = \frac{A_{21}\,A_{32}p_2^3 - A_{22}A_{31}p_2^3}{A_{11}^3A_{22}A_{33}}.$$

$$La_5 = Mu_1^4 - Nu_1^3 + Ou_1^2u_2 - Pu_1^2 - Qu_1u_2 + Ru_1\,u_3 + Su_2^2 + Tu_2 - Uu_3 + Vu_4 + W \tag{5.10}$$

Similarly, we can find expressions for coeffinets $c_n's$ of f^{-1} and below:

$$c_2 = Bv_1. \tag{5.11}$$

$$c_3 = Cv_2 + Dv_1^2 - E. \tag{5.12}$$

$$c_4 = Fv_1^3 - Gv_1^2 + Hv_3 - Iv_2 + J. \tag{5.13}$$

$$Lc_5 = Mv_1^4 - Nv_1^3 + Ov_1^2v_2 - Pv_1^2 - Qv_1v_2 + Rv_1v_3 + Sv_2^2 + Tv_2 - Uv_3 + Vv_4 + W. \tag{5.14}$$

Where

$$L = A_{11}^4A_{22}^2A_{33}A_{45}, \quad M = A_{11}^4A_{22}^2A_{33}p_5 - A_{11}^4A_{33}A_{43}p_3^2, \quad N = A_{44}p_2A_{22}^2A_{11}^3p_4,$$
$$O = p_4A_{11}^4A_{22}^2A_{33} - 2A_{11}^4A_{33}A_{43}p_3p_2,$$
$$P = 2A_{11}^2A_{21}A_{33}A_{43}p_3p_2^2 + A_{44}p_2^2A_{22}A_{11}^2A_{32}p_3 - A_{11}^2A_{22}A_{33}A_{42}p_3p_2^2,$$
$$Q = 2A_{11}^3A_{22}^2A_{44}p_2p_3, \quad R = 2A_{11}^4A_{22}^2A_{33}p_3, \quad S = A_{11}^4A_{22}^2A_{33}p_3 - A_{11}^4A_{33}A_{43}p_2^2,$$
$$T = 2A_{11}^2A_{21}A_{33}A_{43}p_2^3 + A_{11}^2A_{22}A_{32}A_{44}p_2^3 - A_{11}^2A_{22}A_{33}A_{42}p_2^3,$$
$$U = A_{44}p_2^2A_{22}^2A_{11}^3, \quad V = A_{11}^4A_{22}^2A_{33}p_2,$$
$$W = A_{41}p_2^4A_{22}^2A_{33} + A_{21}A_{22}A_{33}A_{42}p_2^4 - A_{21}^2A_{33}A_{43}p_2^4 - A_{44}p_2^4A_{22}A_{21}A_{32} + A_{44}p_2^4A_{22}^2A_{31}.$$

The constants A_{ij} are functions of parameters $\lambda, \mu, \nu, \delta$ which are given as below:

$$
\begin{aligned}
A_{11} =& (\mu+\nu-2\,\delta)\,\lambda+2\delta,\\
A_{21} =& \frac{1}{2}\left(-4\,\delta^2+\mu^2+12\,\delta+(2\nu-1)\,\mu+\nu^2-3\,\nu\right)\lambda+2\,\delta\,\left(\delta-3\right),\\
A_{22} =& (\mu+2\nu-6\delta)\,\lambda+6\delta,\\
A_{31} =& \lambda\left(\frac{7}{3}\nu-\frac{3}{2}\nu^2+\frac{1}{6}\nu^3\right)+\lambda\mu\left(\frac{1}{2}\nu^2-\frac{3}{2}\nu\right)\\
&+\lambda\left(\frac{1}{2}\mu^2-\frac{\mu}{2}\right)\nu+\lambda\left(\frac{\mu}{3}-\frac{1}{2}\mu^2+\frac{1}{6}\mu^3\right)\\
&+(1-\lambda)\left(\frac{4}{3}\delta^3-12\,\delta^2+\frac{56\delta}{3},\right),\\
A_{32} =& \lambda\left(2\,\nu^2-5\,\nu\right)+3\,\lambda\,\mu\,\nu+\lambda\,\left(\mu^2-\mu\right)+(1-\lambda)\left(12\,\delta^2-30\,\delta\right)\\
A_{33} =& 3\lambda\mu+12\,(1-\lambda)\,\delta,\\
A_{41} =& \lambda\left(-\frac{15\nu}{4}+\frac{83\nu^2}{24}-\frac{3}{4}\nu^3+\frac{1}{24}\nu^4\right)+\lambda\mu\left(7/3\nu-3/2\nu^2+1/6\,\nu^3\right)\\
&+\lambda\,\left(1/2\,\mu^2-\mu/2\right)\left(1/2\,\nu^2-3/2\,\nu\right)+\lambda\,\left(\mu/3-1/2\,\mu^2+1/6\,\mu^3\right)\nu\\
&+\lambda\,\left(-\mu/4+\frac{11\,\mu^2}{24}-1/4\,\mu^3+1/24\,\mu^4\right)\\
&+(1-\lambda)\left(-60\,\delta+\frac{166\,\delta^2}{3}-12\,\delta^3+2/3\,\delta^4\right),\\
A_{42} =& \lambda\left(\nu^3-8\,\nu^2+11\,\nu\right)+\lambda\,\mu\,\left(2\,\nu^2-5\,\nu\right)+\lambda\,\mu\,\left(1/2\,\nu^2-3/2\,\nu\right)+2\,\lambda\,\left(1/2\,\mu^2-\mu/2\right)\nu\\
&+\lambda\,\left(\mu^2-\mu\right)\nu+\lambda\,\left(\mu-3/2\,\mu^2+1/2\,\mu^3\right)+(1-\lambda)\left(12\,\delta^3-96\,\delta^2+132\,\delta\right),\\
A_{43} =& \lambda\left(2\,\nu^2-4\,\nu\right)+2\,\lambda\,\mu\,\nu+\lambda\,\left(1/2\,\mu^2-\mu/2\right)+(1-\lambda)\left(18\,\delta^2-36\,\delta\right),\\
A_{44} =& \lambda\left(3\,\nu^2-7\,\nu\right)+4\,\lambda\,\mu\,\nu+\lambda\,\left(\mu^2-\mu\right)+(1-\lambda)\left(24\,\delta^2-56\,\delta\right),\\
A_{45} =& 4\lambda\nu+\lambda\mu+20\,(1-\lambda)\,\delta,\\
A_{51} =& \lambda\,\left(\frac{31\,\nu}{5}-\frac{29\,\nu^2}{4}+\frac{55\,\nu^3}{24}-1/4\,\nu^4+\frac{\nu^5}{120}\right)+\lambda\,\mu\,\left(-\frac{15\,\nu}{4}+\frac{83\,\nu^2}{24}-3/4\,\nu^3+1/24\,\nu^4\right)\\
&+\lambda\,\left(1/2\,\mu^2-\mu/2\right)\left(7/3\,\nu-3/2\,\nu^2+1/6\,\nu^3\right)+\lambda\,\left(\mu/3-1/2\,\mu^2+1/6\,\mu^3\right)\left(1/2\,\nu^2-3/2\,\nu\right)\\
&+\lambda\,\left(-\mu/4+\frac{11\,\mu^2}{24}-1/4\,\mu^3+1/24\,\mu^4\right)\nu+\lambda\,\left(\frac{\mu^5}{120}+\mu/5-\frac{5\,\mu^2}{12}+\frac{7\,\mu^3}{24}-1/12\,\mu^4\right)\\
&+(1-\lambda)\left(\frac{992\,\delta}{5}-232\,\delta^2+\frac{220\,\delta^3}{3}-8\,\delta^4+\frac{4\delta^5}{15}\right),
\end{aligned}
$$

$$
\begin{aligned}
A_{52} =& \lambda\,\left(-23\,\nu+\frac{139\,\nu^2}{6}-11/2\,\nu^3+1/3\,\nu^4\right)+\lambda\,\mu\,\left(\nu^3-8\,\nu^2+11\,\nu\right)\\
&+\lambda\,\mu\,\left(7/3\,\nu-3/2\,\nu^2+1/6\,\nu^3\right)+\lambda\,\left(1/2\,\mu^2-\mu/2\right)\left(2\,\nu^2-5\,\nu\right)\\
&+\lambda\,\left(\mu^2-\mu\right)\left(1/2\,\nu^2-3/2\,\nu\right)+2\,\lambda\,\left(\mu/3-1/2\,\mu^2+1/6\,\mu^3\right)\nu\\
&+\lambda\,\left(\mu-3/2\,\mu^2+1/2\,\mu^3\right)\nu+\lambda\,\left(-\mu+\frac{11\,\mu^2}{6}-\mu^3+1/6\,\mu^4\right)\\
&+(1-\lambda)\left(8\,\delta^4-132\,\delta^3+556\,\delta^2-552\,\delta\right),\\
A_{53} =& \lambda\,\left(2\,\nu^3-14\,\nu^2+17\,\nu\right)+\lambda\,\mu\,\left(2\,\nu^2-4\,\nu\right)+\lambda\,\mu\,\left(2\,\nu^2-5\,\nu\right)+2\,\lambda\,\left(\mu^2-\mu\right)\nu\\
&+\lambda\,\left(1/2\,\mu^2-\mu/2\right)\nu+\lambda\,\left(\mu-3/2\,\mu^2+1/2\,\mu^3\right)+(1-\lambda)\left(36\,\delta^3-252\,\delta^2+306\,\delta\right),
\end{aligned}
$$

$$\begin{aligned} A_{54} =&\lambda\ \left(15\,\nu - 23/2\,\nu^2 + 3/2\,\nu^3\right) + \lambda\,\mu\ \left(3\,\nu^2 - 7\,\nu\right) + 3\,\lambda\ \left(1/2\,\mu^2 - \mu/2\right)\nu \\ &+ \lambda\,\mu\ \left(1/2\,\nu^2 - 3/2\,\nu\right) + \lambda\ \left(\mu^2 - \mu\right)\nu + \lambda\ \left(\mu - 3/2\,\mu^2 + 1/2\,\mu^3\right) \\ &+ (1-\lambda)\left(24\,\delta^3 - 184\,\delta^2 + 240\,\delta\right) \\ A_{55} =&\lambda\ \left(6\,\nu^2 - 11\,\nu\right) + 5\,\lambda\,\mu\,\nu + \lambda\ \left(\mu^2 - \mu\right) + (1-\lambda)\left(72\,\delta^2 - 132\,\delta\right), \\ A_{56} =&\lambda\ \left(4\,\nu^2 - 9\,\nu\right) + 5\,\lambda\,\mu\,\nu + \lambda\ \left(\mu^2 - \mu\right) + (1-\lambda)\,, \left(40\,\delta^2 - 90\,\delta\right) \\ A_{57} =&5\,\lambda\,\nu + \lambda\,\mu + 30\ (1-\lambda)\,\delta. \end{aligned}$$

Theorem 5.2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C^G_{\lambda,\mu,\nu}$ *then*

(i) $|a_2| \leq \min\{|B|, \sqrt{|E| + |C|}\}$,

(ii) $|a_3 - \mu a_2^2| \leq \frac{1}{2}|\mu C| + |(1 - \frac{\mu}{2})C| + \sqrt{D^2 + (\mu-1)^2 E^2}$,

(iii) $\left|a_4 - 5\mu a_2(a_3 - a_2^2)\right| \leq (|\mu| + |\mu - 1|)(|I| + |H|) + \sqrt{(2\mu - 1)(G^2 + J^2) + F^2}$,

(iv) $|La_5 - \mu(3a_3^2 + 6a_2a_4 - 21a_2^2a_3 + 14a_2^4)| \leq (|\mu - 1| + |\mu|)\left(\sqrt{O^2 + Q^2 + T^2} + \sqrt{R^2 + U^2} + |S| + |V|\right)$
$+ \sqrt{(2\mu - 1)^2(M^2 + P^2 + W^2) + N^2}$

Where the constants $A, B, C, D, E, G, H, I, J$ *are given just after equation (5.9) and the constants* L, O, Q, T, U, W *as given just after equation (5.14).*

Proof Since $f(z) = 1 + \sum_{n=2}^{\infty} a_n z^n$, $f^{-1}(w) = 1 + \sum_{n=2}^{\infty} c_n w^n$, and $f(f^{-1})(w) = w$ for all $w \in \mathbb{U}$, we have

$$c_2 = -a_2. \tag{5.15}$$

$$c_3 = 2a_2^2 - a_3. \tag{5.16}$$

$$c_4 = -a_4 + 5a_2a_3 - 5a_2^3. \tag{5.17}$$

$$c_5 = -a_5 + 3a_3^2 + 6a_2a_4 - 21a_2^2a_3 + 14a_2^4. \tag{5.18}$$

From equation (5.7) we get that $|a_2| \leq |B|$.
From equations (5.10) and (5.14) we get $u_1 = -v_1$.
Adding equations (5.8) and (5.12) we get, $2a_2^2 = C(u_2 + v_2) - 2E$.
From lemma (2.1), we get $|u_n| \leq 1 - |u_1|^2 \leq |1 - u_1^2|$.
Therefore we can write $u_n = (1 - u_1^2)x_n$ with $|x_n| \leq 1$ for all $n = 2, 3, 4....$
Therefore we have $2a_2^2 = C(x_2 + y_2)(1 - u_1^2) - 2E$.
By triangle inequality, we get $|a_2^2| \leq |C| + |E|$.
Hence $|a_2| \leq \min\{|B|, \sqrt{|E| + |C|}\}$.
Using $a_2^2 = \frac{C}{2}(x_2 + y_2)(1 - u_1^2) - E$, $u_2 = (1 - u_1^2)x_2$, $v_2 = (1 - u_1^2)y_2$ with $|x_2| \leq 1$, $|y_2| \leq 1$ and from expression of a_3 in equation (5.8) we get that

$$a_3 - \mu a_2^2 = \frac{1}{2}\mu C\left(u_1^2 - 1\right)y_2 + C\left(\left(-1 + \frac{\mu}{2}\right)u_1^2 + 1 - \frac{\mu}{2}\right)x_2 + Du_1^2 + (\mu - 1)E.$$

Hence lemma (2.1) gives that

$$|a_3 - \mu a_2^2| \leq \frac{1}{2}|\mu C| + |(1 - \frac{\mu}{2})C| + \sqrt{D^2 + (\mu - 1)^2 E^2}.$$

Now subtracting the equation (5.9) from the equation (5.12), and using equation (5.16) and lemma (2.1), we get

$$5a_2a_3 - 5a_2^3 = 2Fu_1^3 - 2Gu_1^2 + (H(x_3 + y_3) - I(x_2 + y_2)\left(-u_1{}^2 + 1\right) + 2J.$$

Multiplying above equation by μ and subtracting it from the equation (5.9), we get that

$$a_4 - 5\mu a_2(a_3 - a_2^2) = k_2\, x_2 + k_3\, x_3 + j_2\, y_2 + j_3\, y_3 + F{u_1}^3 + (2\,\mu - 1)\, G{u_1}^2 - (2\,\mu - 1)\, J.$$

Where $k_2 = -I(\mu-1)(u_1-1)(u_1+1), k_3 = H(\mu-1)(u_1-1)(u_1+1),$
$j_2 = -I\mu(u_1-1)(u_1+1), j_3 = H\mu(u_1-1)(u_1+1)$ and μ is any real number.

Therefore by lemma (2.4) and a triangle inequality, we get that

$$\begin{aligned}&\left|a_4 - 5\mu a_2(a_3 - a_2^2)\right|\\ &\leq |k_2| + |k_3| + |j_2| + |j_3| + |F| + |\,(2\mu-1)\, G| + |\,(2\mu-1)\, J|.\\ &\leq (|\mu| + |\mu-1|)(|I| + |H|) + \sqrt{(2\mu-1)(G^2+J^2) + F^2}.\end{aligned}$$

Subtracting La_5 from the equation (5.14), we get

$$\begin{aligned}&3a_3^2 + 6a_2a_4 - 21a_2^2a_3 + 14a_2^4\\ &= V\left(-u_1^2+\right)(x_4+y_4) + \left(Ru_1^3 + Uu_1^2 - Ru_1 - U\right)y_3 + \left(-Ru_1^3 + Uu_1^2 + Ru_1 - U\right)x_3\\ &+ \left(Su_1^4 - 2Su_1^2 + S\right)y_2^2 + \left(-Ou_1^4 - Qu_1^3 + Ou_1^2 - T{u_1}^2 + Qu_1 + T\right)y_2 + 2W + 2Mu_1^4 - 2Pu_1^2\\ &+ \left(Su_1^4 - 2Su_1^2 + S\right)x_2^2 + \left(-Ou_1^4 + Qu_1^3 + Ou_1^2 - Tu_1^2 - Qu_1 + T\right)x_2.\end{aligned}$$

Let μ be any real number. Multiplying above equation by μ and subtracting it from the equation (5.10), we get

$$\begin{aligned}&La_5 - \mu(3a_3^2 + 6a_2a_4 - 21a_2^2a_3 + 14a_2^4)\\ &= k_1x_2 + k_2x_2^2 + k_3x_3 + k_4x_4 + k_5y_2 + k_6y_2^2 + k_7y_3 + k_8x_4 + k_9.\end{aligned}$$

Where $k_1 = (u_1^2-1)(\mu-1)(Ou_1^2 - Qu_1 + T), k_2 = -S(u_1^2-1)^2(\mu-1), k_3 = (u_1^2-1)(\mu-1)(Ru_1 - U),$
$k_4 = V(u_1^2-1)(\mu-1), k_5 = \mu(u_1^2-1)(Ou_1^2 + Qu_1 + T), k_6 = -\mu S(u_1^2-1)^2,$
$k_7 = -\mu(u_1^2-1)(Ru_1+U), k_8 = \mu V(u_1^2-1), k_9 = -u_1^2(2\mu-1)(Mu_1^2 - P) - Nu_1^3 - W(2\mu-1).$

Finally by lemma (2.1), we get

$$|La_5 - \mu(3a_3^2 + 6a_2a_4 - 21a_2^2a_3 + 14a_2^4)| \leq |k_1| + |k_2| + |k_3| + |k_4| + |k_5| + |k_6| + |k_7| + |k_8| + |k_9|$$

We note that by lemma (2.2) we will give bounds on $|k_1|, |k_2|, ..., |k_9|$ as

$$\begin{aligned}&|k_1| \leq |\mu-1|\sqrt{O^2+Q^2+T^2}, \quad |k_2| \leq |S||\mu-1|, \quad |k_3| \leq |\mu-1|\sqrt{R^2+U^2},\\ &|k_4| \leq |\mu-1||V|, \quad |k_5| \leq |\mu|\sqrt{O^2+Q^2+T^2}, \quad |k_6| \leq |\mu||S|, \quad |k_7| \leq |\mu|\sqrt{R^2+U^2},\\ &|k_8| \leq |\mu||V|, \quad |k_9| \leq \sqrt{(2\mu-1)^2(M^2+P^2+W^2)+N^2}.\end{aligned}$$

□

Corollary 5.3. *If $f \in \Sigma^G_{\mu,\nu,1,\delta}$ then*

(i) $|a_2|$

$$\leq \min\left\{\left|\frac{p_2}{\mu+\nu}\right|, \sqrt{\left|\frac{\left(-2\delta^2 + \frac{1}{2}\mu^2 + 6\delta + \left(\nu - \frac{1}{2}\right)\mu + \frac{1}{2}\nu^2 - \frac{3}{2}\nu + 2\delta\,(\delta-3)\right){p_2}^2}{(\mu+\nu)^2\,(\mu+2\,\nu)}\right| + \left|\frac{p_2}{\mu+2\nu}\right|}\right\},$$

(ii) $\left|a_3 - \mu a_2^2\right|$

$$\leq \frac{1}{2}\left|\frac{\mu\, p_2}{\mu+2\,\nu}\right| + \left|\frac{\left(1-\frac{\mu}{2}\right)p_2}{\mu+2\,\nu}\right| + \sqrt{\frac{{p_3}^2}{(\mu+2\nu)^2} + \frac{(\mu-1)^2\left(\frac{1}{2}\mu^2 + \left(\nu-\frac{1}{2}\right)\mu + \frac{1}{2}\nu^2 - \frac{3}{2}\nu\right)^2{p_2}^4}{(\mu+\nu)^4\,(\mu+2\,\nu)^2}},$$

(iii) $\left|a_4 - 5\mu(a_2a_3 - a_2^3)\right|$

$$\leq (|\mu| + |\mu - 1|)\left(\left|\frac{(\mu^2 + 3\mu\nu + 2\nu^2 - \mu - 5\nu)\, p_2{}^2}{(\mu+\nu)(\mu+2\nu)(\mu+3\nu)}\right| + \left|\frac{p_2}{\mu + 3\nu}\right|\right)$$
$$+ \sqrt{\frac{(2\,\mu - 1)\left(\mu^2 + 3\,\mu\,\nu + 2\,\nu^2 - \mu - 5\,\nu\right)^2 p_2{}^2 p_3{}^2}{(\mu+\nu)^2 (\mu + 2\,\nu)^2 (\mu + 3\nu)^2} + \frac{p_4{}^2}{(3\,\nu + \mu)^2}},$$

Corollary 5.4. *If* $f \in \Sigma^G_{\mu,\nu,0,\delta}$ *then*

(i) $|a_2| \leq \min\left\{\left|\frac{p_2}{2\delta}\right|,\ \frac{1}{6}\sqrt{3\left|\frac{(\delta-3)\,p_2{}^2}{\delta^2}\right| + 6\left|\frac{p_2}{\delta}\right|}\right\},$

(ii) $\left|a_3 - \mu a_2^2\right| \leq \frac{1}{12}\left|\frac{\mu p_2}{\delta}\right| + \frac{1}{6}\left|\frac{\left(1 - \frac{\mu}{2}\right) p_2}{\delta}\right| + \frac{1}{12}\sqrt{4\,\frac{p_3{}^2}{\delta^2} + \frac{(\mu-1)^2(\delta-3)^2 p_2{}^4}{\delta^4}},$

(iii) $\left|a_4 - 5\mu(a_2a_3 - a_2^3)\right|$

$$\leq (|\mu| + |\mu - 1|)\left(\frac{1}{144}\left|\frac{\left(12\,\delta^2 - 30\,\delta\right) p_2{}^2}{\delta^3}\right| + \left|\frac{p_2}{\delta}\right|\right) + \sqrt{\frac{(2\,\mu - 1)\left(12\,\delta^2 - 30\,\delta\right)^2 p_2{}^2 p_3{}^2}{20736\,\delta^6} + \frac{p_4{}^2}{144\,\delta^2}},$$

6 Concluding Remark

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathbb{H}(\mathbb{U})$ and $p_n(x)_{n=1}^{\infty}$ be sequence of orthogonal polynomails of real variable x. with generating function $G(x,z) = \sum_{n=1}^{\infty} p_n(x) z^n$. If $f \prec G(x,z)$ then we get upper bounds $b_n(x)$ such that $|a_n| \leq b_n(x)$. Study of the geometric properties of these polynomials $b_n(x)$ and their relation with the f explores more about subclasses of bi-univalent functions.

References

[1] S. Bernstein, Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle, Gauthier-Villars, Paris, 1926, *Reprint: Chelsea, New*, (1970).

[2] D. A. Brannan and J.G. Clunie, Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1–20, 1979,*Academic Press, New York and London*, (1980).

[3] D. A. Brannan and T.S. Taha, On some classes of bi-univalent functions, *Mathematical Analysis and Its Applications*, Elsevier 53–60, (1988).

[4] C. Carathéodory, Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen, *Mathematische Annalen*, **64**(1), 95–115,(1907).

[5] P. L. Duren,*Univalent functions*, Springer Science & Business Media,**259**, (2001).

[6] S. M. El-Deeb and B. M. El-Matary, Subclasses of analytic functions with respect to symmetric and conjugate points connected with q-analogue of the Bessel function, *Publications de l'Institut Mathematique*, **108**(122), 47–61, (2020).

[7] B. A. Frasin and M.K. Aouf, New subclasses of bi-univalent functions, *Applied Mathematics Letters*, **24**(9),1569–1573, (2011).

[8] R. B. Gardner, N. K Govil and S. R. Musukula, Rate of growth of polynomials not vanishing inside a circle, *Rn*,*Citeseer*, **1**(2), (2005).

[9] T. Horzum and E. G. Kocer, On some properties of horadam polynomials, *In Int. Math. Forum*, **4**, 1243–1252, (2009).

[10] F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proceedings of the American Mathematical Society*, **20**(1), 8–12, (1969).

[11] M. Lewin, On a coefficient problem for bi-univalent functions, *Proceedings of the American mathematical society*, **18**(1), 63–68, (1967).

[12] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, *Proceeding of the Conference on Complex Analysis, Z. Li, F. Ren, L. Yang and S. Zhang (Eds), Int. Press*, 157–169, (1994).

[13] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Archive for rational mechanics and analysis*,**32**(2),100–112, (1969).

[14] A. B. Patil and U. H. Naik, Estimates on initial coefficients of certain subclasses of bi-univalent functions associated with quasi-subordination, *Global J. Math. Anal*, **5**(1), 6–10, (2017).

[15] C. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht,Gottingen (1975).

[16] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Applied mathematics letters*, **23**(10), 1188–1192, (2010).

[17] H. M. Srivastava, A. Motamednezhad and S. Salehian. Coefficients of a comprehensive subclass of meromorphic bi-univalent functions associated with the faber polynomial expansion, *Axioms*, Multidisciplinary Digital Publishing Institute, **10**(1), 27, (2021).

[18] H. M. Srivastava, Ş. Altınkaya and S. Yalçın, Certain subclasses of bi-univalent functions associated with the horadam polynomials, *Iranian journal of science and technology, transactions A: Science*,**43**(4), 1873–1879, (2019).

[19] S. R. Swamy, S. Bulut and A. Sailaja, Some special families of holomorphic and Sălăgean type bi-univalent functions associated with Horadam polynomials involving a modified sigmoid activation function, *Hacettepe Journal of Mathematics and Statistics*, **50**(3),710–720.

[20] S. R. Swamy and Y. Sailaja, Special subfamilies of holomorphic and bi-univalent functions related to quasi-subordination,*Earthline Journal of Mathematical Sciences*, **6**(2), 225–237, (2021).

[21] T. S. Taha, Topics in univalent funtion theory. Ph.D. thesis, *King's College London University of London*, (1980).

[22] P. Turan, Ü ber die ableitung von polynomen, *Compositio Mathematica*, **7**, 89–95, (1940).

[23] A.K. Wanas, X. Y. Wang and S. Bulut, Applications of horadam polynomials for a new family of bi-univalent functions associating bazilevic functions with -pseudo functions.

[24] A. K. Wanas and S. Yalçın, Horadam polynomials and their applications tonew family of bi-univalent functions with respect to symmetric conjugate points, *Proyecciones (Antofagasta),Universidad Católica del Norte, Departamento de Matemáticas*, **40**(1), 107–116, (2021).

Author information

Krishnat D. Masalkar, Department of Mathematics, Abasaheb Garware College, Pune, India.
E-mail: krishnatmasalkar@gmail.com

Uday H.Naik, Department of Mathematics, Willingdon College, Sangali, India.
E-mail: udaynaik@gmail.com

Channayya Hiremath, Department of Mathematics, ASP College, Devrukh, India.
E-mail: dspmaspchannayya@gmail.com

Received: August 7th, 2021
Accepted: February 15th, 2022