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REGULARIZATION OF THE CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION ON A TWO-DIMENSIONAL BOUNDED DOMAIN

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Abstract In this paper, the problem of continuation of the ill-posed Cauchy problem solution's is studied for matrix factorizations of the Helmholtz equation in a two-dimensional bounded domains. It is assumed that the solution of the problem exists and also it is continuously differentiable in a closed domain with exactly given Cauchy data. In this case, an explicit formula for the continuation of the solution as well as a regularization formula are established, under the indicated conditions. Their continuous approximations with a given error in the uniform metric are given instead of the Cauchy data. Additionally a stability estimate is obtained for the solution of the Cauchy problem in the classical sense.

1 Introduction

Aim of the paper is to work on the construction of exact and approximate solutions to the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation. Such problems naturally arise in mathematical physics with in various fields of natural science (for example, electro-geological exploration, cardiology, electrodynamics, etc.). In general, the theory of ill-posed problems for elliptic systems of equations has been sufficiently formed using the workings of A.N. Tikhonov, V.K. Ivanov, M.M. Lavrent'ev, N.N. Tarkhanov and many other famous mathematicians. Among them, the most important ones are the so-called conditionally well-posed problems, characterized by stability in the presence of additional information about the nature of the problem data for applications. One of the most effective ways to study such problems is to construct regularizing operators. As an, this can be given as Carleman-type formulas (as in complex analysis) or iterative processes (the Kozlov-Maz'ya-Fomin algorithm, etc.).

This work is devoted to the main problem which is the Cauchy problem for partial differential equations. There are classes of equations for which this problem behaves well named hyperbolic equations. The main attention is paid to the regularization formulas for solutions of the Cauchy problem. The question of the existence of a solution to the problem is not considered but it is assumed a priori. At the same time, it should be noted that any regularization formula leads to an approximate solution of the Cauchy problem for all data, even if there is no solution in the usual classical sense. Moreover one can indicate in what sense the approximate solution turns out to be optimal for explicit regularization formulas. In this sense, exact regularization formulas are very useful for real numerical calculations. There is a good reason to hope that numerous practical applications of regularization formulas are still ahead.

This problem concerns ill-posed problems, i.e., it is unstable. It is known that the Cauchy problem for elliptic equations is unstable relatively small change in the data, it means that it is, incorrect (example Hadamard, see, for instance [24], p. 39). There is a sizable literature on the subject (see, e.g. [26], [27], [4] [36], [37] and [25]). N.N. Tarkhanov [32] published a criterion for the solvability of a larger class of boundary value problems for elliptic systems. In unstable problems, the image of the operator is not is closed. Therefore, the solvability condition can not be is written in the terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique. The problem is solvable for everywhere dense a set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s by the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see, for instance [33]).

While the uniqueness of the solution follows from Holmgren's general theorem (see [3]), the conditional stability of the problem follows from the work of A.N. Tikhonov (see [4]), if we restrict the class of possible solutions to a compactum.

We note that when solving applied problems, one should find the approximate values of $U(x)$ and $\dfrac{\partial U(x)}{\partial x_j},\ x\in G,\ j=1,2.$

In this paper, we construct a family of vector-functions $U(x,f_\delta)=U_{\sigma(\delta)}(x)$ and $\dfrac{\partial U(x,f_\delta)}{\partial x_j}=\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j},\ (j=1,2)$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma=\sigma(\delta)$, at $\delta\to 0$, the family $U_{\sigma(\delta)}(x)$ and $\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ converges in the usual sense to a solution $U(x)$ and its derivative $\dfrac{\partial U(x)}{\partial x_j},\ x\in G$ at a point $x\in G$.

Following A.N. Tikhonov (see [4]), a family of vector-valued functions $U_{\sigma(\delta)}(x)$ and $\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem.

Formulas that allow finding a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carleman type formulas. In [35], Carleman established a formula giving a solution to the Cauchy - Riemann equations in a domain of a special form. Developing his idea, G.M. Goluzin and V.I. Krylov [20] derived a formula for determining the values of analytic functions from data known only on a portion of the boundary, already for arbitrary domains. A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in (see [25]). A formula of the Carleman type, in which the fundamental solution of a differential operator with special properties (the Carleman function) is used, was obtained by M.M. Lavrent'ev (see, for instance [26]-[27]). By Using this method, Sh. Ya. Yarmukhamedov (see, for instance [36]-[39]) constructed the Carleman functions for the Laplace and Helmholtz operators for spatial domains of a special form, when the part of the boundary of the domain where the data is unknown is a conical surface or a hyper surface $\{x_3=0\}$. In [33], an integral formula was proved for systems of equations of elliptic type of the first order, with constant coefficients in a bounded domain. Using the methodology of works [36]-[39], Ikehata [28] considered the probe method and Carleman functions for the Laplace and Helmholtz equations in the three-dimensional domain. Considering exponentially growing solutions, Ikehata [29] obtained a formula for solving the Helmholtz equation with a variable coefficient for regions in space where the unknown data are located on a section of the hypersurface $\{x\cdot s=t\}$. Carleman type formulas for various elliptic equations and systems were also obtained in works [19], [20], [28]-[29], [5]-[18]. In [19] it was considered the Cauchy problem for the Helmholtz equation in an arbitrary bounded plane domain with Cauchy data, known only on the region boundary. The solvability criterion of the Cauchy problem for the Laplace equation in the space $\mathbb{R}^m$ was considered by Shlapunov in [1]. In work [21], the continuation of the problem for the Helmholtz equation was investigated and the results of numerical experiments are presented. The construction of the Carleman matrix for elliptic systems was carried out by: Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov, D.A. Juraev and others (see, for instance [36]-[39], [1]-[2], [22]-[23], [5]-[18]). The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain (see, for instance [33]).

In many well-posed problems for systems of equations of elliptic type of the first order with constant coefficients that factorize the Helmholtz operator, it is not possible to calculate the values of the vector function on the entire boundary. Therefore, the problem of reconstructing the solution of systems of equations of first order elliptic type with constant coefficients, factorizing the Helmholtz operator (see, for instance [5]-[18]), is one of the topical problems in the theory of differential equations.

For the last decades, interest in classical ill-posed problems of mathematical physics has remained. This direction in the study of the properties of solutions of the Cauchy problem for the Laplace equation was started in [26]-[27], [36]-[39] and subsequently developed in [19]-[20], [30]-[33], [28]-[29], [22]-[23], [1]-[2], [5]-[18].

Let $\mathbb{R}^2$ be a two-dimensional real Euclidean space,

$$x = (x_1, x_2) \in \mathbb{R}^2,\ y = (y_1, y_2) \in \mathbb{R}^2.$$

$G \subset \mathbb{R}^2$ is a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_2 = 0$ and some smooth curve S lying in the half-space $y_2 > 0$, i.e., $\partial G = S \bigcup T$.

We introduce the following notation:

$$r = |y-x|\,, \alpha = |y_1 - x_1|\,, w = i\sqrt{u^2+\alpha^2} + y_2,\ u \geq 0,$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^T, \frac{\partial}{\partial x} \to \xi^T, \xi^T = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ be a transposed vector } \xi,$$

$$U(x) = (U_1(x), ..., U_n(x))^T,\ u^0 = (1, ..., 1) \in \mathbb{R}^n,\ n = 2^m,\ m = 2,$$

$$E(z) = \left\| \begin{matrix} z_1 \ldots 0 \\ \ldots\ldots. \\ 0 \ldots z_n \end{matrix} \right\| - \text{diagonal matrix}, z = (z_1, ..., z_n) \in \mathbb{R}^n.$$

Let $D(\xi^T)$ be a $(n \times n)-$dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$ and λ is a real number.

We consider a system of differential equations in the region G

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{1.1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by class of vector functions by $A(G)-$ in the domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (1.1).

2 Construction of the Carleman matrix and the Cauchy problem

Formulation of the problem. Suppose, that $U(y) \in A(G)$ and

$$U(y)|_S = f(y),\ y \in S. \tag{2.1}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

If $U(y) \in A(G)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int\limits_{\partial G} N(y, x; \lambda)U(y)ds_y,\ \ x \in G, \tag{2.2}$$

where
$$N(y,\,x;\lambda)=\left(E\left(\varphi_2(\lambda r)u^0\right)D^*\left(\frac{\partial}{\partial x}\right)\right)D(t^T).$$

Also $t=(t_1,\,t_2)-$is the unit exterior normal, drawn at a point y, the curve ∂G and $\varphi_2(\lambda r)-$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$, where $\varphi_2(\lambda r)$ defined by the following formula (see [34]):
$$\varphi_2(\lambda r)=-\frac{i}{4}H_0^{(1)}(\lambda r). \tag{2.3}$$

It is defined that $K(w)$ is an entire function taking real values for real w, ($w=u+iv,\ \ u,v-$real numbers) and satisfying the following conditions:
$$K(u)\neq 0,\ \sup_{v\geq 1}\left|v^pK^{(p)}(w)\right|=B(u,p)<\infty,$$
$$-\infty<u<\infty,\ p=0,\,1,\,2. \tag{2.4}$$

Also, it can be defined the function $\Phi(y,\,x;\lambda)$ at $y\neq x$ by the following equality
$$\Phi(y,\,x;\lambda)=-\frac{1}{2\pi K(x_2)}\int\limits_0^\infty \operatorname{Im}\left[\frac{K(w)}{w-x_2}\right]\frac{u\,I_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du, \tag{2.5}$$
where $I_0(\lambda u)=J_0(i\lambda u)-$is the Bessel function of the first kind of zero order (see [3]).

In the formula (2.5), choosing
$$K(w)=\exp(\sigma w),\ K(x_2)=\exp(\sigma x_2),\ \sigma>0, \tag{2.6}$$
we get
$$\Phi_\sigma(y,\,x;\lambda)=-\frac{e^{-\sigma x_2}}{2\pi}\int\limits_0^\infty \operatorname{Im}\left[\frac{\exp(\sigma w)}{w-x_2}\right]\frac{u\,I_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du. \tag{2.7}$$

The formula (2.2) is true if we instead $\varphi_2(\lambda r)$ of substituting the function
$$\Phi_\sigma(y,\,x;\lambda)=\varphi_2(\lambda r)+g_\sigma(y,\,x;\lambda), \tag{2.8}$$
where $g_\sigma(y,\,x)-$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y=x$.

Then, the integral formula has the form:
$$U(x)=\int\limits_{\partial G}N_\sigma(y,\,x;\lambda)U(y)ds_y,\ \ x\in G, \tag{2.9}$$
where
$$N_\sigma(y,\,x;\lambda)=\left(E\left(\Phi_\sigma(y,\,x;\lambda)u^0\right)D^*\left(\frac{\partial}{\partial x}\right)\right)D(t^T).$$

3 The continuation formula and regularization according to M.M. Lavrent'ev's

Theorem 3.1. *Assume that $U(y)\in A(G)$ satisfies the inequality*
$$|U(y)|\leq 1,\ y\in T. \tag{3.1}$$

If
$$U_\sigma(x)=\int\limits_S N_\sigma(y,\,x;\lambda)U(y)ds_y,\ \ x\in G, \tag{3.2}$$

then the following estimates are true:

$$|U(x) - U_\sigma(x)| \le C(\lambda, x)\sigma e^{-\sigma x_2},\ \ \sigma > 1,\ \ x \in G, \tag{3.3}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j}\right| \le C(\lambda, x)\sigma e^{-\sigma x_2},\ \ \sigma > 1,\ \ x \in G,\ \ j = 1, 2. \tag{3.4}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C(\lambda, x)$.

Proof. Let us estimate inequality (3.3) firstly. Using the integral formula (2.9) and the equality (3.2), we obtain

$$U(x) = \int\limits_S N_\sigma(y, x; \lambda)U(y)ds_y + \int\limits_T N_\sigma(y, x; \lambda)U(y)ds_y =$$

$$= U_\sigma(x) + \int\limits_T N_\sigma(y, x; \lambda)U(y)ds_y,\ \ x \in G.$$

Taking into account the inequality (3.1), we estimate the following

$$|U(x) - U_\sigma(x)| \le \left|\int\limits_T N_\sigma(y, x; \lambda)U(y)ds_y\right| \le$$
$$\le \int\limits_T |N_\sigma(y, x; \lambda)|\,|U(y)|\,ds_y \le \int\limits_T |N_\sigma(y, x; \lambda)|\,ds_y,\ \ \ x \in G. \tag{3.5}$$

We estimate the integrals $\int\limits_T |\Phi_\sigma(y, x; \lambda)|\,ds_y$, $\int\limits_T \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_1}\right| ds_y$ and $\int\limits_T \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_2}\right| ds_y$ on the part T of the plane $y_2 = 0$.

Separating the imaginary part of (2.7), we get

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{\sigma(y_2 - x_2)}}{2\pi}\left[\int\limits_0^\infty \frac{\cos\sigma\sqrt{u^2 + \alpha^2}}{u^2 + r^2}\ uI_0(\lambda u)\,du - \right.$$
$$\left. - \int\limits_0^\infty \frac{(y_2 - x_2)\sin\sigma\sqrt{u^2 + \alpha^2}}{u^2 + r^2}\frac{uI_0(\lambda u)}{\sqrt{u^2 + \alpha^2}}du\right],\ \ x_2 > 0. \tag{3.6}$$

From (3.6) and the inequality

$$I_0(\lambda u) \le \sqrt{\frac{2}{\lambda\pi u}}, \tag{3.7}$$

we have

$$\int\limits_T |\Phi_\sigma(y, x; \lambda)|\,ds_y \le C(\lambda, x)\sigma e^{-\sigma x_2},\ \ \sigma > 1,\ \ x \in G, \tag{3.8}$$

To estimate the second integral, we use the equality

$$\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_1} = \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s}\frac{\partial s}{\partial y_1} = 2(y_1 - x_1)\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s},$$
$$s = \alpha^2. \tag{3.9}$$

Considering equality (3.6), inequality (3.7) and equality (3.9), we obtain

$$\int\limits_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_1} \right| ds_y \le C(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma > 1,\ x \in G, \tag{3.10}$$

Now, we estimate the integral $\int\limits_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_2} \right| ds_y$.

Taking into account equality (3.6) and inequality (3.7), we have

$$\int\limits_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_2} \right| ds_y \le C(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma > 1,\ x \in G, \tag{3.11}$$

From inequalities (3.7), (3.10) and (3.11), bearing in mind (3.5), we get an estimate (3.3).

Now let us prove inequality (3.4). To do this, we take the derivatives from equalities (2.9) and (3.2) with respect to $x_j,\ (j=1,2)$. Then, we obtain the following:

$$\begin{aligned} \frac{\partial U(x)}{\partial x_j} &= \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y + \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y, \\ \frac{\partial U_\sigma(x)}{\partial x_j} &= \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y, \quad x \in G,\ j=1,2. \end{aligned} \tag{3.12}$$

Taking into account the (3.12) and inequality (3.1), we estimate the following

$$\begin{aligned} \left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial_\sigma U(x)}{\partial x_j} \right| &\le \left| \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y \right| \le \\ \le \int\limits_T \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| |U(y)|\, ds_y &\le \int\limits_T \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| ds_y, \\ x \in G,\ j&=1,2. \end{aligned} \tag{3.13}$$

To do this, we estimate the integrals $\int\limits_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1} \right| ds_y$ and $\int\limits_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2} \right| ds_y$ on the part T of the plane $y_2 = 0$.

To estimate the first integrals, we use the equality

$$\begin{aligned} \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1} &= \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial s} \frac{\partial s}{\partial x_1} = -2(y_1 - x_1) \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial s}, \\ s &= \alpha^2. \end{aligned} \tag{3.14}$$

Given equality (3.6), inequality (3.7) and equality (3.14), we obtain

$$\int\limits_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1} \right| ds_y \le C(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma > 1,\ x \in G. \tag{3.15}$$

Now, we estimate the integral $\int\limits_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2} \right| ds_y$.

Taking into account equality (3.6) and inequality (3.7), we obtain

$$\int\limits_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2} \right| ds_y \le C(\lambda, x)\sigma e^{-\sigma x_2},\ \sigma > 1,\ x \in G. \tag{3.16}$$

From inequalities (3.15) and (3.16), bearing in mind (3.13), we get an estimate (3.4).
So, theorem 3.1 is proved. □

Corollary 3.2. *The limiting equality For each $x \in G$, the following equalities are true*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x),\ \lim_{\sigma\to\infty} \frac{\partial U_\sigma(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j},\ j = 1,2.$$

We denote by $\overline{G}_\varepsilon$ the set

$$\overline{G}_\varepsilon = \left\{ (x_1, x_2) \in G,\ a > x_2 \ge \varepsilon,\ a = \max_T \psi(x_1),\ 0 < \varepsilon < a \right\}.$$

It is easy to see that the set $\overline{G}_\varepsilon \subset G$ is compact.

Corollary 3.3. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_\sigma(x)\}$ and $\left\{ \frac{\partial U_\sigma(x)}{\partial x_j} \right\}$ converge uniformly for $\sigma \to \infty$. It mean that following converges are satisfied:*

$$U_\sigma(x) \rightrightarrows U(x),\ \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j},\ j = 1,2.$$

It should be noted that the set $E_\varepsilon = G \backslash \overline{G}_\varepsilon$ serves as a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

4 Estimation of the stability of the solution to the Cauchy problem

Suppose that the curve S is given by the equation

$$y_2 = \psi(y_1),\ y_1 \in \mathbb{R},$$

where $\psi(y_1)$ is a single-valued function satisfying the Lyapunov conditions.
We put

$$a = \max_T \psi(y_1),\ b = \max_T \sqrt{1 + \psi'^2(y_1)}.$$

Theorem 4.1. *Suppose that $U(y) \in A(G)$ satisfies condition (3.10), and on a smooth curve S the inequality*

$$|U(y)| \le \delta,\ 0 < \delta < 1. \tag{4.1}$$

Then, the following estimates are true

$$|U(x)| \le C(\lambda, x)\sigma\delta^{\frac{x_2}{a}},\ \sigma > 1,\ x \in G. \tag{4.2}$$

$$\left| \frac{\partial U(x)}{\partial x_j} \right| \le C(\lambda, x)\sigma\delta^{\frac{x_2}{a}},\ \sigma > 1,\ x \in G,\ j = 1,2. \tag{4.3}$$

Proof. It is estimated inequality (4.2) firstly. Using the integral formula (2.9), we have

$$U(x) = \int\limits_S N_\sigma(y,x;\lambda)U(y)ds_y + \int\limits_T N_\sigma(y,x;\lambda))U(y)ds_y,\ x \in G. \tag{4.4}$$

We estimate the following

$$|U(x)| \leq \left| \int\limits_S N_\sigma(y,x;\lambda)U(y)ds_y \right| + \left| \int\limits_T N_\sigma(y,x;\lambda)U(y)ds_y \right|, \ x \in G. \tag{4.5}$$

Given inequality (4.1), we estimate the first integral of inequality (4.5).

$$\begin{gathered} \left| \int\limits_S N_\sigma(y,x;\lambda)U(y)ds_y \right| \leq \int\limits_S |N_\sigma(y,x;\lambda)|\,|U(y)|\,ds_y \leq \\ \leq \delta \int\limits_S |N_\sigma(y,x;\lambda)|\,ds_y, \ x \in G. \end{gathered} \tag{4.6}$$

We estimate the integrals $\int\limits_S |\Phi_\sigma(y,x;\lambda)|\,ds_y$, $\int\limits_S \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_1}\right| ds_y$, and $\int\limits_S \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_2}\right| ds_y$ on a smooth curve S.

Given equality (3.6) and the inequality (3.7), we have

$$\int\limits_S |\Phi_\sigma(y,x;\lambda)|\,ds_y \leq C(\lambda,x)\sigma e^{\sigma(a-x_2)}, \ \sigma > 1, \ x \in G. \tag{4.7}$$

To estimate the second integral, using equalities (3.6) and (3.9) as well as inequality (3.7), we get

$$\int\limits_S \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_1}\right| ds_y \leq C(\lambda,x)\sigma e^{\sigma(a-x_2)}, \ \sigma > 1, \ x \in G. \tag{4.8}$$

Also, to estimate the integral $\int\limits_S \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_2}\right| ds_y$, using equality (3.6) and inequality (3.7), we obtain

$$\int\limits_S \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_2}\right| ds_y \leq C(\lambda,x)\sigma e^{\sigma(a-x_2)}, \ \sigma > 1, \ x \in G. \tag{4.9}$$

From (4.7)-(4.9), bearing in mind (4.6), we obtain

$$\left| \int\limits_S N_\sigma(y,x;\lambda)U(y)ds_y \right| \leq C(\lambda,x)\sigma\delta\, e^{\sigma(a-x_2)}, \ \sigma > 1, \ x \in G. \tag{4.10}$$

By the way following inequality is known

$$\left| \int\limits_T N_\sigma(y,x;\lambda)U(y)ds_y \right| \leq C(\lambda,x)\sigma e^{-\sigma x_2}, \ \sigma > 1, \ x \in G. \tag{4.11}$$

Now, taking into account (4.10)-(4.11), bearing in mind (4.5), we have

$$|U(x)| \leq \frac{C(\lambda,x)\sigma}{2}(\delta\, e^{\sigma a} + 1)e^{-\sigma x_2}, \ \sigma > 1, \ x \in G. \tag{4.12}$$

Choosing σ from the equality

$$\sigma = \frac{1}{a}\ln\frac{1}{\delta}, \tag{4.13}$$

we obtain an estimate (4.2).

Now let us prove inequality (4.3). To do this, we find the partial derivative from the integral formula (2.9) with respect to the variable $x_j,\ j=1,2$:

$$\frac{\partial U(x)}{\partial x_j}=\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y+\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y=$$
$$=\frac{\partial U_\sigma(x)}{\partial x_j}+\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y,\ x\in G,\ j=1,2. \tag{4.14}$$

Here

$$\frac{\partial U_\sigma(x)}{\partial x_j}=\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y. \tag{4.15}$$

We estimate the following

$$\left|\frac{\partial U(x)}{\partial x_j}\right|\le\left|\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|+\left|\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\le$$
$$\le\left|\frac{\partial U_\sigma(x)}{\partial x_j}\right|+\left|\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|,\ x\in G,\ j=1,2. \tag{4.16}$$

Given inequality (4.1), we estimate the first integral of inequality (4.16).

$$\left|\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\le\int\limits_S\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right||U(y)|\,ds_y\le$$
$$\le\delta\int\limits_S\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y,\ x\in G,\ j=1,2. \tag{4.17}$$

To do this, we estimate the integrals $\int\limits_S\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1}\right|ds_y$, and $\int\limits_S\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2}\right|ds_y$ on a smooth curve S.

Given equality (3.6), inequality (3.7) and equality (4.14), we obtain

$$\int\limits_S\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1}\right|ds_y\le C(\lambda,x)\sigma e^{\sigma(a-x_2)},\ \sigma>1,\ x\in G, \tag{4.18}$$

Now, we estimate the integral $\int\limits_S\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2}\right|ds_y$.

Taking into account equality (3.6) and inequality (3.7), we obtain

$$\int\limits_S\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2}\right|ds_y\le C(\lambda,x)\sigma e^{\sigma(a-x_2)},\ \sigma>1,\ x\in G, \tag{4.19}$$

From (4.18)-(4.19), bearing in mind (4.17), we obtain

$$\left|\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\le C(\lambda,x)\sigma\delta e^{-\sigma x_2},\ \sigma>1,\ x\in G,$$
$$j=1,2. \tag{4.20}$$

The following is known

$$\left|\int\limits_{T} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| \leq C(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma>1,\ x\in G, \qquad (4.21)$$
$$j=1,2.$$

Now, taking into account (4.20)-(4.21), bearing in mind (4.16), we have

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq \frac{C(\lambda,x)\sigma}{2}(\delta e^{\sigma a}+1)e^{-\sigma x_2},\ \sigma>1,\ x\in G, \qquad (4.22)$$
$$j=1,2.$$

Choosing σ from the equality (4.13), we obtain an estimate (4.3).
Thus, theorem 4.1 is proved. □

Let $U(y)\in A(G)$ and instead $U(y)$ on S with its approximation $f_\delta(y)$, respectively, with an error $0<\delta<1$,

$$\max_S |U(y)-f_\delta(y)| \leq \delta. \qquad (4.23)$$

We put

$$U_{\sigma(\delta)}(x) = \int\limits_S N_\sigma(y,x;\lambda) f_\delta(y) ds_y,\ x\in G. \qquad (4.24)$$

Theorem 4.2. *Let $U(y)\in A(G)$ on the part of the plane $y_2=0$ satisfy condition (3.1)*

Then, the following estimates are true

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \leq C(\lambda,x)\sigma\delta^{\frac{x_2}{a}},\ \sigma>1,\ x\in G. \qquad (4.25)$$

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq C(\lambda,x)\sigma\delta^{\frac{x_2}{a}},\ \sigma>1,\ x\in G,\ j=1,2. \qquad (4.26)$$

Proof. From the integral formulas (2.9) and (4.24), we have

$$U(x)-U_{\sigma(\delta)}(x) = \int\limits_{\partial G} N_\sigma(y,x;\lambda)U(y)ds_y -$$

$$-\int\limits_S N_\sigma(y,x;\lambda) f_\delta(y) ds_y = \int\limits_S N_\sigma(y,x;\lambda)U(y)ds_y +$$

$$+\int\limits_T N_\sigma(y,x;\lambda)U(y)ds_y - \int_S N_\sigma(y,x;\lambda) f_\delta(y) ds_y =$$

$$=\int\limits_S N_\sigma(y,x;\lambda)\left\{U(y)-f_\delta(y)\right\}ds_y + \int\limits_T N_\sigma(y,x;\lambda)U(y)ds_y.$$

and

$$\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \int\limits_{\partial G} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y -$$

$$- \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} f_\delta(y) ds_y = \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y +$$

$$+ \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y - \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} f_\delta(y) ds_y =$$

$$= \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \{U(y) - f_\delta(y)\} ds_y + \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y,$$

$$j = 1, 2.$$

Using conditions (3.1) and (4.23), we estimate the following:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| = \left|\int\limits_S N_\sigma(y,x;\lambda)\{U(y) - f_\delta(y)\} ds_y\right| +$$

$$+ \left|\int\limits_T N_\sigma(y,x;\lambda) U(y) ds_y\right| \le \int\limits_S |N_\sigma(y,x;\lambda)| \, |\{U(y) - f_\delta(y)\}| \, ds_y +$$

$$+ \int\limits_T |N_\sigma(y,x;\lambda)| \, |U(y)| \, ds_y \le \delta \int_S |N_\sigma(y,x;\lambda)| \, ds_y +$$

$$+ \int\limits_T |N_\sigma(y,x;\lambda)| \, ds_y.$$

and

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| = \left|\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \{U(y) - f_\delta(y)\} ds_y\right| +$$

$$+ \left|\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| \le \int\limits_S \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| |\{U(y) - f_\delta(y)\}| \, ds_y +$$

$$+ \int\limits_T \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| |U(y)| \, ds_y \le \delta \int_S \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y +$$

$$+ \int\limits_T \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y, \ j = 1, 2.$$

Now, repeating the proof of Theorems 3.1 and 4.1, we obtain

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le \frac{C(\lambda,x)\sigma}{2} (\delta \, e^{\sigma a} + 1) e^{-\sigma x_2}.$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{U_{\sigma(\delta)}(x)}{\partial x_j}\right| \le \frac{C(\lambda,x)\sigma}{2} (\delta \, e^{\sigma a} + 1) e^{-\sigma x_2}, \ j = 1, 2.$$

From here, choosing σ from equality (4.13), we have an estimates (4.25) and (4.26).
So, theorem 4.2 is proved. □

Corollary 4.3. *The following equalities are true*

$$\lim_{\delta\to 0} U_{\sigma(\delta)}(x) = U(x),\ \lim_{\delta\to 0} \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j},\ j = 1, 2.$$

for each $x \in G$*,*

Corollary 4.4. *If* $x \in \overline{G}_\varepsilon$*, then the families of functions* $\{U_{\sigma(\delta)}(x)\}$ *and* $\left\{\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ *converge uniformly for* $\delta \to 0$*. It mean that following converges are satisfied:*

$$U_{\sigma(\delta)}(x) \rightrightarrows U(x),\ \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j},\ j = 1, 2.$$

5 Regularization of the Cauchy problem for a domain of the type of a curvilinear triangle

Let $\mathbb{R}^2$ be a two dimensional real Euclidean space,

$$x = (x_1, x_2) \in \mathbb{R}^2,\ y = (y_1, y_2) \in \mathbb{R}^2.$$

We introduce the following notation:

$$r = |y - x|\,, \alpha = |y' - x'|\,, w = i\tau\sqrt{u^2 + \alpha^2} + \beta,\ w_0 = i\tau\alpha + \beta,$$

$$\beta = \tau y_2,\ \tau = tg\frac{\pi}{2\rho},\ \rho > 1,\ u \geq 0,\ s = \alpha^2,$$

$$G_\rho = \{y:\ |y_1| < \tau y_2,\ y_2 > 0\}\,,\ \partial G_\rho = \{y:\ |y_1| = \tau y_2,\ y_2 > 0\}\,,$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^T,\ \frac{\partial}{\partial x} = \xi^T,\ \xi^T = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ - transposed vector } \xi,$$

$$U(x) = (U_1(x), \ldots, U_n(x))^T,\ u^0 = (1, \ldots, 1) \in \mathbb{R}^n,\ n = 2^m,\ m = 2,$$

$$E(z) = \left\| \begin{matrix} z_1 \ldots 0 \\ \ldots\ldots. \\ 0 \ldots z_n \end{matrix} \right\| - \text{diagonal matrix},\ z = (z_1, \ldots, z_n) \in \mathbb{R}^n.$$

$G_\rho \subset \mathbb{R}^2$ is a bounded simply connected domain whose boundary consists of segments of rays

$$|y_1| = \tau y_2, \quad 0 < y_2 \leq y_0 < \infty,$$

with the beginning at zero and the arc S of a smooth curve lying inside the angle of width $\frac{\pi}{\rho}$, i.e., $\partial G_\rho = S \bigcup T,\ T = \partial G_\rho \backslash S$.

We assume that $(0, x_2) \in G_\rho,\ \ x_2 > 0$. $G_\rho-$is called a domain of the type of a curvilinear triangle.

We consider a system of differential equations in the region G_ρ

$$D\left(\frac{\partial}{\partial x}\right) U(x) = 0, \tag{5.1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by $A(G_\rho)$the class of vector functions in the domain G_ρ continuous on $\overline{G}_\rho = G_\rho \bigcup \partial G_\rho$ and satisfying system (5.1).

Formulation of the problem. Suppose $U(y) \in A(G_\rho)$ and

$$U(y)|_S = f(y), \ y \in S. \tag{5.2}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain G_ρ, based on it's values $f(y)$ on S.

If $U(y) \in A(G_\rho)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int\limits_{\partial G_\rho} N(y, x; \lambda) U(y) ds_y, \quad x \in G_\rho, \tag{5.3}$$

where

$$N(y, x; \lambda) = \left(E\left(\varphi_2(\lambda r) u^0\right) D^* \left(\frac{\partial}{\partial x}\right)\right) D(t^T).$$

In the formula (2.5), choosing

$$K(w) = E_\rho(\sigma^{1/\rho} w), \quad K(x_2) = E_\rho(\sigma^{1/\rho}\gamma), \quad \gamma = \tau x_2, \quad \sigma > 0, \tag{5.4}$$

we get

$$\mathbf{\Phi}_\sigma(y, x; \lambda) = -\frac{E_\rho^{-1}(\sigma^{1/\rho}\gamma)}{2\pi} \int\limits_0^\infty \operatorname{Im}\left[\frac{E_\rho(\sigma^{1/\rho} w)}{w - x_m}\right] \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du. \tag{5.5}$$

Here $E_\rho(\sigma^{1/\rho} w)-$ is the entire Mittag-Leffler function.

Then the integral formula has the form:

$$U(x) = \int\limits_{\partial G_\rho} N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G_\rho, \tag{5.6}$$

where

$$N_\sigma(y, x; \lambda) = \left(E\left(\mathbf{\Phi}_\sigma(y, x; \lambda) u^0\right) D^* \left(\frac{\partial}{\partial x}\right)\right) D(t^T).$$

Recall the basic properties of the Mittag-Leffler function. The entire function of Mittag-Leffler is defined by a series:

$$\sum_{n=1}^{\infty} \frac{w^n}{\Gamma(1 + \rho^{-1} n)} = E_\rho(w), \ w = u + iv,$$

where $\Gamma(s)-$ is the Euler gamma function.

We denote by $\gamma_\varepsilon(\beta_0) (\varepsilon > 0, \ 0 < \beta_0 < \pi)$ the contour in the complex plane ζ, run in the direction of non-decreasing $\arg \zeta$ and consisting of the following parts:

1. The beam $\arg \zeta = -\beta_0, \ |\zeta| \geq \varepsilon$;
2. The arc $-\beta_0 < \arg \zeta < \beta_0$ of circle $|\zeta| = \varepsilon$;
3. The beam $\arg \zeta = \beta_0, \ |\zeta| \geq \varepsilon$.

The contour $\gamma_\varepsilon(\beta_0)$ divides the plane ζ into two unbounded simply connected domains G_ρ^- and G_ρ^+ lying to the left and to the right of $\gamma_\varepsilon(\beta_0)$, respectively.

Let $\rho > 1, \ \frac{\pi}{2\rho} < \beta_0 < \frac{\pi}{\rho}$.

Denote

$$\psi_\rho(w) = \frac{1}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^\rho)}{\zeta - w} d\zeta, \tag{5.7}$$

Then the following integral representations are valid:

$$E_\rho(w) = \psi_\rho(w), \quad z \in G_\rho^-, \tag{5.8}$$

$$E_\rho(w) = \rho \exp(w^\rho) + \psi_\rho(w), \quad z \in G_\rho^+, \tag{5.9}$$

From these formulas we find

$$\left. \begin{array}{l} |E_\rho(w)| \leq \rho \exp(\mathrm{Re}\, w^\rho) + |\psi_\rho(w)|\,, \quad |\arg w| \leq \frac{\pi}{2\rho} + \eta_0, \\ \\ |E_\rho(w)| \leq |\psi_\rho(w)|\,, \quad \frac{\pi}{2\rho} + \eta_0 \leq |\arg w| \leq \pi, \quad \eta_0 > 0 \end{array} \right\} \tag{5.10}$$

$$|\psi_\rho(w)| \leq \frac{M}{1+|w|},\ M = const \tag{5.11}$$

$$E_\rho(w) \approx \rho \exp(w^\rho), \quad w > 0, \quad w \to \infty, \tag{5.12}$$

Further, since $E_\rho(w)$ is real with real w, then

$$\mathrm{Re}\, \psi_\rho(w) = \frac{\rho}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{2\zeta - \mathrm{Re}\, w}{(\zeta - w)\zeta - \overline{w})} \exp(\zeta^\rho) d\zeta,$$

$$\mathrm{Im}\, \psi_\rho(w) = \frac{\rho \mathrm{Im}\,(w)}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^\rho)}{(\zeta - w)\zeta - \overline{w})} d\zeta,$$

The information given here concerning the function $E_\rho(w)$ is taken from (see, for instance [10] and [13]).

In what follows, to prove the main theorems, we need the following estimates for the function $\Phi_\sigma(y, x; \lambda)$.

Lemma 5.1. *Let* $x = (x_1, x_2) \in G_\rho,\ y \neq x,\ \sigma \geq \lambda + \sigma_0,\ \sigma_0 > 0$, *then*

1) at $\beta \leq \alpha$ *inequalities are satisfied*

$$|\Phi_\sigma(y, x; \lambda)| \leq C(\rho, \lambda) E_\rho^{-1}(\sigma^{1/\rho} w) \ln \frac{1+r^2}{r^2}, \quad x \in G_\rho, \tag{5.13}$$

$$\left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} \right| \leq C(\rho, \lambda) \frac{E_\rho^{-1}(\sigma^{1/\rho} w)}{r}, \quad x \in G_\rho,\ j = 1, 2, \tag{5.14}$$

$$\left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_j} \right| \leq C(\rho, \lambda) \frac{E_\rho^{-1}(\sigma^{1/\rho} w)}{r}, \quad x \in G_\rho,\ j = 1, 2. \tag{5.15}$$

2) at $\beta > \alpha$ *inequalities are satisfied*

$$|\Phi_\sigma(y, x; \lambda)| \leq C(\rho, \lambda) E_\rho^{-1}(\sigma^{1/\rho} w) \left(\ln \frac{1+r^2}{r^2} \right) \exp(\sigma \mathrm{Re}\, w_0^\rho), \quad x \in G_\rho, \tag{5.16}$$

$$\left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} \right| \leq C(\rho, \lambda) E_\rho^{-1}(\sigma^{1/\rho} w) \frac{1}{2} \exp(\sigma \mathrm{Re}\, w_0^\rho), \quad x \in G_\rho,\ j = 1, 2, \tag{5.17}$$

$$\left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_j} \right| \leq C(\rho, \lambda) E_\rho^{-1}(\sigma^{1/\rho} w) \frac{1}{2} \exp(\sigma \mathrm{Re}\, w_0^\rho), \quad x \in G_\rho,\ j = 1, 2. \tag{5.18}$$

Here $C(\rho, \lambda)$ is the function depending on ρ and λ.

For a fixed $x \in G_\rho$ we denote by S^* the part of S on which $\beta \geq \alpha$. If $x \in G_\rho$, then $S = S^*$ (in this case, $\beta = \tau y_2$ and the inequality $\beta \geq \alpha$ means that y lies inside or on the a curvilinear triangle).

Theorem 5.2. *Let* $U(y) \in A(G_\rho)$ *it satisfy the inequality*

$$|U(y)| \leq 1,\ y \in T. \tag{5.19}$$

If

$$U_\sigma(x) = \int\limits_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y,\ x \in G_\rho, \tag{5.20}$$

then the following estimates are true

$$|U(x)-U_\sigma(x)| \le C_\rho(\lambda,x)\sigma\exp(-\sigma\gamma^\rho),\ \ \sigma>1,\ x\in G_\rho. \tag{5.21}$$

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial U_\sigma(x)}{\partial x_j}\right| \le C_\rho(\lambda,x)\sigma\exp(-\sigma\gamma^\rho),\ \ \sigma>1,\ x\in G_\rho,\ j=1,2. \tag{5.22}$$

Here and below functions bounded on compact subsets of the domain G_ρ, we denote by $C_\rho(\lambda,x)$.

Proof. Let us first estimate inequality (5.21). Using the integral formula (5.6) and the equality (5.20), we obtain

$$U(x)=\int\limits_{S^*}N_\sigma(y,x;\lambda)U(y)ds_y+\int\limits_{\partial G_\rho\backslash S^*}N_\sigma(y,x;\lambda)U(y)ds_y=$$

$$=U_\sigma(x)+\int\limits_{\partial G_\rho\backslash S^*}N_\sigma(y,x;\lambda)U(y)ds_y,\ \ x\in G_\rho.$$

Taking into account the inequality (5.19), we estimate the following

$$|U(x)-U_\sigma(x)|\le\left|\int\limits_{\partial G_\rho\backslash S^*}N_\sigma(y,x;\lambda)U(y)ds_y\right|\le$$
$$\le\int\limits_{\partial G_\rho\backslash S^*}|N_\sigma(y,x;\lambda)|\,|U(y)|\,ds_y\le\int\limits_{\partial G_\rho\backslash S^*}|N_\sigma(y,x;\lambda)|\,ds_y,\ \ x\in G_\rho. \tag{5.23}$$

To do this, we estimate the integrals $\int\limits_{\partial G_\rho\backslash S^*}|\Phi_\sigma(y,x;\lambda)|\,ds_y$, $\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_1}\right|ds_y$ and $\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_2}\right|ds_y$ on the part $\partial G_\rho\backslash S^*$ of the plane $y_2=0$.

Separating the imaginary part of (5.5), we obtain

$$\Phi_\sigma(y,x;\lambda)=\frac{E_\rho^{-1}(\sigma^{1/\rho}\gamma)}{2\pi}\left[\int\limits_0^\infty\frac{(y_m-x_m)\mathrm{Im}E_\rho(\sigma^{1/\rho}w)}{u^2+r^2}\ \frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du-\right.$$
$$\left.-\int\limits_0^\infty\frac{u\mathrm{Re}\,E_\rho(\sigma^{1/\rho}w)}{u^2+r^2}I_0(\lambda u)\,du\right],\ y\ne x,\ x_2>0. \tag{5.24}$$

Given (5.24) and the inequality

$$I_0(\lambda u)\le\sqrt{\frac{2}{\lambda\pi u}}, \tag{5.25}$$

we have

$$\int\limits_{\partial G_\rho\backslash S^*}|\Phi_\sigma(y,x;\lambda)|\,ds_y\le C_\rho(\lambda,x)\sigma\exp(-\sigma\gamma^\rho),\ \ \sigma>1,\ x\in G_\rho, \tag{5.26}$$

To estimate the second integral, we use the equality

$$\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_1}=\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial y_1}=2(y_1-x_1)\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s},$$
$$s=\alpha^2. \tag{5.27}$$

Given equality (5.24), inequality (5.25) and equality (5.27), we obtain

$$\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_1} \right| ds_y \le C_\rho(\lambda,x)\sigma \exp(-\sigma\gamma^\rho), \;\; \sigma > 1, \;\; x \in G_\rho. \tag{5.28}$$

Now, we estimate the integral $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_2} \right| ds_y$.

Taking into account equality (5.24) and inequality (5.25), we obtain

$$\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_2} \right| ds_y \le C_\rho(\lambda,x)\sigma \exp(-\sigma\gamma^\rho), \;\; \sigma > 1, \;\; x \in G_\rho. \tag{5.29}$$

From inequalities (5.26), (5.28) and (5.29), we obtain an estimate (5.21).

Now let us prove inequality (5.22). To do this, we take the derivatives from equalities (5.6) and (5.20) with respect to x_j, $(j = 1,2)$, then we obtain the following:

$$\begin{gathered} \frac{\partial U(x)}{\partial x_j} = \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y + \int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y, \\ \frac{\partial U_\sigma(x)}{\partial x_j} = \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y, \quad x \in G_\rho, \;\; j = 1,2. \end{gathered} \tag{5.30}$$

Taking into account the (5.30) and inequality (5.19), we estimate the following

$$\begin{gathered} \left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial_\sigma U(x)}{\partial x_j} \right| \le \left| \int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y \right| \le \\ \le \int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| |U(y)| \, ds_y \le \int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| ds_y, \\ x \in G_\rho, \;\; j = 1,2. \end{gathered} \tag{5.31}$$

To do this, we estimate the integrals $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1} \right| ds_y$, and $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2} \right| ds_y$ on the part $\partial G_\rho \backslash S^*$ of the plane $y_2 = 0$.

To estimate the first integrals, we use the equality

$$\begin{gathered} \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1} = \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial s} \frac{\partial s}{\partial x_1} = -2(y_1 - x_1) \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial s}, \\ s = \alpha^2. \end{gathered} \tag{5.32}$$

Given equality (5.24), inequality (5.25) and equality (5.32), we obtain

$$\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1} \right| ds_y \le C_\rho(\lambda,x)\sigma \exp(-\sigma\gamma^\rho), \;\; \sigma > 1, \;\; x \in G_\rho. \tag{5.33}$$

Now, we estimate the integral $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2} \right| ds_y$.

Taking into account equality (5.24) and inequality (5.25), we obtain

$$\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2} \right| ds_y \leq C_\rho(\lambda,x)\sigma \exp(-\sigma\gamma^\rho), \ \ \sigma > 1, \ \ x \in G_\rho. \tag{5.34}$$

From inequalities (5.31), (5.33) and (5.34), we obtain an estimate (5.22).
Theorem 5.2 is proved. □

Corollary 5.3. *For each $x \in G_\rho$, the equalities are true*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x), \ \ \lim_{\sigma\to\infty} \frac{\partial U_\sigma(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \ \ j = 1, 2.$$

We denote by $\overline{G}_\varepsilon$ the set

$$\overline{G}_\varepsilon = \left\{ (x_1, x_2) \in G_\rho, \ a > x_2 \geq \varepsilon, \ \ a = \max_T \psi(x_1), \ \ 0 < \varepsilon < a \right\}.$$

Here, $\psi(x_1)$ - is a curve. It is easy to see that the set $\overline{G}_\varepsilon \subset G_\rho$ is compact.

Corollary 5.4. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_\sigma(x)\}$ and $\left\{ \frac{\partial U_\sigma(x)}{\partial x_j} \right\}$ converge uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x), \ \ \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j}, \ \ j = 1, 2.$$

It should be noted that the set $E_\varepsilon = G_\rho \backslash \overline{G}_\varepsilon$ serves as a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

Suppose that the curve S is given by the equation

$$y_2 = \psi(y_1), \ \ y_1 \in \mathbb{R},$$

where $\psi(y_1)$ is a single-valued function satisfying the Lyapunov conditions.
We put

$$a = \max_T \psi(y_1), \ \ b = \max_T \sqrt{1 + \psi'^2(y_1)}.$$

Theorem 5.5. *Let $U(y) \in A(G_\rho)$ satisfy condition (5.19), and on a smooth curve S the inequality*

$$|U(y)| \leq \delta, \ \ 0 < \delta < 1. \tag{5.35}$$

Then the following estimates are true

$$|U(x)| \leq C_\rho(\lambda,x)\sigma\delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho. \tag{5.36}$$

$$\left| \frac{\partial U(x)}{\partial x_j} \right| \leq C_\rho(\lambda,x)\sigma\delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \quad j = 1, 2. \tag{5.37}$$

Here is $a^\rho = \max\limits_{y\in S} \operatorname{Re} w_0^\rho$.

Proof. Let us first estimate inequality (5.36). Using the integral formula (5.6), we have

$$U(x) = \int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y,x;\lambda))U(y)ds_y, \ \ x \in G_\rho. \tag{5.38}$$

We estimate the following

$$|U(x)| \leq \left| \int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y \right| + \left| \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y,x;\lambda)U(y)ds_y \right|, \ x \in G_\rho. \tag{5.39}$$

Given inequality (5.35), we estimate the first integral of inequality (5.39).

$$\left|\int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| \leq \int\limits_{S^*} |N_\sigma(y,x;\lambda)|\,|U(y)|\,ds_y \leq$$
$$\leq \delta\int\limits_{S^*} |N_\sigma(y,x;\lambda)|\,ds_y,\ x\in G_\rho. \tag{5.40}$$

We estimate the integrals $\int\limits_{S^*}|\Phi_\sigma(y,x;\lambda)|\,ds_y$, $\int\limits_{S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_1}\right|ds_y$, and $\int\limits_{S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_2}\right|ds_y$ on a smooth curve S.

Given equality (5.24) and the inequality (5.25), we have

$$\int\limits_{S^*}|\Phi_\sigma(y,x;\lambda)|\,ds_y \leq C_\rho(\lambda,x)\sigma\exp\sigma(\tau^\rho a^\rho-\gamma^\rho),\quad \sigma>1,\quad x\in G_\rho. \tag{5.41}$$

To estimate the second integral, using equalities (5.24) and (5.27) as well as inequality (5.25), we obtain

$$\int\limits_{S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_1}\right|ds_y \leq C_\rho(\lambda,x)\sigma\exp\sigma(\tau^\rho a^\rho-\gamma^\rho),\ \sigma>1,\ x\in G_\rho. \tag{5.42}$$

To estimate the integral $\int\limits_{S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right|ds_y$, using equality (5.24) and inequality (5.25), we obtain

$$\int\limits_{S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_2}\right|ds_y \leq C_\rho(\lambda,x)\sigma\exp\sigma(\tau^\rho a^\rho-\gamma^\rho),\ \sigma>1,\ x\in G_\rho. \tag{5.43}$$

From (5.41) - (5.43), we obtain

$$\left|\int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| \leq C_\rho(\lambda,x)\sigma\delta\exp\sigma(\tau^\rho a^\rho-\gamma^\rho),\ \sigma>1,\ x\in G_\rho. \tag{5.44}$$

The following is known

$$\left|\int\limits_{\partial G_\rho\backslash S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| \leq C_\rho(\lambda,x)\sigma\exp(-\sigma\gamma^\rho),\ \sigma>1,\ x\in G_\rho. \tag{5.45}$$

Now taking into account (5.44) - (5.45), we have

$$|U(x)| \leq \frac{C_\rho(\lambda,x)\sigma}{2}(\delta\exp(\sigma\tau^\rho a^\rho)+1)\exp(-\sigma\gamma^\rho),\ \sigma>1,\ x\in G_\rho. \tag{5.46}$$

Choosing σ from the equality

$$\sigma=\frac{1}{a^\rho}\ln\frac{1}{\delta}, \tag{5.47}$$

we obtain an estimate (5.36).

Now let us prove inequality (5.37). To do this, we find the partial derivative from the integral formula (5.6) with respect to the variable $x_j,\ j=1,2.$:

$$\frac{\partial U(x)}{\partial x_j}=\int\limits_{S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y+\int\limits_{\partial G_\rho\backslash S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y+$$
$$+\frac{\partial U_\sigma(x)}{\partial x_j}+\int\limits_{\partial G_\rho\backslash S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y,\ x\in G_\rho,\ j=1,2. \tag{5.48}$$

Here
$$\frac{\partial U_\sigma(x)}{\partial x_j} = \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y. \tag{5.49}$$

We estimate the following
$$\begin{gathered}\left|\frac{\partial U(x)}{\partial x_j}\right| \leq \left|\int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| + \left|\int\limits_{\partial G_\rho \setminus S^*} \frac{\partial N_\sigma(y,x;\lambda))}{\partial x_j} U(y) ds_y\right| \leq \\ \leq \left|\frac{\partial U_\sigma(x)}{\partial x_j}\right| + \left|\int\limits_{\partial G_\rho \setminus S^*} \frac{\partial N_\sigma(y,x;\lambda))}{\partial x_j} U(y) ds_y\right|, \; x \in G_\rho, \; j = 1,2.\end{gathered} \tag{5.50}$$

Given inequality (5.35), we estimate the first integral of inequality (5.50).
$$\begin{gathered}\left|\int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| \leq \int\limits_{S^*} \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| |U(y)|\, ds_y \leq \\ \leq \delta \int\limits_{S^*} \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y, \; x \in G_\rho, \; j = 1,2.\end{gathered} \tag{5.51}$$

To do this, we estimate the integrals $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1}\right| ds_y$, and $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2}\right| ds_y$ on a smooth curve S.

Given equality (5.24), (5.25) and equality (5.32), we obtain
$$\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_1}\right| ds_y \leq C_\rho(\lambda,x)\sigma \exp\sigma(\tau^\rho a^\rho - \gamma^\rho), \; \sigma > 1, \; x \in G_\rho. \tag{5.52}$$

Now, we estimate the integral $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2}\right| ds_y$.

Taking into account equality (5.24) and inequality (5.25), we obtain
$$\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_2}\right| ds_y \leq C_\rho(\lambda,x)\sigma\delta \exp\sigma(\tau^\rho a^\rho - \gamma^\rho), \; \sigma > 1, \; x \in G_\rho. \tag{5.53}$$

From (5.52) - (5.53), we obtain
$$\begin{gathered}\left|\int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y)\right| \leq C_\rho(\lambda,x)\sigma\delta \, \exp\sigma(\tau^\rho a^\rho - \gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho, \\ j = 1,2.\end{gathered} \tag{5.54}$$

The following is known
$$\begin{gathered}\left|\int\limits_{\partial G_\rho \setminus S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| \leq C_\rho(\lambda,x)\sigma \exp(-\sigma\gamma^\rho), \; \sigma > 1, \; x \in G_\rho, \\ j = 1,2.\end{gathered} \tag{5.55}$$

Now taking into account (5.54) - (5.55), we have
$$\begin{gathered}\left|\frac{\partial U(x)}{\partial x_j}\right| \leq \frac{C_\rho(\lambda,x)\sigma}{2}(\delta \exp(\sigma\tau^\rho a^\rho) + 1) \exp(-\sigma\gamma^\rho), \; \sigma > 1, \; x \in G_\rho, \\ j = 1,2.\end{gathered} \tag{5.56}$$

Choosing σ from the equality (5.47) we obtain an estimate (5.37).
Theorem 5.5 is proved. □

Let $U(y) \in A(G_\rho)$ and instead $U(y)$ on S with its approximation $f_\delta(y)$, respectively, with an error $0 < \delta < 1$,

$$\max_S |U(y) - f_\delta(y)| \le \delta. \tag{5.57}$$

We put

$$U_{\sigma(\delta)}(x) = \int\limits_{S^*} N_\sigma(y, x; \lambda) f_\delta(y) ds_y, \ x \in G_\rho. \tag{5.58}$$

Theorem 5.6. *Let $U(y) \in A(G_\rho)$ on the part of the plane $y_2 = 0$ satisfy condition (5.19). Then the following estimates is true*

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le C_\rho(\lambda, x)\sigma\delta^{\left(\frac{\gamma}{a}\right)^\rho}, \ \sigma > 1, \ x \in G_\rho. \tag{5.59}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \le C_\rho(\lambda, x)\sigma\delta^{\left(\frac{\gamma}{a}\right)^\rho}, \ \sigma > 1, \ x \in G_\rho, \ j = 1, 2. \tag{5.60}$$

Proof. From the integral formulas (5.6) and (5.58), we have

$$U(x) - U_{\sigma(\delta)}(x) = \int\limits_{\partial G_\rho} N_\sigma(y, x; \lambda) U(y) ds_y - \int\limits_{S^*} N_\sigma(y, x; \lambda) f_\delta(y) ds_y =$$

$$= \int\limits_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y - \int\limits_{S^*} N_\sigma(y, x; \lambda) f_\delta(y) ds_y =$$

$$= \int\limits_{S^*} N_\sigma(y, x; \lambda) \left\{U(y) - f_\delta(y)\right\} ds_y + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y.$$

and

$$\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \int\limits_{\partial G_\rho} \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y - \int\limits_{S^*} \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} f_\delta(y) ds_y =$$

$$= \int\limits_{S^*} \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y + \int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y - \int\limits_{S^*} \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} f_\delta(y) ds_y =$$

$$= \int\limits_{S^*} \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} \left\{U(y) - f_\delta(y)\right\} ds_y + \int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y, \ j = 1, 2.$$

Using conditions (5.19) and (5.57), we estimate the following:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| = \left|\int\limits_{S^*} N_\sigma(y, x; \lambda) \left\{U(y) - f_\delta(y)\right\} ds_y\right| +$$

$$+ \left|\int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y\right| \le \int\limits_{S^*} \left|N_\sigma(y, x; \lambda)\right| \left|\left\{U(y) - f_\delta(y)\right\}\right| ds_y +$$

$$+ \int\limits_{\partial G_\rho \backslash S^*} \left|N_\sigma(y, x; \lambda)\right| \left|U(y)\right| ds_y \le \delta \int\limits_{S^*} \left|N_\sigma(y, x; \lambda)\right| ds_y + \int\limits_{\partial G_\rho \backslash S^*} \left|N_\sigma(y, x; \lambda)\right| ds_y.$$

and

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right|=\left|\int\limits_{S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\left\{U(y)-f_\delta(y)\right\}ds_y\right|+$$

$$+\left|\int\limits_{\partial G_\rho\backslash S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\leq\int\limits_{S^*}\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|\left|\left\{U(y)-f_\delta(y)\right\}\right|ds_y+$$

$$+\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right||U(y)|\,ds_y\leq\delta\int\limits_{S^*}\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y+$$

$$+\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y,\ j=1,2.$$

Now, repeating the proof of Theorems 5.4 and 5.5, we obtain

$$\left|U(x)-U_{\sigma(\delta)}(x)\right|\leq\frac{C_\rho(\lambda,x)\sigma}{2}(\delta\,\exp(\sigma\tau^\rho a^\rho)+1)\exp(-\sigma\gamma^\rho),$$

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{U_{\sigma(\delta)}(x)}{\partial x_j}\right|\leq\frac{C_\rho(\lambda,x)\sigma}{2}(\delta\,\exp(\sigma\tau^\rho a^\rho)+1)\exp(-\sigma\gamma^\rho)\quad j=1,2.$$

From here, choosing σ from equality (5.47), we obtain an estimates (5.59) and (5.60).
Theorem 5.6 is proved. □

Corollary 5.7. *For each $x\in G_\rho$, the equalities are true*

$$\lim_{\delta\to 0}U_{\sigma(\delta)}(x)=U(x),\ \lim_{\delta\to 0}\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}=\frac{\partial U(x)}{\partial x_j},\ j=1,2.$$

Corollary 5.8. *If $x\in\overline{G}_\varepsilon$, then the families of functions $\{U_{\sigma(\delta)}(x)\}$ and $\left\{\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ converge uniformly for $\delta\to 0$, i.e.:*

$$U_{\sigma(\delta)}(x)\rightrightarrows U(x),\ \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\rightrightarrows\frac{\partial U(x)}{\partial x_j},\ j=1,2.$$

6 Conclusion

Following results are obtained in the article:

Using the Carleman function, a formula is obtained for the continuation of the solution of linear elliptic systems of the first order with constant coefficients in a spatial bounded domains $\mathbb{R}^2$. The resulting formula is an analogue of the classical formula of B. Riemann, W. Voltaire and J. Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation is given. The problem it is considered when, instead of the exact data of the Cauchy problem, their approximations with a given deviation in the uniform metric are given and under the assumption that the solution of the Cauchy problem is bounded on part T, of the boundary of the domains G and G_ρ, an explicit regularization formula is obtained.

Thus, functionals $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ determines the regularization of the solution of problems (1.1) - (2.1) and (5.1) - (5.2).

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