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ON MEROMORPHIC STARLIKE AND CONVEX FUNCTIONS

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Abstract Using the technique of differential subordination, we define a certain class of functions in punctured unit disk. We obtain certain sufficient conditions for meromorphic starlike and convex functions. As special cases, we also mention some corollaries and obvious consequences of the main results.

1 Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \ (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $\mathbb{E}_0 = \mathbb{E} \setminus \{0\}$, where $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \Sigma_p$ is said to be meromorphic p-valent starlike of order α if $f(z) \neq 0$ for $z \in \mathbb{E}_0$ and

$$-\Re \frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}). \tag{1.2}$$

The class of all such meromorphic p-valent starlike functions is denoted by $\mathcal{MS}_p^*(\alpha)$.
A function $f \in \Sigma_p$ is called meromorphic p-valent convex of order α if $f'(z) \neq 0$ and

$$-\Re \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad (\alpha < 1; z \in \mathbb{E}).$$

The class of all meromorphic p-valent convex functions defined above is denoted by $\mathcal{MK}_p(\alpha)$.
In particular $\mathcal{MS}_p^* = \mathcal{MS}_P^*(0), \mathcal{MS}^* = \mathcal{MS}_1^*(0)$ and $\mathcal{MK}_p = \mathcal{MK}_p(0), \mathcal{MK} = \mathcal{MK}_1(0)$.

The class $\mathcal{MS}^*(\alpha)$ was introduced and studied by Pommerenke [1], Miller [3], Mogra et al. [4], Cho[7] and Aouf [5,6].
Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \longrightarrow \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0), \tag{1.3}$$

then p is called a solution of the differential subordination (1.3). The univalent function q is called a dominant of differential subordination (1.3) if $p \prec q$ for all p satisfying (1.3). A dominant $\tilde{q} \prec q$ for all dominants q of (1.3), is said to be the best dominant of (1.3).
In 2003, Irmak et al. [2] introduced and studied the subclass $\mathcal{T}_\lambda(p;\alpha)$. According to them a function $f \in \mathcal{T}(p)$ is said to be in $\mathcal{T}_\lambda(p;\alpha)$ if it satisfies the inequality

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right) > \alpha$$

for $z \in \mathbb{E},\ 0 \leq \lambda \leq 1, 0 \leq \alpha < p; p \in \mathbb{N}$.
Analogous to the class $\mathcal{T}_\lambda(p;\alpha)$, we define the class $\mathcal{MT}_p(\lambda;\alpha)$ as follows:
Let $\mathcal{MT}_p(\lambda;\alpha)$ be the subclass of Σ_p consisting of functions $f(z)$ of form (1.1) and satisfying the inequality

$$-\Re\left(\frac{(1+2\lambda)zf'(z)+\lambda z^2 f''(z)}{(1+\lambda)f(z)+\lambda zf'(z)}\right) > \alpha$$

for $z \in \mathbb{E},\ -1 \leq \lambda \leq 0, 0 \leq \alpha < p; p \in \mathbb{N}$. We note that for suitable choices of α and λ, we obtain the following subclasses of $\mathcal{MT}_p(\lambda;\alpha)$.

$$\mathcal{MT}_1(\lambda;\alpha) = \mathcal{MT}(\lambda;\alpha),\ -1 \leq \lambda \leq 0, 0 \leq \alpha < 1,$$

$$\mathcal{MT}_p(0;\alpha) = \mathcal{MS}_p^*(\alpha), 0 \leq \alpha < p; p \in \mathbb{N},$$

$$\mathcal{MT}_p(-1;\alpha) = \mathcal{MK}_p(\alpha), 0 \leq \alpha < p; p \in \mathbb{N},$$

$$\mathcal{MT}(0;\alpha) = \mathcal{MS}_1^*(\alpha) = \mathcal{MS}^*(\alpha) \subseteq \mathcal{MS}^*(0) = \mathcal{MS}^*, 0 \leq \alpha < 1,$$

$$\mathcal{MT}(-1;\alpha) = \mathcal{MK}_1(\alpha) = \mathcal{MK}(\alpha) \subseteq \mathcal{MK}(0) = \mathcal{MK}, 0 \leq \alpha < 1,$$

In this paper, we define above class of $\mathcal{MT}_p(\lambda;\alpha)$ functions in punctured unit disk and use the technique of differential subordination to obtain certain new criteria for starlikeness and convexity of meromorphic functions.

2 Preliminaries

We shall use the following lemma to prove our results.

Lemma 2.1. *([8])Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)],\ h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*
(i) h is convex, or
(ii) Q_1 is starlike.
In addition, assume that
(iii) $\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0$ for all z in $\mathbb{E}$.
If p is analytic in $\mathbb{E}$, with $p(0) = q(0),\ p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],\ z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

3 Main Results

Theorem 3.1. *Let q be univalent in $\mathbb{E}$ with $q(z) \neq \{0, m\}$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \beta\frac{zq'(z)}{q(z)} - \gamma\frac{zq'(z)}{q(z)-m}\right) > 0,\ z \in \mathbb{E}.$$

Suppose that $f \in \Sigma_p$. Define a function $F(z)$ as $F(z) = (1+\lambda)z^2 f(z) + \lambda z^3 f'(z)$, $-1 \leq \lambda \leq 0$ and if F is such that $2 - \frac{zF'(z)}{F(z)} \neq \{0, m\}$ satisfies

$$1 + \frac{z\left(2 - \frac{zF'(z)}{F(z)}\right)'}{\left(2 - \frac{zF'(z)}{F(z)}\right)^\beta \left(2 - \frac{zF'(z)}{F(z)} - m\right)^\gamma} \prec 1 + \frac{zq'(z)}{q^\beta(z)(q(z)-m)^\gamma} \tag{3.1}$$

where the complex powers in (3.1) take their principal values only.Then,

$$2 - \frac{zF'(z)}{F(z)} \prec q(z),\ z \in \mathbb{E},$$

and q is the best dominant.

Proof. For $f \in \Sigma_p$ and $F(z) = (1+\lambda)z^2 f(z) + \lambda z^3 f'(z)$. Hence

$$2 - \frac{zF'(z)}{F(z)} = -\frac{(1+2\lambda)zf'(z) + \lambda z^2 f''(z)}{(1+\lambda)f(z) + \lambda z f'(z)}.$$

Let $u(z) = 2 - \dfrac{zF'(z)}{F(z)}$. Therefore (3.1) reduces to

$$1 + \frac{zu'(z)}{u^{\beta}(z)(u(z)-m)^{\gamma}} \prec 1 + \frac{zq'(z)}{q^{\beta}(z)(q(z)-m)^{\gamma}}$$

Define θ and ϕ as

$$\theta(w) = 1 \quad \text{and} \quad \phi(w) = \frac{1}{w^{\beta}(w-m)^{\gamma}}.$$

Clearly, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0, m\}$ and $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0, m\}$, where $m \in \mathbb{N}_0$. Therefore

$$Q_1(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q^{\beta}(z)(q(z)-m)^{\gamma}}$$

and $h(z) = \theta(q(z)) + Q_1(z) = 1 + Q_1(z)$. A little calculation yields

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \beta\frac{zq'(z)}{q(z)} - \gamma\frac{zq'(z)}{q(z)-m}$$

and

$$\frac{zh'(z)}{Q_1(z)} = \frac{zQ_1'(z)}{Q_1(z)}$$

In view of the given conditions, we have

$$\Re\left(\frac{zQ_1'(z)}{Q_1(z)}\right) > 0.$$

Therefore, $Q_1(z)$ is starlike in $\mathbb{E}$ and

$$\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0,\ z \in \mathbb{E}.$$

The proof, now, follows from Lemma 2.1. □

For $\beta = 1 = \gamma$ and $m = p$, Theorem 3.1 reduces to the following result:

Theorem 3.2. *Let q be univalent in $\mathbb{E}$ with $q(z) \neq \{0, p\}$, $p \in \mathbb{N}$ and satisfying the following condition*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} - \frac{zq'(z)}{q(z)-p}\right) > 0,\ z \in \mathbb{E}.$$

For $f \in \Sigma_p$, define a function $F(z)$ as $F(z) = (1+\lambda)z^2 f(z) + \lambda z^3 f'(z),\ -1 \leq \lambda \leq 0$ and if F is such that $2 - \dfrac{zF'(z)}{F(z)} \neq \{0, p\}$ satisfies the differential subordination

$$1 + \frac{z\left(2 - \frac{zF'(z)}{F(z)}\right)'}{\left(2 - \frac{zF'(z)}{F(z)}\right)\left(2 - \frac{zF'(z)}{F(z)} - p\right)} \prec 1 + \frac{zq'(z)}{q(z)(q(z)-p)},$$

then,

$$2-\frac{zF'(z)}{F(z)}\prec q(z),\ z\in\mathbb{E},$$

and q is the best dominant.

On selecting $q(z)=1+\frac{2}{3}z^2$ as a dominant in the Theorem 3.2, we observe that for a natural number p, $p\geq 2$,

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}-\frac{zq'(z)}{q(z)-p}\right)=\Re\left(\frac{18(1-p)-8z^4}{9(1-p)+6z^2(2-p)+4z^4}\right)>0.$$

Thus, we have the following result:

Theorem 3.3. *For natural number p, $p\geq 2$ and $-1\leq\lambda\leq 0$, let $f\in\Sigma_p$ and define $F(z)=(1+\lambda)z^2f(z)+\lambda z^3f'(z)$. If $F(z)$ with $2-\frac{zF'(z)}{F(z)}\neq\{0,p\}$, satisfies the condition*

$$1+\frac{z\left(2-\frac{zF'(z)}{F(z)}\right)'}{\left(2-\frac{zF'(z)}{F(z)}\right)\left(2-\frac{zF'(z)}{F(z)}-p\right)}\prec 1+\frac{12z^2}{(3+2z^2)(3+2z^2-3p)},$$

then

$$2-\frac{zF'(z)}{F(z)}\prec 1+\frac{2}{3}z^2,\ z\in\mathbb{E},$$

$$i.e.\ \Re\frac{1}{p}\left(2-\frac{zF'(z)}{F(z)}\right)>0,\ z\in\mathbb{E},$$

hence $f\in\mathcal{MT}_p(\lambda,0)$.

On taking $\lambda=0$ in Theorem 3.3 , we have:

Corollary 3.4. *If $f\in\Sigma_p$, $\frac{-zf'(z)}{f(z)}\neq\{0,p\}$, where $p\in\mathbb{N}$ and $p\geq 2$, satisfies*

$$\frac{1+\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}-p}{\frac{-zf'(z)}{f(z)}-p}\prec 1+\frac{12z^2}{(3+2z^2)(3+2z^2-3p)},$$

then

$$-\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{3}z^2,\ z\in\mathbb{E},$$

i.e.

$$-\Re\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E},$$

hence $f\in\mathcal{MS}_p^*$.

Taking $p=2$ in the above corollary, we have:

Corollary 3.5. *If $f\in\Sigma_2$, $\frac{-zf'(z)}{f(z)}\neq\{0,2\}$, satisfies*

$$\frac{1+2\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}}{2+\frac{zf'(z)}{f(z)}}\prec 1+\frac{12z^2}{(3+2z^2)(3+2z^2-3p)},$$

then

$$-\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{3}z^2,\ z\in\mathbb{E},$$

hence $f\in\mathcal{MS}_2^*$.

By selecting $\lambda = -1$ in Theorem 3.3, we get the following result:

Corollary 3.6. *If* $f \in \Sigma_p$, $-\left(1+\dfrac{zf''(z)}{f'(z)}\right) \neq \{0, p\}$, *where* $p \in \mathbb{N}$ *and* $p \geq 2$, *satisfies*

$$\frac{(1+p)\left(1+\dfrac{zf''(z)}{f'(z)}\right)+z^2\left(\dfrac{2f''^2(z)}{f'^2(z)}-\dfrac{f'''(z)}{f'(z)}\right)}{\left(1+\dfrac{zf''(z)}{f'(z)}\right)\left(1+\dfrac{zf''(z)}{f'(z)}+p\right)} \prec 1+\frac{12z^2}{(3+2z^2)(3+2z^2-3p)},$$

then

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{2}{3}z^2,\ z \in \mathbb{E},$$

i.e.

$$-\Re\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{E},$$

hence $f \in \mathcal{MK}_p$.

On taking $p = 2$ in the above corollary, we have:

Corollary 3.7. *If* $f \in \Sigma_2$, $\dfrac{-zf''(z)}{f'(z)} \neq 3$, *satisfies*

$$\frac{3\left(1+\dfrac{zf''(z)}{f'(z)}\right)+z^2\left(\dfrac{2f''^2(z)}{f'^2(z)}-\dfrac{f'''(z)}{f'(z)}\right)}{\left(1+\dfrac{zf''(z)}{f'(z)}\right)\left(3+\dfrac{zf''(z)}{f'(z)}\right)} \prec 1+\frac{12z^2}{4z^4-9},\ z \in \mathbb{E},$$

then

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{2}{3}z^2,\ z \in \mathbb{E},$$

hence $f \in \mathcal{MK}_2$.

On putting $\beta = 1$ and $\gamma = 0$ in Theorem 3.1, we have the following result:

Theorem 3.8. *Let* q *be univalent in* $\mathbb{E}$, *with* q *satisfy*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right) > 0.$$

Suppose that $f \in \Sigma_p, p \in \mathbb{N}$. *Define a function* $F(z)$ *as* $F(z) = (1+\lambda)z^2 f(z) + \lambda z^3 f'(z),\ -1 \leq \lambda \leq 0$ *and if F is such that* $2-\dfrac{zF'(z)}{F(z)} \neq 0$ *satisfies*

$$\frac{\dfrac{-zF'(z)}{F(z)}\left(1+\dfrac{zF''(z)}{F'(z)}-\dfrac{zF'(z)}{F(z)}\right)}{2-\dfrac{zF'(z)}{F(z)}} \prec \frac{zq'(z)}{q(z)},\ z \in \mathbb{E},$$

then,

$$2-\frac{zF'(z)}{F(z)} \prec q(z),\ z \in \mathbb{E}$$

and q *is the best dominant.*

While selecting $q(z) = \dfrac{1+z}{1-z}$ as a dominant in the above theorem, we can easily check that

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{1+z^2}{1-z^2}\right) > 0.$$

Hence we have the following result:

Theorem 3.9. *Let $f \in \Sigma_p, p \in \mathbb{N}$ and $F(z) = (1+\lambda)z^2 f(z) + \lambda z^3 f'(z), -1 \le \lambda \le 0$, such that $\frac{zF'(z)}{F(z)} \ne 2$. If F satisfies*

$$\frac{\frac{-zF'(z)}{F(z)}\left(1+\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}\right)}{2-\frac{zF'(z)}{F(z)}} \prec \frac{2z}{1-z^2},\ z \in \mathbb{E},$$

then,

$$2-\frac{zF'(z)}{F(z)} \prec \frac{1-z}{1+z},\ z \in \mathbb{E},$$

hence $f \in \mathcal{MT}_p(\lambda, 0)$.

On taking $\lambda = 0$ in Theorem 3.9, we have:

Corollary 3.10. *If $f \in \Sigma_p$, $\frac{-zf'(z)}{f(z)} \ne \{0, p\}$, where $p \in \mathbb{N}$ satisfies the following*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{2z}{1-z^2},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

i.e.

$$-\Re\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{E}$$

and hence $f(z) \in \mathcal{MS}_p^*$.i.e. f is p-valently meromorphic starlike.

By putting $p = 1$ in the above corollary, we conclude:

Corollary 3.11. *If $f \in \Sigma$, $\frac{-zf'(z)}{f(z)} \ne 0$, satisfies the differential subordination*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{2z}{1-z^2},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

and hence $f(z) \in \mathcal{MS}^*$.i.e. f is meromorphic starlike.

Selecting $\lambda = -1$ in Theorem 3.9, we get:

Corollary 3.12. *If $f \in \Sigma_p, p \in \mathbb{N}$ satisfy*

$$\frac{\frac{zf''(z)}{f'(z)}\left(1+\frac{zf'''(z)}{f''(z)}-\frac{zf''(z)}{f'(z)}\right)}{1+\frac{zf''(z)}{f'(z)}} \prec \frac{2z}{1-z^2},$$

then

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

i.e.

$$-\Re\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{E}$$

and hence $f(z) \in \mathcal{MK}_p$.

For $p=1$ in the above corollary, we get:

Corollary 3.13. *If $f\in\Sigma$ satisfy*

$$\frac{\dfrac{zf''(z)}{f'(z)}\left(1+\dfrac{zf'''(z)}{f''(z)}-\dfrac{zf''(z)}{f'(z)}\right)}{1+\dfrac{zf''(z)}{f'(z)}}\prec\frac{2z}{1-z^2},$$

then

$$-\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+z}{1-z},\ z\in\mathbb{E}$$

and hence $f(z)\in\mathcal{MK}$.

Selecting $\beta=\gamma=1$ and $m=0$ in Theorem 3.1, we get the following result:

Theorem 3.14. *Let $q(\neq 0)$ be a univalent function satisfying*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-2\frac{zq'(z)}{q(z)}\right)>0, z\in\mathbb{E}.$$

Suppose $f\in\Sigma_p, p\in\mathbb{N}$ and $F(z)$ as $F(z)=(1+\lambda)z^2f(z)+\lambda z^3f'(z),\ -1\leq\lambda\leq 0$ and if F, $2-\dfrac{zF'(z)}{F(z)}\neq 0$ satisfies

$$1+\frac{z\left(2-\dfrac{zF'(z)}{F(z)}\right)'}{\left(2-\dfrac{zF'(z)}{F(z)}\right)^2}\prec 1+\frac{zq'(z)}{q^2(z)}, z\in\mathbb{E},$$

then

$$2-\frac{zF'(z)}{F(z)}\prec q(z)$$

and q is the best dominant.

Selecting $q(z)=\dfrac{1+z}{1-z}$ as a dominant in the above theorem, we can easily check that

$$\Re\left(1+\frac{zq''(z)}{q'(z)}-2\frac{zq'(z)}{q(z)}\right)=\Re\left(\frac{1-z}{1+z}\right)>0.$$

Therefore, from Theorem 3.14, we obtain:

Theorem 3.15. *Let $f\in\Sigma_p, p\in\mathbb{N}$ and $F(z)=(1+\lambda)z^2f(z)+\lambda z^3f'(z), -1\leq\lambda\leq 0$, such that $2-\dfrac{zF'(z)}{F(z)}\neq 0$ satisfies*

$$1+\frac{z\left(2-\dfrac{zF'(z)}{F(z)}\right)'}{\left(2-\dfrac{zF'(z)}{F(z)}\right)^2}\prec\frac{1+4z+z^2}{(1+z)^2}, z\in\mathbb{E},$$

then

$$2-\frac{zF'(z)}{F(z)}\prec\frac{1+z}{1-z},\ z\in\mathbb{E},$$

and hence $f(z)\in\mathcal{MT}_p^*(\lambda,0)$.

For $\lambda=0$ in Theorem 3.15, we have:

Corollary 3.16. *If* $f \in \Sigma_p$, $\dfrac{-zf'(z)}{f(z)} \neq \{0, p\}$, *where* $p \in \mathbb{N}$ *satisfies*

$$\frac{1+\dfrac{zf''(z)}{f'(z)}-\dfrac{2zf'(z)}{f(z)}}{\dfrac{-zf'(z)}{f(z)}} \prec \frac{1+4z+z^2}{(1+z)^2}, z \in \mathbb{E},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

and hence $f(z) \in \mathcal{MS}_p^*$.

Putting $p = 1$ in the above corollary, we have:

Corollary 3.17. *If* $f \in \Sigma$, $\dfrac{-zf'(z)}{f(z)} \neq 0$, *satisfies*

$$\frac{1+\dfrac{zf''(z)}{f'(z)}-\dfrac{2zf'(z)}{f(z)}}{\dfrac{-zf'(z)}{f(z)}} \prec \frac{1+4z+z^2}{(1+z)^2}, z \in \mathbb{E},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

and hence $f(z) \in \mathcal{MS}^*$.

Taking $\lambda = -1$ in Theorem 3.15, we have:

Corollary 3.18. *If* $f \in \Sigma_p$, *where* $p \in \mathbb{N}$ *satisfies*

$$\frac{1+\dfrac{zf''(z)}{f'(z)}\left(1+\dfrac{2zf''(z)}{f'(z)}-\dfrac{zf'''(z)}{f''(z)}\right)}{\left(1+\dfrac{zf''(z)}{f'(z)}\right)^2} \prec \frac{1+4z+z^2}{(1+z)^2}, z \in \mathbb{E},$$

then

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

and hence $f(z) \in \mathcal{MK}_p$.

On substituting $p = 1$ in the above result, we get:

Corollary 3.19. *If* $f \in \Sigma$ *satisfies*

$$\frac{1+\dfrac{zf''(z)}{f'(z)}\left(1+\dfrac{2zf''(z)}{f'(z)}-\dfrac{zf'''(z)}{f''(z)}\right)}{\left(1+\dfrac{zf''(z)}{f'(z)}\right)^2} \prec \frac{1+4z+z^2}{(1+z)^2}, z \in \mathbb{E},$$

then

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z},\ z \in \mathbb{E},$$

and hence $f(z) \in \mathcal{MK}$.

Selecting $\beta = 0 = \gamma$ in Theorem 3.1, we have the following result:

Theorem 3.20. *Let q be a convex univalent function and let $f \in \Sigma_p, p \in \mathbb{N}$. Define $F(z) = (1+\lambda)z^2 f(z) + \lambda z^3 f'(z), \ -1 \leq \lambda \leq 0$ and if F is such that $-\dfrac{zF'(z)}{F(z)} \neq 0$ satisfies*

$$\frac{-zF'(z)}{F(z)}\left(1+\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}\right) \prec \ zq'(z), \ z \in \mathbb{E},$$

then,

$$2-\frac{zF'(z)}{F(z)} \prec q(z), \ z \in \mathbb{E},$$

and q is the best dominant.

Taking $q(z) = 1 + \dfrac{2}{3}z^2$ as a dominant in the above theorem. A little calculation yields that $\Re\left(1+\dfrac{zq''(z)}{q'(z)}\right) = 2 > 0$. Hence , we get the following result:

Theorem 3.21. *Let $f \in \Sigma_p, p \in \mathbb{N}$.If $F(z) = (1+\lambda)z^2 f(z) + \lambda z^3 f'(z), \ -1 \leq \lambda \leq 0$ and $2-\dfrac{zF'(z)}{F(z)} \neq 0$ satisfies*

$$\frac{-zF'(z)}{F(z)}\left(1+\frac{zF''(z)}{F'(z)}-\frac{zF'(z)}{F(z)}\right) \prec \frac{4}{3}z^2, z \in \mathbb{E},$$

then

$$2-\frac{zF'(z)}{F(z)} \prec 1+\frac{2}{3}z^2,$$

hence $f \in \mathcal{MT}_p(\lambda, 0)$.

For $\lambda = 0$ in Theorem 3.21, we get:

Corollary 3.22. *For $f \in \Sigma_p, p \in \mathbb{N}$.If f satisfies*

$$\frac{-zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{4}{3}z^2, z \in \mathbb{E},$$

then

$$-\frac{zf'(z)}{f(z)} \prec 1+\frac{2}{3}z^2,$$

hence $f \in \mathcal{MS}_p^*$.

For $p = 1$ in the above corollary, we have:

Corollary 3.23. *For $f \in \Sigma, -\dfrac{zf'(z)}{f(z)} \neq 0$, satisfies*

$$\frac{-zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{4}{3}z^2, z \in \mathbb{E},$$

then

$$-\frac{zf'(z)}{f(z)} \prec 1+\frac{2}{3}z^2,$$

hence $f \in \mathcal{MS}^*$.

Putting $\lambda = -1$ in Theorem 3.21, we have:

Corollary 3.24. *For* $f \in \Sigma_p, -\left(1+\frac{zf''(z)}{f'(z)}\right) \neq 0, p \in .$ *If f satisfies*

$$\frac{-zf''(z)}{f'(z)}\left(1+\frac{zf'''(z)}{f''(z)}-\frac{zf''(z)}{f'(z)}\right) \prec \frac{4}{3}z^2, z \in \mathbb{E},$$

then

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{2}{3}z^2,$$

hence $f \in \mathcal{MK}_p$.

For $p = 1$ in the above corollary, we get:

Corollary 3.25. *If* $f \in \Sigma$ *satisfies*

$$\frac{-zf''(z)}{f'(z)}\left(1+\frac{zf'''(z)}{f''(z)}-\frac{zf''(z)}{f'(z)}\right) \prec \frac{4}{3}z^2, z \in \mathbb{E},$$

then

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{2}{3}z^2,$$

hence $f \in \mathcal{MK}$.

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