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α- Wronskian- Theory and Existence results

Ajay Dixit and Amit Ujlayan

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Abstract The paper deals with the concept of α Wronskian for the following fractional differential equation

$$(D^{\alpha}D^{\alpha}+P(s)D^{\alpha}+Q(s))y(s)=R(s).$$

We establish the relation between proposed Wronskian and ordinary Wronskian and obtain some important results including Abel's formula for fractional differential equations. The method of variation of constants has been briefly explained to solve the problems. The procedure is very clear to understand how to observe one of the solutions of the basis of the differential equation, then to find other independent solutions and therefore to reach the complete solution. The fractional derivative used is the UD derivative.

1 Introduction

A solution of fractional order differential equations is now one of the emerging areas of present research as nowadays the research is being done in fractional space. To model, a physical phenomena differential equation of arbitrary order plays a major role to study. Various studies show their frequent appearances in different applications such as plasma physics, biomathematics, theology fluid mechanics, control theory, nanoscience engineering, signal processing, etc. Generally, one faces the problem to find the solutions of concerned fractional differential systems [3, 4, 10].
Though several fractional derivatives have been suggested by many mathematicians with their different approaches like Grunwald-Letnicov presented the fractional derivative in discrete form using a backward difference operator, Riemann-Liouville introduced the fractional derivative in continuous form by defining fractional integral first, Caputo made some modifications in the R-L definition [2, 5, 7], etc. Besides these, series solution (Mittag-Leffler) and numerical schemes (ADM) [1, 8, 12] are also there. The method of variation of constants is not applicable always [9] due to some lack of basic properties. One has to ensure that which definition is conformable with the applied process.
Recall that the concept of Wronskian exhibits a prerequisite role in the study of differential equations. Here we work with the UD derivative [6, 11] which is also defined in classical form, but with some different approach. Proposed Wronskian is an extension of the usual Wronskian in the natural sense. Some problems have been discussed to illustrate the application of the work.

2 Basic definitions

Definition 2.1. Let $h(s)$ be a real valued function defined on $[0,\infty)$, then the UD derivative [6] is defined in the following manner

$$D^{\alpha}h(s)=\alpha h'(s)+(1-\alpha)h(s),\ \ where\ \ 0\leqslant\alpha\leqslant 1, \tag{2.1}$$

provided $h(s)$ is differentiable.

Definition 2.2. The UD -Integral [6] of a continuous function $h(s)$ for positive real numbers is defined as:

$$I^{\alpha}(h(s))=\frac{1}{\alpha}e^{-\frac{(1-\alpha)}{\alpha}s}\int h(s).e^{\frac{(1-\alpha)}{\alpha}s}ds+Ae^{-\frac{(1-\alpha)}{\alpha}s};\ \alpha\in(0,1], \tag{2.2}$$

where A is constant.

Definition 2.3. Consider the second order linear fractional differential equation

$$(D^{\alpha} D^{\alpha} + P(s) D^{\alpha} + Q(s))y = R(s). \tag{2.3}$$

If u, v be its two basic solutions, then α- Wronskian W_{α} is presented as

$$\begin{aligned} W_{\alpha}[u,v] &= \begin{vmatrix} u & v \\ D^{\alpha}u & D^{\alpha}v \end{vmatrix} \\ &= \alpha(uv' - vu') \\ &= \alpha W[u,v] \end{aligned}$$

where W denotes the ordinary Wronskian of the linear differential equation of second order.

Remark 2.4. It is to be noted that if u, v are dependent(or independent) for the ordinary DE then they will be the same for the corresponding fractional differential equation except α is not equal to zero.

3 Some results of α- Wronskian

Theorem 3.1. *Let u, v be two basis solutions of (2.3) and if the W_{α} is zero then*

$$\frac{d^{\alpha}}{ds^{\alpha}}\left(\frac{u}{v}\right) = (1-\alpha)\frac{u}{v} \tag{3.1}$$

Proof. With UD derivative,

$$\begin{aligned} ((1-\alpha) + \alpha D)\left(\frac{u}{v}\right) &= \frac{(1-\alpha)uv + \alpha(uv' - vu')}{v^2} \\ &= (1-\alpha)\frac{u}{v}. \end{aligned}$$

Here it is clear that if $W = 0$ instead of $W_{\alpha} = 0$, we get the same result.

Corollary 3.2. *If u, v are constant multiple of each other that is both are dependent then*

$$\frac{d^{\alpha}}{ds^{\alpha}}\left(\frac{u}{v}\right) = (1-\alpha)\frac{u}{v}; \alpha \in [0,1].$$

Theorem 3.3.

$$(2(1-\alpha) + P(s))W_{\alpha} + \alpha^2 DW = 0, \tag{3.2}$$

where D and W denotes the ordinary differentiation and ordinary Wronskian respectively.

Proof. We have

$$\begin{aligned} D^{\alpha}\left[W_{\alpha}(u,v)\right] &= \alpha((1-\alpha) + \alpha D)(uv' - vu') \\ &= \alpha\left[(1-\alpha)(uv' - vu') + \alpha(uv'' - vu'')\right] \\ &= \alpha(1-\alpha)W[u.v] + \alpha^2 DW[u.v]. \end{aligned}$$

Also,

$$\begin{aligned} D^{\alpha}\left[W_{\alpha}(u,v)\right] &= uD^{\alpha}D^{\alpha}v - vD^{\alpha}D^{\alpha}u - (1-\alpha)uv + (1-\alpha)vD^{\alpha}u \\ &= u(-PD^{\alpha} - Q)v - v(-PD^{\alpha} - D^{\alpha})u - (1-\alpha)uv + (1-\alpha)vD^{\alpha}u \\ &= (P + (1-\alpha))(vD^{\alpha}u - uD^{\alpha}v) \\ &= -(P + (1-\alpha))W_{\alpha}. \end{aligned}$$

Therefore,

$$\alpha(1-\alpha)W + \alpha^2 DW + (P + (1-\alpha))W_{\alpha} = 0. \tag{3.3}$$

Theorem 3.4.

$$W_\alpha[u,v] = e^{-\frac{1}{\alpha}\int (P+2(1-\alpha))ds}. \tag{3.4}$$

Proof. From the above theorem we have

$$D^\alpha(W_\alpha) = -(P+(1-\alpha))W_\alpha$$
$$\Rightarrow \frac{D(W_\alpha)}{W_\alpha} = -\frac{P+2(1-\alpha)}{\alpha}$$

solving we get

$$W_\alpha[u,v] = Ae^{-\frac{1}{\alpha}\int (P+2(1-\alpha))dt}.$$

Theorem 3.5. *If one solution of second order linear fractional differential equation*

$$(D^\alpha D^\alpha + P(s)D^\alpha + Q(s))y = 0 \tag{3.5}$$

is u then the other independent solution is given by

$$v = \frac{u}{\alpha}\int \frac{e^{-\frac{1}{\alpha}\int (P+2(1-\alpha))ds}}{u^2}ds. \tag{3.6}$$

Proof. Let the second solution is $v = u.r$ where u is known solution and r is a function of s. Then,

$$\begin{aligned} W_\alpha &= \alpha(uv' - vu') \\ &= \alpha[u(ur' + ru') - u'ru] \\ &= \alpha r' u^2. \end{aligned}$$

Now

$$\begin{aligned} W_\alpha[u,v] &= Ae^{-\frac{1}{\alpha}\int (P+2(1-\alpha))ds} \\ &\Rightarrow \alpha u^2 r' = Ae^{-\frac{1}{\alpha}\int (P+2(1-\alpha))ds} \\ &\Rightarrow s = C\int \frac{e^{-\frac{1}{\alpha}\int (P+2(1-\alpha))ds}}{u^2}ds. \end{aligned}$$

and hence, complete solution of homogeneous part will be

$$y = c_1 u + c_2 u\int \frac{e^{-\frac{1}{\alpha}\int (P+2(1-\alpha))ds}}{u^2}ds. \tag{3.7}$$

Theorem 3.6. *In (3.5)*

(a) $y = s^m$ *is the one of the part of the basis if*

$$((1-\alpha)^2 + \alpha^2 m(m-1) + 2\alpha(1-\alpha)m) + s(\alpha + (1-\alpha)m)P(s) + s^2 Q(s) = 0.$$

(b) $y = e^{ms}$ *is the one of the part of the basis if*

$$((1-\alpha)^2 + \alpha^2 m^2 + 2\alpha(1-\alpha)m) + ((1-\alpha) + \alpha m)P(s) + Q(s) = 0.$$

Proof. Using the definition of UD derivative to get

$$((1-\alpha) + \alpha D)^2 s^m + P(s)((1-\alpha) + \alpha D)s^m + s^m Q(s) = 0.$$

and

$$((1-\alpha) + \alpha D)^2 e^{ms} + P(s)((1-\alpha) + \alpha D)e^{ms} + e^{ms} Q(s) = 0.$$

respectively.

4 Technique to solve the concerned problems

Consider the first order fractional linear DE

$$(D^\alpha + p(s))y(s) = r(s); D^\alpha \equiv ((1-\alpha) + \alpha D); \alpha \in (0,1] \tag{4.1}$$

let $u(s)$ be the solution of homogeneous part of (4.1).
Assume that particular solution is $y_p = v(s)u(s)$ such that y_p satisfy (4.1). We get

$$v'(s) = \frac{r(s)}{\alpha u(s)}$$

and so general solution is given by

$$y(s) = c_1 u + \frac{1}{\alpha}\int \frac{r(s)}{u(s)} ds. \tag{4.2}$$

which is same as

$$y(s) = c_1 e^{-\frac{(p(s)+(1-\alpha))s}{\alpha}} + \frac{1}{\alpha}\int r(s) e^{\frac{(p(s)+(1-\alpha))s}{\alpha}} ds$$

as

$$u(s) = \frac{1}{\alpha} e^{\frac{-(p(s)+(1-\alpha))s}{\alpha}}.$$

Now consider the second order linear FDE

$$(D^\alpha D^\alpha + P(s)D^\alpha + Q(s))y(s) = R(s);\ \alpha \in (0,1] \tag{4.3}$$

If two basis solutions of (4.3) are u and v then we try to write its particular solution $y_p = Au + Bv$.

where A and B are functions depends on s with the following condition:

$$A'u + B'v = 0. \tag{4.4}$$

Since, $D^\alpha y = (1-\alpha)(Au + Bv) + \alpha(Au' + Bv')$ and

$$D^\alpha D^\alpha y = (1-\alpha)[(1-\alpha)(Au+Bv) + \alpha(Au'+Bv')] + \alpha[(1-\alpha)(Au'+Bv') +$$
$$\alpha(A'u' + B'v' + Au'' + Bv'')].$$

Putting the values of $D^\alpha D^\alpha y, D^\alpha y, y$ with (4.3) using (4.4), we obtain

$$\alpha^2(A'u' + B'v') = R \tag{4.5}$$

solving (4.4) and (4.5) for A and B, we get

$$A = -\frac{1}{\alpha}\int \frac{Rv}{W_\alpha} ds \ \ and \ \ B = \frac{1}{\alpha}\int \frac{Ru}{W_\alpha} ds \tag{4.6}$$

and therefore complete solution is

$$y(s) = c_1 u + c_2 v + Au + Bv.$$

5 Solved examples

The section demonstrates, some solved problems, based on the discussed theory.

Example 5.1. Consider the example

$$(D^{2/3}D^{2/3} - 4D^{2/3} + 3)y = \sin e^{-s}$$

obviously the independent solutions of the homogeneous part are e^s and e^{4s} using Theorem 3.6 so that,

$$W_{2/3} = 2e^{5s}.$$

$$\begin{aligned} A &= -\frac{1}{\alpha}\int \frac{Rv}{W_\alpha} ds \\ &= -\frac{3}{2}\int \frac{e^{4s}\sin e^{-s}}{2e^{5s}} ds \\ &= \frac{3}{4}\cos e^{-s}, \end{aligned}$$

and

$$\begin{aligned} B &= \frac{1}{\alpha}\int \frac{Ru}{W_\alpha} ds \\ &= \frac{3}{2}\int \frac{e^{s}\sin e^{-s}}{2e^{5s}} ds \\ &= \frac{3}{4}(e^{-3s}\cos e^{-s} - 3e^{-2s}\sin e^{-s} - 6e^{-s}\cos e^{-s} + 6\sin e^{-s}). \end{aligned}$$

Therefore,

$$y = c_1 e^s + c_2 e^{4s} + Ae^s + Be^{4s}.$$

Example 5.2. Consider the example

$$s^2 \frac{d^{1/2}}{ds^{1/2}}\left(\frac{d^{1/2}y}{ds^{1/2}}\right) - (s^2+2s)\frac{d^{1/2}y}{ds^{1/2}} + \left(s+\frac{5}{4}\right) y = s^6 e^s$$

we may write

$$\left(D^{1/2}D^{1/2} - \left(1+\frac{2}{s}\right)D^{1/2} + \frac{1}{s^2}\left(s+\frac{5}{4}\right)\right) y = s^4 e^s$$

we observe that $y = s$ satisfy the homogeneous part of above equation so $u = s$ and get the second solution using Theorem 3.5

$$\begin{aligned} v &= \frac{u}{\alpha}\int \frac{e^{-\frac{1}{\alpha}\int (P+2(1-\alpha))ds}}{u^2} ds \\ &= 2\int \frac{e^{\int -2\left(1+\frac{2}{s}\right)ds}}{s^2} ds \\ &= \frac{2}{3}s^4. \end{aligned}$$

Solution set of the basis is $\left\{s, \frac{2}{3}s^4\right\}$ so that $W_{\frac{1}{2}} = 4s^4$.
Now

$$\begin{aligned} A &= -\frac{1}{\alpha}\int \frac{Rv}{W_\alpha} ds \\ &= -2\int \frac{s^4 e^s\left(\frac{2}{3}s^4\right)}{4s^4} ds \\ &= -\frac{1}{3}(s^4 - 4s^3 + 12s^2 - 24s + 24)e^s \end{aligned}$$

and

$$\begin{aligned} B &= \frac{1}{\alpha}\int \frac{Ru}{W_\alpha} ds \\ &= 2\int \frac{s^4 e^s\,(s)}{4s^4} ds \\ &= \frac{1}{2}(s-1)e^s. \end{aligned}$$

Therefore,

$$y = c_1 s + c_2 s^4 + As + Bs^4.$$

6 Conclusion

We investigate the α Wronskian with its properties and develop the method of variation of constants using the UD derivative. Establishing various results here the algorithm is used to solve some fractional differential equation which coincides with classical differential equation at α=1. A new approach for fractional derivative has been used which produces analytic results but is different from the classical derivative and Conformable derivative.

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Author information

Ajay Dixit, Department of Applied Sciences, KIET Group of Institutions Ghaziabad 201206., India.
E-mail: `hellodxtajay@gmail.com`

Amit Ujlayan, School of Vocational Studies and Applied Sciences Gautam Buddha University, Gr. Noida., India.
E-mail: iitk.amit@gmail.com

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