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NUMERICAL SOLUTION OF VARIABLE ORDER FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS USING ORTHONORMAL FUNCTIONS

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Abstract In this article, orthonormal functions and their operational matrices are proposed to solve the variable order fractional integro-differential equations. Operational matrices for the integer order derivative, the variable order derivative and the integration of these functions are derived here. Using orthonormal functions with the operational matrices, the proposed integral equation is transformed into a system of algebraic equations after choosing suitable collocation points. Existence and uniqueness of the solution are presented. Illustrative examples are provided to demonstrate the accuracy and the applicability of the present method.

1 Introduction

Many physical processes appear to exhibit fractional order behavior that may vary with time or space. Fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary non-integer order. The problem of fractional differential calculus raised by Leibniz dates back to 1695, and it has been an ongoing research for the past three centuries. Many models in interdisciplinary fields can be elegantly described with the help of fractional derivative, such as dynamics of earthquakes, viscoelastic systems, diffusion model, wave propagation, and partial bed-load transport. The continuum of order in fractional calculus allows the order of the fractional operator to be considered as a variable. Some practical examples of variable order fractional differential operators are: anomalous diffusion modeling [1], mechanical applications [2], multifractional Gaussian noises [3], FIR filters [4]. Moreover a physical experimental study of variable-order fractional operators has been considered in [5].

Due to the fractional order exponents in differential operators, analytic solutions of FDEs are usually difficult to obtain. Indeed, it is too tough task to compute an exact solution of a wide class of the differential equations of fractional order. In the past years, different kind of vigorous techniques has been introduced to find an analytical solution of fractional differential equations, such as homotopy analysis method [6], homotopy perturbation method [7], variational iteration method [8], Adomian decomposition method [9], and hybridization of Fourier and power series method [10]. Sahu et al. have solved fractional integro-differetial equations by sinc-Galerkin method [11] and Legendre wavelet method [12]. A compact difference scheme has been applied to solve a class of space-time fractional differential equations [13]. Chebyshev wavelets method has been applied to find numerical solution of fractional partial differential equation of parabolic type [14]. The analytical method for solving variable order fractional differential equation is very scarce in literature. Some numerical techniques are used to solve fractional differential equation of variable order. Shen et al. [15] have given an approximate scheme for the variable order time fractional diffusion equation. Chen et al. [16, 17] paid their attention to Bernstein polynomials to solve variable order linear cable equation and variable order time fractional diffusion equation. Liu et al. [18] have applied an operational matrix technique for solving variable order fractional differential-integral equation based on the second kind of Chebyshev polynomials.

In this article, we have considered the variable order fractional integro-differential equation

(VOFIDE) given by

$$D^{\alpha(t)}y(t) + a(t)\int_0^t y(\tau)d\tau + b(t)y'(t) + c(t)y(t) = f(t) \tag{1.1}$$

with initial condition $y(0) = y_0,\ \ t \in [0,1]$ and $D^{\alpha(t)}y(t)$ is the fractional derivative of $y(t)$ in Caputo's sense. The functions $a(t)$, $b(t)$, $c(t)$ and $f(t)$ are real continuous functions. Equation 1.1 has been solved by orthonormal function approximation (OFA) based on Legendre polynomials.

The rest of the paper has been organized as follows: in section 2 and 3, we introduce the preliminaries of fractional calculus and orthonormal functions, respectively. Section 4 describes the existence and uniqueness of the solution. In section 5 and 6, we discuss the function approximations and operational matrices, respectively. Numerical scheme has been provided in section 7. Section 8 deals with the illustrative examples which show the efficiency and the accuracy of the present method.

2 Preliminaries

In this section, we provide some important definitions and results of fractional calculus.

Definition 2.1. (*Riemann-Liouville fractional Integral*)
The Riemann-Liouville fractional integral [19] of order $\alpha > 0$ of a function f is defined as

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau,\ t>0,\ \alpha\in\mathbb{R}^+,$$

Definition 2.2. (*Caputo fractional derivative*)
The fractional derivative, introduced by Caputo [20, 21] in the late sixties, is called Caputo fractional derivative. The fractional derivative of $f(t)$ in the Caputo sense is defined by

$$\begin{aligned} D_t^{\alpha}f(t) &= J^{m-\alpha}D_t^m f(t) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{d^m f(\tau)}{d\tau^m}d\tau, \\ \qquad m-1<\alpha<m,\ \ m\in N, \\ \frac{d^m f(t)}{dt^m},\ \ \alpha = m, \quad m\in N, \end{cases} \end{aligned}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α has been considered.

For the Caputo derivative, we have

$$D^{\alpha}C = 0, \qquad (C\ \ is\ \ a\ \ constant)$$

$$D^{\alpha}t^{\beta} = \begin{cases} 0, & \beta \le \alpha-1, \\ \frac{\Gamma(\beta+1)t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, & \beta > \alpha-1. \end{cases}$$

Also,

$$^{C}D_t^{\alpha}J^{\alpha}f(t) = f(t)$$

and

$$J^{\alpha}[\,^{C}D_t^{\alpha}f(t)] = f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0+), \quad t>0.$$

3 Orthonormal functions

In this paper, the orthonormal functions are constructed from Legendre polynomials. These functions can be constructed by dilation and translation of Legendre polynomials as

$$\psi_{k,n,m}(t) = |a|^{k/2} P_m(a^k t - nb), n = 1, 3, ..., 2^k - 1, \tag{3.1}$$

where a and b are dilation and translation parameters, respectively. Here $P_m(t)$ is Legendre polynomials of order m, which are orthogonal with respect to weight function $w(t) = 1$ on the interval $[-1, 1]$. This can be determined from the following recurrence formulae

$$\begin{aligned}
&P_0(t) = 1, \\
&P_1(t) = t, \\
&P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t)
\end{aligned}$$

for $m = 1, 2, 3, ...$ and the orthogonal property of Legendre polynomials on the interval $-1 \leq t \leq 1$ as

$$\int_{-1}^{1} P_m(t) P_n(t) dx = \frac{2}{2n+1} \delta_{mn}, \tag{3.2}$$

where δ_{mn} is the Kronecker delta function which takes the value 1, when $m = n$ and 0, when $m \neq n$.

In particular, when $a = 2$ and $b = 1$, then $\psi_{k,n,m}(t)$ in eq. (3.1) can be written as

$$\psi_{k,n,m}(t) = 2^{k/2} P_m(2^k t - n). \tag{3.3}$$

The orthogonal functions defined in eq. (3.3) form a basis for $L^2[0, 1)$. Choosing $k = 1$, eq. (3.3) can be expressed as

$$\psi_m(t) = \sqrt{m + \frac{1}{2}}\; 2^{k/2} P_m(2t - 1), \tag{3.4}$$

where $m = 0, 1, ..., M - 1$. The coefficient $\sqrt{m + \frac{1}{2}}$ is for orthonormality.

4 Existence and uniqueness

Let us consider the eq. (1.1) in operator form as

$$D^\alpha y(t) + a(t)\mathcal{K}y(t) + \mathcal{F}y(t) = 0, \tag{4.1}$$

where $\mathcal{K}y(t) = \int_0^t y(\tau) d\tau$ and $\mathcal{F}y(t) = b(t)y'(t) + c(t)y(t) - f(t)$. Applying J^α both sides of eq. (4.1), we have

$$y(t) = h(t) - J^\alpha [a(t)\mathcal{K}y(t) + \mathcal{F}y(t)], \tag{4.2}$$

where $h(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y^{(k)}(0+)$. Eq. (4.2) can be written as fixed point equation form $\mathcal{A}y = y$, where $\mathcal{A}$ is defined as

$$\mathcal{A}y(t) = h(t) - J^\alpha [a(t)\mathcal{K}y(t) + \mathcal{F}y(t)].$$

Let $(C[0, 1], \|.\|_\infty)$ be the Hilbert space of all continuous functions with infinity norm. Also, the operator $\mathcal{F}$ satisfies the Lipschitz condition on $[0, 1]$ as $|\mathcal{F}\tilde{y} - \mathcal{F}y| \leq L|\tilde{y} - y|$, where L is the Lipschitz constant. Then, we proceed to prove the uniqueness of the solution of the eq. (1.1). Let $\mathcal{A} : C[0, 1] \to C[0, 1]$ such that

$$\mathcal{A}y(t) = h(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t - z)^{\alpha - 1} [a(z)\mathcal{K}y(z) + \mathcal{F}y(z)] dz.$$

Let $\tilde{y}, y \in C[0,1]$ and

$$\begin{aligned}
\mathcal{A}\tilde{y}(t) - \mathcal{A}y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1}[a(z)[\mathcal{K}\tilde{y}(z) - \mathcal{K}y(z)] \\
&\quad + [\mathcal{F}\tilde{y}(z) - \mathcal{F}y(z)]]dz \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1}[(a(z)|\mathcal{K}| + L)|\tilde{y} - y|]dz \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-z)^{\alpha-1}(\|a(z)\|_\infty \|\mathcal{K}\|_\infty + L)\|\tilde{y} - y\|_\infty dz \\
&\leq (\|a\|_\infty \|\mathcal{K}\|_\infty + L)\|\tilde{y} - y\|_\infty \frac{|t|^\alpha}{\Gamma(\alpha+1)} \\
&\leq (\|a\|_\infty \|\mathcal{K}\|_\infty + L)\frac{1}{\Gamma(\alpha+1)}\|\tilde{y} - y\|_\infty.
\end{aligned}$$

For $(\|a\|_\infty \|\mathcal{K}\|_\infty + L)\frac{1}{\Gamma(\alpha+1)} < 1$ and by contraction mapping theorem, the eq. (1.1) has an unique solution.

5 Function approximation

A function $f(t)$ defined over $[0,1)$ can be expressed by the orthonormal functions as

$$f(t) = \sum_{m=0}^{\infty} c_m \psi_m(t), \tag{5.1}$$

where $c_m = \langle f(t), \psi_m(t)\rangle$, in which $\langle ., . \rangle$ denotes the inner product. If the infinite series in eq. (5.1) is truncated, then eq. (5.1) can be written as

$$f(t) \cong \sum_{m=0}^{M-1} c_m \psi_m(t) = C^T \Psi(t), \tag{5.2}$$

where C and $\Psi(t)$ are $(M \times 1)$ matrices given by

$$C = [c_0, c_1, ..., c_{M-1}]^T, \tag{5.3}$$

$$\Psi(t) = [\psi_0(t), \psi_1(t), ..., \psi_{M-1}(t)]^T. \tag{5.4}$$

Theorem 5.1. *If $y(t) \in L^2(\mathbb{R})$ be a continuous function defined on $[0,1]$ and $\|y(t)\| \leq \mathcal{M}_y$, then the orthonormal functions expansion of $y(t)$ defined in eq. (5.2) converges uniformly and also*

$$\|c_m\| \leq \sqrt{2}\mathcal{M}_y.$$

Proof: Any function $y(t) \in L^2[0,1]$ can be expressed by orthonormal functions as

$$y(t) \cong \sum_{m=0}^{M-1} c_m \psi_m(t),$$

where the coefficients c_m can be determined as

$$c_m = \langle y(t), \psi_m(t)\rangle.$$

Now for $m \geq 0$,

$$\begin{aligned}
c_m &= \langle y(t), \psi_m(t)\rangle \\
&= \int_0^1 y(t)\psi_m(t)dt \\
&= \sqrt{2}\sqrt{m + \frac{1}{2}}\int_0^1 y(t)P_m(2t-1)dt.
\end{aligned}$$

Now, changing the variable $2t - 1 = z$, we have

$$c_m = \sqrt{m + \frac{1}{2}} 2^{-1/2} \int_{-1}^{1} y(\frac{z+1}{2}) P_m(z) dz.$$

Now,

$$\begin{aligned}
\|c_m\| &\leq \sqrt{m + \frac{1}{2}} 2^{-1/2} \int_{-1}^{1} \|y(\frac{z+1}{2})\| \|P_m(z)\| dz \\
&\leq \sqrt{m + \frac{1}{2}} 2^{-1/2} \mathcal{M}_y \int_{-1}^{1} \|P_m(z)\| dz \\
&= \sqrt{m + \frac{1}{2}} 2^{-1/2} \mathcal{M}_y \sqrt{\frac{2}{2m+1}} \int_{-1}^{1} dz \\
&= 2^{1/2} \mathcal{M}_y.
\end{aligned}$$

This means that the series $\sum_{m=0}^{M-1} c_m$ is absolutely convergent and hence the series $\sum_{m=0}^{M-1} c_m \psi_m(t)$ is uniformly convergent.

6 Operational Matrices

Consider eq. (5.4), we obtain, $\Psi(t) = A \cdot T_{m-1}(t)$, where,

$$T_{m-1}(t) = [1, t, t^2, ..., t^{m-1}]^T \tag{6.1}$$

and A matrix can be defined as

$$A = P^{-1} \quad \text{and} \quad P = [p_{i,j}], \tag{6.2}$$

where,

$$p_{i,j} = \sqrt{2j+1} \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \binom{j+k}{k} \left(\frac{1}{i+k+1} \right)$$

for $0 \leq i \leq M-1$ and $0 \leq j \leq M-1$.

Operational matrix of derivative: Operational matrix of derivative of orthonormal functions defined in the eq. (5.4) is defined as

$$\Psi'(t) = \frac{d}{dt} \Psi(t) = A \cdot \frac{d}{dt} T_{m-1}(t) = A \cdot D \cdot T_{m-1}(t), \tag{6.3}$$

where

$$D = \begin{bmatrix} 0 & 0 & 0 & . & . & & . & 0 \\ 1 & 0 & 0 & . & . & & . & 0 \\ 0 & 2 & 0 & . & . & & . & 0 \\ . & . & . & . & & & & . \\ . & . & . & & . & & & . \\ . & . & . & & & & . & . \\ 0 & 0 & 0 & . & . & & M-1 & 0 \end{bmatrix}.$$

Operational matrix of integration: Operational matrix of integration of orthonormal functions defined in the eq. (5.4) is defined as

$$\begin{aligned}
\int_0^t \Psi(t') dt' &= A \int_0^t T_{m-1}(t') dt' \\
&= A \cdot J \cdot T_{m-1}(t) + \frac{t^M}{M} I_M,
\end{aligned} \tag{6.4}$$

where

$$J = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & \frac{1}{2} & . & . & . & 0 \\ 0 & 0 & 0 & \frac{1}{3} & . & . & 0 \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & \frac{1}{M-1} \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}$$

and I_M is the last column of the identity matrix of order M. We can approximate eq. (6.4) as

$$\int_0^t \Psi(t')dt' \cong A \cdot J \cdot T_{m-1}(t). \tag{6.5}$$

Operational matrix of fractional derivative: Operational matrix of fractional derivative of orthonormal functions defined in the eq. (5.4) is defined as

$$\begin{aligned} {}^C D_0^{\alpha(t)} \Psi(t) = {}^C D_0^{\alpha(t)} A \cdot T_{M-1}(t) = A \cdot {}^C D_0^{\alpha(t)} T_{M-1}(t) \\ = A \cdot M_\alpha(t) \cdot T_{M-1}(t), \end{aligned} \tag{6.6}$$

where

$$M_\alpha(t) = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & \frac{\Gamma(2)t^{-\alpha(t)}}{\Gamma(2-\alpha(t))} & 0 & ... & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{-\alpha(t)}}{\Gamma(3-\alpha(t))} & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & ... & \frac{\Gamma(M)t^{-\alpha(t)}}{\Gamma(M-\alpha(t))} \end{bmatrix}.$$

7 Numerical method

Consider the variable order fractional integro-differential equation given in eq. (1.1) as

$$D^{\alpha(t)} y(t) + p(t) \int_0^t y(\tau)d\tau + q(t)y'(t) + r(t)y(t) = f(t). \tag{7.1}$$

Using eq. (5.2), approximate the unknown function $y(t)$ as

$$y(t) = C^T \Psi(t) = C^T \cdot A \cdot T_{M-1}(t). \tag{7.2}$$

From eqs. (6.3), (6.5), (6.6) and (7.2), we have

$$y'(t) = C^T \cdot A \cdot D \cdot T_{M-1}(t), \tag{7.3}$$

$$\int_0^t y(\tau)d\tau = C^T \cdot A \cdot J \cdot T_{M-1}(t), \tag{7.4}$$

and

$$D^{\alpha(t)} y(t) = C^T \cdot A \cdot M_\alpha(t) \cdot T_{M-1}(t). \tag{7.5}$$

Applying eqs. (7.2)-(7.5) in eq. (7.1), we have

$$\begin{aligned} C^T \cdot A \cdot M_\alpha(t) \cdot T_{M-1}(t) + p(t)(C^T \cdot A \cdot J \cdot T_{M-1}(t)) \\ + q(t)(C^T \cdot A \cdot D \cdot T_{M-1}(t)) \\ + r(t)(C^T \cdot A \cdot T_{M-1}(t)) = f(t) \end{aligned} \tag{7.6}$$

Table 1. Approximate solutions and absolute errors by OFA method for Example 1

x	Exact	OFA	Error
0	1	1	0
0.1	0.91	0.91	2.77556e-15
0.2	0.84	0.84	3.3067e-15
0.3	0.79	0.79	3.5271e-15
0.4	0.76	0.76	3.77476e-15
0.5	0.75	0.75	3.88578e-15
0.6	0.76	0.76	3.88578e-15
0.7	0.79	0.79	3.66374e-15
0.8	0.84	0.84	3.21965e-15
0.9	0.91	0.91	2.88658e-15

$$\Longrightarrow C^T \cdot A \cdot (M_\alpha(t) + p(t)J + q(t)D + r(t)I) \cdot T_{M-1}(t) \\ = f(t), \tag{7.7}$$

where I is an identity matrix of order M. Applying the collocation points $t_i = \frac{2i-1}{2M}, \quad i = 1, 2, ..., M$ in eq. (7.7), we have

$$C^T \cdot A \cdot (M_\alpha(t_i) + p(t_i)J + q(t_i)D + r(t_i)I) \cdot T_{M-1}(t_i) \\ = f(t_i). \tag{7.8}$$

Eq. (7.8) gives a system of linear equations with M-unknowns, i.e., $c_0, c_1, ..., c_{M-1}$. Solving this system by Newton's method, we will get the unknowns $c_m, \quad m = 0, 1, ..., M-1$ and hence the approximate solution of $y(t)$ can be determined as $y(t) = C^T \cdot \Psi(t)$.

8 Illustrative examples

Example 1.
Let us consider the variable order fractional integro-differential equation [18]

$$D^{\alpha(t)} y(t) + e^t \int_0^t y(\tau)d\tau \\ + \sin(t)y'(t) + (t-1)y(t) = f(t), \quad y(0) = 1,$$

where

$$f(t) = \frac{3t^{1-t/3}(7t-6)}{(18-9t+t^2)\Gamma(1-t/3)} + e^t\left(t - \frac{t^2}{2} + \frac{t^3}{3}\right) \\ +(2t-1)\sin(t) + (t-1)(t^2-t+1)$$

and $\alpha(t) = \frac{t}{3}$. The exact solution of this problem is $t^2 - t + 1$. This problem has been solved by OFA method which reduces the VFIDE to a system of algebraic equations. The results obtained by present method for $M = 4$ and their absolute errors have been shown in Table 1. Figure 1 shows the error graphs for $M = 4, 5, 6$. From the graph, it is clear that the present method is stable and the error converges to zero for increasing the value of M.

Example 2.
Let us consider the variable order fractional integro-differential equation [18]

$$D^{\alpha(t)} y(t) + 6\int_0^t y(\tau)d\tau + 2ty'(t) + y(t) = f(t), \; y(0) = 0,$$

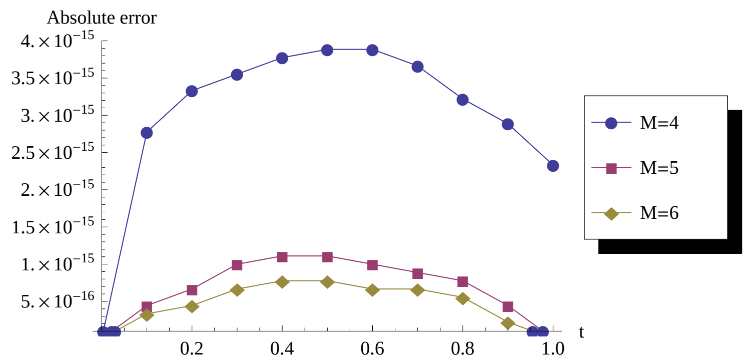

Figure 1. Absolute errors of Example 1 for $M = 4, 5, 6$

Table 2. Approximate solutions and absolute errors by OFA method for Example 2

x	Exact	OFA	Error
0	0	3.47853e-11	3.47853e-11
0.1	1.55	1.55	2.91585e-11
0.2	3.2	3.2	2.39306e-11
0.3	4.95	4.95	1.91003e-11
0.4	6.8	6.8	1.4692e-11
0.5	8.75	8.75	1.06350e-11
0.6	10.8	10.8	6.99885e-12
0.7	12.95	12.95	3.76232e-12
0.8	15.2	15.2	9.21929e-13
0.9	17.55	17.55	1.51701e-12

where

$$f(t) = \frac{10}{\Gamma(3-\alpha(t))}t^{2-\alpha(t)} + \frac{15}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} \\ +10t^3 + 70t^2 + 45t$$

and $\alpha(t) = \frac{3}{5}(\sin t + \cos t)$. The exact solution of this problem is $5t^2 + 15t$. This problem has been solved by OFA method (for $M = 4$) which reduces the VFIDE to a system of algebraic equations. The results obtained by present method and their absolute errors have been shown in Table 2. Figure 2 shows the error graphs for $M = 5, 6$. From the graph, it is clear that the present method is stable and the error converges to zero for increasing the value of M.

Example 3.
Let us consider the variable order fractional integro-differential equation

$$D^{\alpha(t)}y(t) + e^t \int_0^t y(\tau)d\tau + y(t) = f(t),$$

where $f(t) = (2t+3)(t+6)$ and $\alpha(t) = \frac{\sin t + \cos t}{3}$. The exact solution of this problem is not known. This problem has been solved by OFA method for $M = 4, 5, 6, 7, 8$ and shown in Figure 1. From the figure, it is clear that the approximate solution curves converge to a single curve while increasing the value of M. So the present method is stable and applicable to solve this type of problem.

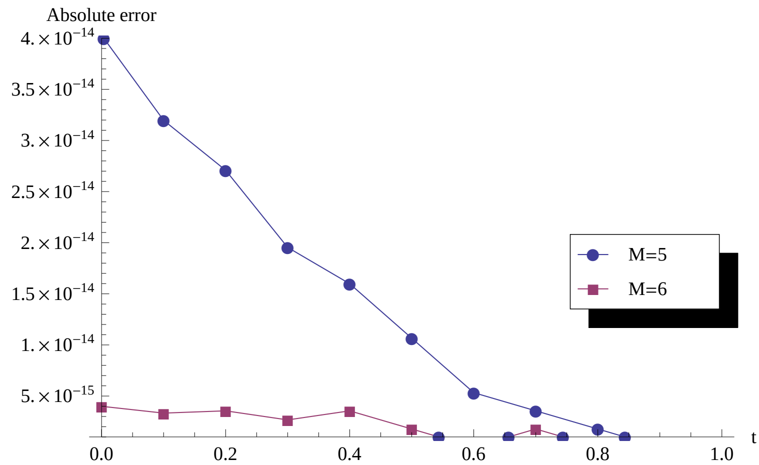

Figure 2. Absolute errors of Example 2 for $M = 5, 6$

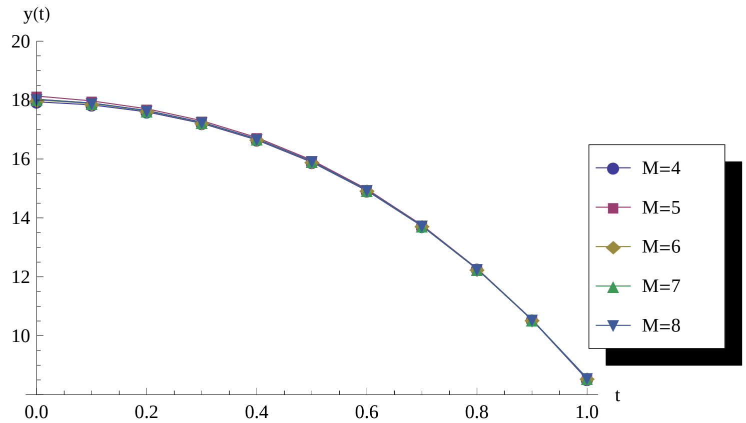

Figure 3. Numerical solutions of Example 3 for $M = 4, 5, 6, 7, 8$

9 Conclusion

In this work, orthonormal functions based on Legendre polynomials have been applied to solve variable order fractional integro-differential equations. Operational matrices are constructed for easily find the operations on unknown functions. This method reduces the integral equations to a system of algebraic equations and then the obtained algebraic system has been solved by Newton's method. The authors have solved three test problems to show the applicability of the present method. The illustrative examples have been included to demonstrate the validity and applicability of the proposed technique. Undoubtedly, these examples also exhibit the accuracy and efficiency of the present method.

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