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ON EXACT SEQUENCES OF MODULE BUNDLES OVER ALGEBRA BUNDLES

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Abstract A (right) module bundle over an algebra bundle $\xi = (\xi, p, X)$ is a vector bundle $\mathscr{M} = (\mathscr{M}, q, X)$ together with a morphism $\theta : \mathscr{M} \oplus \xi \to \mathscr{M}$ which induces a right ξ_x-module structure on $\mathscr{M}_x$ for each $x \in X$. In this paper, we prove some important results involving exact sequences of ξ-module bundles, mainly, the snake lemma.

1 Introduction

The concepts of algebra bundle and module bundle are studied widely in [2, 4, 6, 7]. An algebra bundle [4] is a vector bundle $\xi = (\xi, p, X)$ in which each fibre ξ_x is an algebra and for each x in X, there is an open neighbourhood U of x, an algebra A and a homeomorphism $\phi : U \times A \to p^{-1}(U)$ such that for each y in U, $\phi_y : A \to p^{-1}(y)$ is an algebra isomorphism.

A vector bundle $\mathscr{M} = (\mathscr{M}, q, X)$ is a right ξ-module bundle or simply a ξ-module bundle if there exists a morphism $\theta : \mathscr{M} \oplus \xi \to \mathscr{M}$ which induces a right ξ_x-module structure on $\mathscr{M}_x$ for each $x \in X$. A vector bundle morphism $\phi : \mathscr{M} \to \mathscr{N}$ of a ξ-module bundle $\mathscr{M}$ into a ξ-module bundle $\mathscr{N}$ is called a ξ-module bundle morphism or simply ξ-morphism if for each $x \in X, \phi_x : \mathscr{M}_x \to \mathscr{N}_x$ is a ξ_x-module homomorphism. If ϕ is a homeomorphism then it is called ξ-isomorphism [2].

Let $\mathscr{M}, \mathscr{M}'$ be ξ-module bundles and $f : \mathscr{M} \to \mathscr{M}'$ be a ξ-morphism. We say that f is of locally constant rank if for each $x \in X$, there is a neighborhood U of x in X such that $\dim \operatorname{Im} f_y$ is constant for $y \in U$.

All underlying vector spaces are real and finite dimensional and all algebras considered in the paper are finite dimensional associative algebras. All module bundles, algebra bundles and submodule bundles have same base space X. All exact sequences of module bundles have morphisms with locally constant rank. Throughout this paper ξ denotes an associative algebra bundle. In the next section, we prove some important results involving exact sequences of ξ-module bundles including the snake lemma.

2 Results

Proposition 2.1. *Let $\mathscr{M}, \mathscr{M}'$ be ξ-module bundles and $f : \mathscr{M} \to \mathscr{M}'$ be a ξ-morphism with locally constant rank. Then,* $\ker f$*, Im* f *are ξ-submodule bundles of $\mathscr{M}, \mathscr{M}'$, respectively.*

Proof. By [8, Proposition 1], $\ker f, \operatorname{Im} f$ are vector subbundles of $\mathscr{M}, \mathscr{M}'$ respectively. For each $x \in X$, $\ker f_x, \operatorname{Im} f_x$ are submodules of $\mathscr{M}_x, \mathscr{M}'_x$ respectively. Hence the proof. □

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Definition 2.2. Let $\mathscr{M}' \xrightarrow{u} \mathscr{M} \xrightarrow{v} \mathscr{M}''$ be a sequence of ξ-module bundles and ξ-morphisms with locally constant rank. We say that this sequence is exact at $\mathscr{M}$ if $\mathrm{Im}\, u = Ker\ v$. A short exact sequence is a sequence $0 \to \mathscr{M}' \xrightarrow{u} \mathscr{M} \xrightarrow{v} \mathscr{M}'' \to 0$ which is exact at each $\mathscr{M}', \mathscr{M}, \mathscr{M}''$. This means that u is injective, v is surjective and $\mathrm{Im}\, u = Ker\ v$.

Remark 2.3. Let $\mathscr{M}' \xrightarrow{u} \mathscr{M} \xrightarrow{v} \mathscr{M}''$ be a sequence of ξ-module bundles and ξ-morphisms with locally constant rank. Then this sequence is exact if and only if the for each $x \in X$, $\mathscr{M}'_x \xrightarrow{u_x} \mathscr{M}_x \xrightarrow{v_x} \mathscr{M}''_x$ is an exact sequence of ξ_x-modules and ξ_x-morphisms.

Lemma 2.4. *Let X be a topological space and $l : X \to Hom(\mathbb{R}^n, \mathbb{R}^m)$ be a continuous function. Then the map*

$$F : X \times Hom(\mathbb{R}^m, \mathbb{R}^k) \to X \times Hom(\mathbb{R}^n, \mathbb{R}^k)$$

defined by $(x, f) \mapsto (x, f \circ l(x))$ is continuous.

Proof. It is enough to show that the map $(x, f) \mapsto f \circ l(x)$ of $X \times Hom(\mathbb{R}^m, \mathbb{R}^k)$ into $Hom(\mathbb{R}^n, \mathbb{R}^k)$ is continuous. Let $(x_\lambda, f_\lambda)_{\lambda \in \Lambda}$ be a net in $X \times Hom(\mathbb{R}^m, \mathbb{R}^k)$ converging to (x, f). Then $x_\lambda \to x$ in X and $f_\lambda \to f$ in $Hom(\mathbb{R}^m, \mathbb{R}^k)$. Since l is continuous, $l(x_\lambda) \to l(x)$ in $Hom(\mathbb{R}^m, \mathbb{R}^n)$. So $\{\|l(x_\lambda\|)\}$ is bounded. Now,

$$\begin{aligned} \|f_\lambda \circ l(x_\lambda) - f \circ l(x)\| &= \|(f_\lambda \circ l(x_\lambda) - f \circ l(x_\lambda)) + (f \circ l(x_\lambda) - f \circ l(x))\| \\ &\le \|f_\lambda - f\|\|l(x_\lambda)\| + \|f\|\|l(x_\lambda) - l(x)\|. \end{aligned}$$

Since $\{\|l(x_\lambda\|)\}$ is bounded, the above inequality implies that $f_\lambda \circ l(x_\lambda) \to f \circ l(x)$ in $Hom(\mathbb{R}^m, \mathbb{R}^k)$. Thus, the map $(x, f) \mapsto f \circ l(x)$ is continuous, as desired. □

Lemma 2.5. *Let $u : \mathscr{M} \to \mathscr{M}'$ be a ξ-morphism with locally constant rank. Let $\mathscr{N}$ be any ξ-module bundle. Define*

$$\overline{u} : Hom(\mathscr{M}', \mathscr{N}) \to Hom(\mathscr{M}, \mathscr{N})$$

by $\overline{u}_x(f_x) = f_x \circ u_x \forall\, f_x \in Hom(\mathscr{M}'_x, \mathscr{N}_x), x \in X$. Then $\overline{u}$ is a ξ-morphism with locally constant rank.

Proof. Clearly each $\overline{u}_x$ is a ξ_x-homorphism. We need to show that $\overline{u}$ is continuous. Let

$$p : \mathscr{M} \to X,\ p' : \mathscr{M}' \to X,\ p_1 : \mathscr{N} \to X,$$

$$q : Hom(\mathscr{M}, \mathscr{N}) \to X \text{ and } q' : Hom(\mathscr{M}', \mathscr{N}) \to X$$

be the bundle projections. Let $x \in X$ and

$$\phi_1 : p^{-1}(U) \to U \times \mathbb{R}^n, \phi_2 : p'^{-1}(U) \to U \times \mathbb{R}^m, \phi_3 : p_1^{-1}(U) \to U \times \mathbb{R}^k$$

be the local trivializations at x of $\mathscr{M}, \mathscr{M}', \mathscr{N}$ respectively. Then

$$h_1 : q^{-1}(U) \to U \times Hom(\mathbb{R}^m, \mathbb{R}^k)$$

given by $h_1(f_y) = (y, \phi_{3y} \circ f_y \circ (\phi_{1y})^{-1})\ \forall y \in U, f_y \in Hom(\mathscr{M}_y, \mathscr{N}_y)$ is a local trivialization of $Hom(\mathscr{M}, \mathscr{N})$ at x. Similarly, a local trivialization at x of $Hom(\mathscr{M}', \mathscr{N})$ is given by

$$h_2 : q'^{-1}(U) \to U \times Hom(\mathbb{R}^n, \mathbb{R}^k),$$
$$h_2(g_y) = (y, \phi_{3y} \circ g_y \circ (\phi_{2y})^{-1})\ \forall y \in U, g_y \in Hom(\mathscr{M}'_y, \mathscr{N}_y).$$

Define $\psi : Hom(\mathbb{R}^m, \mathbb{R}^k) \to Hom(\mathbb{R}^n, \mathbb{R}^k))$ by $\psi(y, f) = (y, f \circ \phi_{2y} \circ u_y \circ (\phi_{1y})^{-1})$. Since $y \mapsto u_y, y \mapsto (\phi_{1y})^{-1}, y \mapsto \phi_{2y}$ are continuous, the map $y \mapsto \phi_{2y} \circ u_y \circ (\phi_{1y})^{-1}$ is continuous. By the Lemma 2.4, it follows that ψ is continuous. Now, we have the following diagram.

$$\begin{array}{ccc} q'^{-1}(U) & \xrightarrow{h_2} & U \times Hom(\mathbb{R}^m, \mathbb{R}^k) \\ \Big\downarrow \overline{u}\big|_{q'^{-1}(U)} & & \Big\downarrow \psi \\ q^{-1}(U) & \xrightarrow{h_1} & U \times Hom(\mathbb{R}^n, \mathbb{R}^k) \end{array}$$

For any $y \in U$ and $g_y \in Hom(\mathscr{M}'_y, \mathscr{N}_y)$, we have

$$\begin{aligned} \psi h_2(g_y) &= \psi(y, \phi_{3y} \circ g_y \circ (\phi_{2y})^{-1}) \\ &= (y, \phi_3 y \circ g_y \circ (\phi_{2y})^{-1} \circ \phi_{2y} \circ u_y \circ (\phi_{1y})^{-1}) \\ &= (y, \phi_{3y} \circ g_y \circ u_y \circ (\phi_{1y})^{-1}) \end{aligned}$$

and

$$\begin{aligned} h_1\overline{u}(g_y) &= h_1\overline{u}_y(g_y) \\ &= h_1(g_y \circ u_y) \\ &= (y, \phi_{3y} \circ g_y \circ u_y \circ (\phi_{1y})^{-1}) \end{aligned}$$

Thus, the above diagram is commutative. Hence $\psi \circ h_2 = h_1 \circ \overline{u}\big|_{q'^{-1}(U)}$. So $\overline{u}\big|_{q'^{-1}(U)} = (h_1)^{-1} \circ \psi \circ h_2$ which is a continuous function. As x varies over X, the sets $q'^{-1}(U)$ forms an open cover of $Hom(\mathscr{M}', \mathscr{N})$. Therefore $\overline{u}$ is continuous on $Hom(\mathscr{M}', \mathscr{N})$. Hence $\overline{u}$ is a ξ-morphism.

Now we show that $\overline{u}$ has locally constant rank. Let $x \in X$. We have

$$\overline{u}_x : Hom(\mathscr{M}'_x, \mathscr{N}_x) \to Hom(\mathscr{M}_x, \mathscr{N}_x)$$

given by, $u_x(f_x) = f_x \circ u_x$. It follows that Rank $\overline{u}_x$ = Rank u_x·dim $\mathscr{N}_x$. Since u is of locally constant rank, Rank u_y is constant for all y in some neighborhood of x. Also, dim $\mathscr{N}_y$ is constant in some neighborhood of x. Thus, Rank $\overline{u}_y$ is constant in some neighborhood of x. □

Theorem 2.6. *Let*

$$\mathscr{M}' \xrightarrow{u} \mathscr{M} \xrightarrow{v} \mathscr{M}'' \to 0$$

be a sequence of ξ-module bundles and ξ-momorphisms with locally constant rank. Then this sequence is exact if and only if for all ξ- module bundles $\mathscr{N}$, the sequence

$$0 \to Hom(\mathscr{M}'', \mathscr{N}) \xrightarrow{\overline{v}} Hom(\mathscr{M}, \mathscr{N}) \xrightarrow{\overline{u}} Hom(\mathscr{M}', \mathscr{N})$$

is exact, where

$$\begin{aligned} \overline{v}_x \ : \ Hom(\mathscr{M}''_x, \mathscr{N}_x) &\to Hom(\mathscr{M}_x, \mathscr{N}_x) \\ f_x &\mapsto f_x \circ v_x \end{aligned}$$

and

$$\begin{aligned} \overline{u}_x \ : \ Hom(\mathscr{M}_x, \mathscr{N}_x) &\to Hom(\mathscr{M}'_x, \mathscr{N}_x) \\ g_x &\mapsto g_x \circ u_x. \end{aligned}$$

Proof. By Lemma 2.5 $\overline{u}, \overline{v}$ are ξ-morphisms with locally constant rank. Let $x \in X$. By the theory of modules, the sequence

$$\mathscr{M}'_x \xrightarrow{u_x} \mathscr{M}_x \xrightarrow{v} \mathscr{M}''_x \to 0$$

is exact if and only if the sequence

$$0 \to \mathrm{Hom}(\mathscr{M}''_x, \mathscr{N}_x) \xrightarrow{\overline{v}_x} \mathrm{Hom}(\mathscr{M}_x, \mathscr{N}_x) \xrightarrow{\overline{u}_x} \mathrm{Hom}(\mathscr{M}'_x, \mathscr{N}_x)$$

is exact for every ξ-module $\mathscr{N}_x$. Now the theorem follows from the Remark 2.3. □

Theorem 2.7. *Let*

$$0 \to \mathscr{N}' \xrightarrow{u} \mathscr{N} \xrightarrow{v} \mathscr{N}''$$

be a sequence of ξ-module bundles and ξ-momorphisms with locally constant rank. Then this sequence is exact if and only if for all ξ- module bundles $\mathscr{M}$, the sequence

$$0 \to Hom(\mathscr{M}, \mathscr{N}') \xrightarrow{\overline{u}} Hom(\mathscr{M}, \mathscr{N}) \xrightarrow{\overline{v}} Hom(\mathscr{M}, \mathscr{N}'')$$

is exact, where

$$\begin{array}{rcl}\overline{u}_x : Hom(\mathscr{M}_x, \mathscr{N}'_x) & \to & Hom(\mathscr{M}_x, \mathscr{N}_x) \\ f_x & \mapsto & u_x \circ f_x\end{array}$$

and

$$\begin{array}{rcl}\overline{v}_x : Hom(\mathscr{M}_x, \mathscr{N}_x) & \to & Hom(\mathscr{M}_x, \mathscr{N}''_x) \\ g_x & \mapsto & v_x \circ g_x.\end{array}$$

Proof. The proof is similar to the proof of Theorem 2.6. □

Definition 2.8. A short exact sequence $0 \to \mathscr{M}' \xrightarrow{u} \mathscr{M} \xrightarrow{v} \mathscr{M}'' \to 0$ is called split if there is a ξ-isomorphism $\theta : \mathscr{M} \to \mathscr{M}' \oplus \mathscr{M}''$ such that the following diagram commutes where the maps in the second row $\mathscr{M}' \to \mathscr{M}' \oplus \mathscr{M}''$ and $\mathscr{M}' \oplus \mathscr{M}'' \to \mathscr{M}''$ are standard embedding and projection respectively.

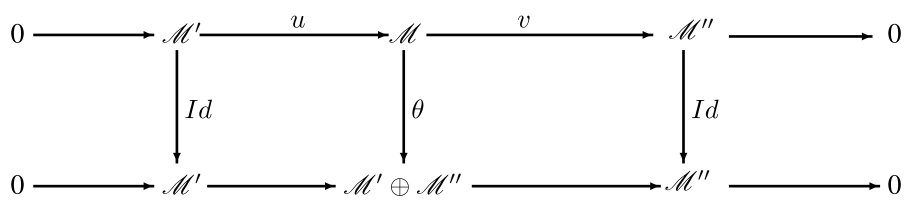


Figure 1: Diagram

The next theorem is analogous to modules which characterizes split short exact sequence.

Theorem 2.9. *For a short exact sequence*

$$0 \to \mathscr{M}' \xrightarrow{u} \mathscr{M} \xrightarrow{v} \mathscr{M}'' \to 0$$

the following conditions are equivalent.

(i) The above short exact sequence splits.

(ii) There is a ξ-morphism $f : \mathscr{M} \to \mathscr{M}'$ such that $f \circ u = Id_{\mathscr{M}'}$.

(iii) There is a ξ-morphism $g : \mathscr{M}'' \to \mathscr{M}$ such that $v \circ g = Id_{\mathscr{M}''}$.

Proof. (i) $\implies$ (ii): We have a ξ-isomorphism $\theta : \mathscr{M} \to \mathscr{M}' \oplus \mathscr{M}''$ such that the diagram in Figure 1 commutes. By the commutativity of second square we get $\theta(m) = (f, v)$ for some ξ-morphism $f : \mathscr{M} \to \mathscr{M}'$. By the commutativity of first square we have $\theta(u(m')) = (m', 0)$ for $m' \in \mathscr{M}'$. So $(f(u(m'), v(u(m')) = (m', 0)$ which gives $f(u(m')) = m'$.
(ii) $\implies$ (i): Define $\theta : \mathscr{M} \to \mathscr{M}' \oplus \mathscr{M}''$ by, $\theta = (f, v)$. Then θ is ξ-morphism such that the diagram in Figure 1 commutes. Since θ_x are isomorphisms θ is ξ-isomorphism. The equivalence of (i) and (iii) follows in similar lines. □

In [7], we have shown that every semisimple module bundle can be expressed as a direct sum of simple module bundles uniquely up to isomorphism. This implies the following result:

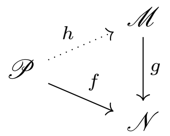
Corollary 2.10. *If $\mathscr{M}$ is semisimple ξ-module bundle then every short exact sequence*

$$0 \to \mathscr{M}' \xrightarrow{u} \mathscr{M} \xrightarrow{v} \mathscr{M}'' \to 0$$

splits.

Proof. Since $\mathscr{M}$ is semisimple, $\operatorname{Im} u = Ker\ v$ is a submodule bundle of $\mathscr{M}$ [7, Proposition 5.3, 5.4]. By [7, Theorem 5.11], there is a submodule bundle $\mathscr{N}$ of $\mathscr{M}$ such that $\mathscr{M} = \operatorname{Im} u \oplus \mathscr{N}$. Define $f : \mathscr{M}' \to \mathscr{M}$ as follows. Any element of $\mathscr{M}$ is of the form $(u(m), n)$ for some $m \in \mathscr{M}, n \in \mathscr{N}$. Define $f(u(m), n) = m$. Then f is a ξ-morphism such that $f \circ u = Id_{\mathscr{M}'}$. By Theorem 2.9, every such exact sequence splits. □

Definition 2.11. A ξ-module bundle $\mathscr{P}$ is called projective if for every ξ-epimorphism $g : \mathscr{M} \to \mathscr{N}$ and every ξ-morphism $f : \mathscr{P} \to \mathscr{N}$ there is a ξ-morphism $h : \mathscr{P} \to \mathscr{M}$ such that the following diagram commutes.



Theorem 2.12. *A ξ-module bundle $\mathscr{P}$ is projective if and only if every exact sequence*

$$0 \to \mathscr{M} \xrightarrow{u} \mathscr{N} \xrightarrow{v} \mathscr{P} \to 0$$

splits.

Proof. Let $\mathscr{P}$ be projective. Consider the exact sequence

$$0 \to \mathscr{M} \xrightarrow{u} \mathscr{N} \xrightarrow{v} \mathscr{P} \to 0.$$

Let $Id : \mathscr{P} \to \mathscr{P}$ be the identity morphism. Since the sequence is exact, $v : \mathscr{N} \to \mathscr{P}$ is an epimorphism. As $\mathscr{P}$ is projective, there is a ξ-morphism $h : \mathscr{P} \to \mathscr{N}$ such that $v \circ h = Id$. By the Theorem 2.9, the exact sequence splits.

Conversely, suppose every such exact sequence splits. Let $g : \mathscr{M} \to \mathscr{N}$ be ξ-epimorphism and $f : \mathscr{P} \to \mathscr{N}$ be a ξ-morphism. Let $\mathscr{L} = \{(p, m) \in \mathscr{P} \oplus \mathscr{M} : f(p) = g(m)\}$. Then $\mathscr{L}$ is a submodule bundle of $\mathscr{P} \oplus \mathscr{M}$. Consider $\pi_1 : \mathscr{L} \to \mathscr{P}$, the projection onto first factor. Then π_1 is ξ-epimorphism. So we have the exact sequence

$$0 \to \ker \pi_1 \to \mathscr{L} \xrightarrow{\pi_1} \mathscr{P} \to 0$$

which splits by our assumption. So, by the Theorem 2.9 there is a ξ-morphism $h : \mathscr{P} \to \mathscr{L}$ such that $\pi_1 \circ h = Id$. Let $h_1 = \pi_2 \circ h$, where $\pi_2 : \mathscr{L} \to \mathscr{M}$ is the projection onto second factor. Then $h_1 : \mathscr{P} \to \mathscr{M}$ is a ξ-morphism and for $p \in \mathscr{P}$, $g(h_1(p)) = g(\pi_2(h(p))) = g(\pi_2(p, m))$ for some $m \in \mathscr{M}$ with $f(p) = g(m)$. So $g(h_1(p)) = g(m) = f(p)$. Thus $g \circ h = f$ as desired. □

Theorem 2.13 (Schanuel's Lemma for Module Bundles). *Let* $0 \to \mathscr{M} \xrightarrow{u} \mathscr{P} \xrightarrow{v} \mathscr{N} \to 0$ *and* $0 \to \mathscr{M}' \xrightarrow{u'} \mathscr{P}' \xrightarrow{v'} \mathscr{N} \to 0$ *be two exact sequences, where $\mathscr{P}, \mathscr{P}'$ are projective. Then $\mathscr{P} \oplus \mathscr{M}' \cong \mathscr{P}' \oplus \mathscr{M}$.*

Proof. Since $\mathscr{P}$ is projective, there is a ξ-morphism $h : \mathscr{P} \to \mathscr{P}'$ such that $v' \circ h = v$. This h induces a ξ-morphism $h' : \mathscr{M} \to \mathscr{M}'$. So we have the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{M} & \xrightarrow{u} & \mathscr{P} & \xrightarrow{v} & \mathscr{N} & \longrightarrow & 0 \\
 & & \downarrow h' & & \downarrow h & & \downarrow Id & & \\
0 & \longrightarrow & \mathscr{M}' & \xrightarrow{u'} & \mathscr{P}' & \xrightarrow{v'} & \mathscr{N} & \longrightarrow & 0
\end{array}$$

Define $\alpha : \mathscr{P} \oplus \mathscr{M}' \to \mathscr{P}'$ and $\beta : \mathscr{M} \to \mathscr{P} \oplus \mathscr{M}'$ by $\alpha(p, m') = h(p) - u'(m')$ for $(p, m') \in \mathscr{P} \oplus \mathscr{M}'$ and $\beta(m) = (u(m), h'(m))$ for $m \in \mathscr{M}$. Then α, β are ξ-morphisms. So we have an exact sequence, $0 \to \mathscr{M} \xrightarrow{\beta} \mathscr{P} \oplus \mathscr{M}' \xrightarrow{\alpha} \mathscr{P}' \to 0$. Since $\mathscr{P}'$ is projective, this exact sequence splits by the Theorem 2.12. Hence $\mathscr{P}' \oplus \mathscr{M} \cong \mathscr{P} \oplus \mathscr{M}'$. □

Remark 2.14. Let

$$\begin{array}{ccc}
\mathscr{M} & \xrightarrow{u} & \mathscr{M}' \\
\downarrow f & & \downarrow f' \\
\mathscr{N} & \xrightarrow{v} & \mathscr{N}'
\end{array}$$

be a commutative diagram of ξ-modules and ξ-morphisms with locally constant rank. Then $u(\ker f) \subseteq \ker f'$ and $v(Im\ f) \subseteq Im\ f'$. So u and v induce the maps, $\overline{u} : \ker f \to \ker f'$

and $\widetilde{v} : \mathscr{N}/\text{Im} f \to \mathscr{N}'/\text{Im} f'$ given by, $\overline{u}(m) = u(m)\ \forall\ m \in \ker f$ (- restriction of u), and $\widetilde{v}([n]) = [v(n)]\ \forall\ [n] \in \mathscr{N}/\text{Im} f$.

Lemma 2.15. *The induced maps $\overline{u}$ and $\widetilde{v}$ given in the above remark are ξ-morphisms with locally constant rank.*

Proof. Let $x \in X$. We have rank $\overline{u}_x = \dim\ker f_x - \dim(\ker u_x \cap \ker f_x)$. Since u, f have locally constant rank, it follows that $\overline{u}$ also has locally constant rank. Similarly, the restricted map, $\overline{v} : \text{Im} f \to \text{Im} f'$ has locally constant rank. Now, rank $\widetilde{v}_x = \text{rank } v_x - \text{rank } \overline{v}_x$. Since $v, \overline{v}$ are with locally constant rank, it follows that $\widetilde{v}$ has locally constant rank. Clearly, each $\overline{u}_x, \widetilde{v}_x$ are ξ_x-homomorphisms. Also, $\overline{u}$ being a restriction of a continuous map, is continuous. So it remains to show that $\widetilde{v}$ is continuous. Consider the following diagram:

$$\begin{array}{ccc}
\mathscr{N} & \xrightarrow{\ v\ } & \mathscr{N}' \\
\Big\downarrow q & & \Big\downarrow q' \\
\mathscr{N}/\text{Im} f & \xrightarrow{\ \widetilde{v}\ } & \mathscr{N}'/\text{Im} f'
\end{array}$$

where, q, q' are the quotient maps. For $n \in \mathscr{N}$, we have $(\widetilde{v} \circ q)(n) = \widetilde{v}([n]) = [v(n)] = q'(v(n))$. Hence the above diagram is commutative. If G is any open set in $\mathscr{N}'/\text{Im} f'$ then by the definition of quotient topology, $q'^{-1}(G)$ is open in $\mathscr{N}'$. Since v is continuous, $v^{-1}(q'^{-1}(G))$ is open in $\mathscr{N}$. But $v^{-1}(q'^{-1}(G)) = (q' \circ v)^{-1}(G) = (\widetilde{v} \circ q)^{-1}(G) = q^{-1}(\widetilde{v}^{-1}(G))$. Thus, $q^{-1}(\widetilde{v}^{-1}(G))$ is open in $\mathscr{N}$. Again, by the definition of quotient topology, $\widetilde{v}^{-1}(G)$ is open in $\mathscr{N}/\text{Im} f$. Thus $\widetilde{v}$ is continuous, as desired. □

Theorem 2.16. *Let*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{M}' & \xrightarrow{\ u\ } & \mathscr{M} & \xrightarrow{\ v\ } & \mathscr{M}'' & \longrightarrow & 0 \\
 & & \Big\downarrow f' & & \Big\downarrow f & & \Big\downarrow f'' & & \\
0 & \longrightarrow & \mathscr{N}' & \xrightarrow{\ u'\ } & \mathscr{N} & \xrightarrow{\ v'\ } & \mathscr{N}'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of ξ-module bundles and ξ-morphisms having locally constant rank, with the rows exact. Then the following sequences

$$0 \to \ker f' \xrightarrow{\overline{u}} \ker f \xrightarrow{\overline{v}} \ker f''$$

and

$$\mathscr{N}'/Im\, f' \xrightarrow{\widetilde{u'}} \mathscr{N}/Im\, f \xrightarrow{\widetilde{v'}} \mathscr{N}''/Im\, f'' \to 0$$

are exact, where $\overline{u}$, $\overline{v}$ are restrictions of u, v, respectively, and $\widetilde{u'}$, $\widetilde{v'}$ are induced by u', v', respectively.

Proof. By Lemma 2.15, the induced sequences are sequences of ξ-module bundles and ξ-morphisms with constant rank. Now the theorem follows by observing exactness of fiber sequences and applying the Lemma 2.3 □

Remark 2.17. Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{M}' & \xrightarrow{\ u\ } & \mathscr{M} & \xrightarrow{\ v\ } & \mathscr{M}'' & \longrightarrow & 0 \\
 & & \Big\downarrow f' & & \Big\downarrow f & & \Big\downarrow f'' & & \\
0 & \longrightarrow & \mathscr{N}' & \xrightarrow{\ u'\ } & \mathscr{N} & \xrightarrow{\ v'\ } & \mathscr{N}'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of ξ-module bundles and ξ-morphisms having locally constant rank, with the rows exact. Then by Theorem 2.16, the following sequences

$$0 \to \ker f' \xrightarrow{\overline{u}} \ker f \xrightarrow{\overline{v}} \ker f'',$$

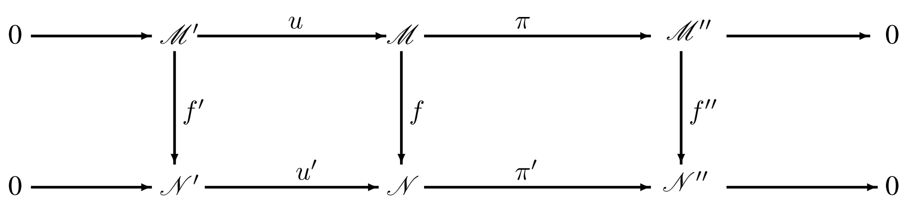
$$\mathscr{N}'/\operatorname{Im} f' \xrightarrow{\widetilde{u'}} \mathscr{N}/\operatorname{Im} f \xrightarrow{\widetilde{v'}} \mathscr{N}''/\operatorname{Im} f'' \to 0$$

are exact. Now we define $\partial : \ker f" \to {}'/\operatorname{Im} f'$ as follows. Let $x \in X$. Consider any $m'' \in (\ker f'')_x$. Since v_x is surjective, there exists a $m_x \in \mathscr{M}_x$ such that $m'' = v_x(m)$. Now

$$\begin{aligned} v'_x(f_x(m)) &= v'_x \circ f_x(m) \\ &= f''_x \circ v_x(m) \ (\because\ v' \circ f = f'' \circ v) \\ &= f''_x(v_x(m)) = f''_x(m'') \ (\because\ v(m) = m'') \\ v'_x(f_x(m)) &= 0_x \ (\because\ m'' \in \ker f''). \end{aligned}$$

Therefore $f_x(m) \in \ker v'_x \subseteq \operatorname{Im} u'_x$. Thus we can find a $n' \in N'_x$ such that $f_x(m) = u'_x(n')$. We define $\partial_x(m'') = n' + \operatorname{Im} f'_x$. Then ∂ is a ξ-morphism with locally constant rank. ∂ is called boundary morphism or connecting morphism.

Theorem 2.18 (Snake lemma). *Let $\mathscr{M}, \mathscr{N}$ be ξ-module bundles, $\mathscr{M}', \mathscr{N}'$ be submodule bundles of $\mathscr{M}, \mathscr{N}$ respectively and $\mathscr{M}'' = \mathscr{M}/\mathscr{M}', \mathscr{N}'' = \mathscr{N}/\mathscr{N}'$ be corresponding quotient module bundles. Let*



be a commutative diagram of ξ-module bundles and ξ-morphisms, where u, u' are inclusion maps, π, π' are quotient maps. Then the sequence

$$0 \to \ker f' \xrightarrow{\overline{u}} \ker f \xrightarrow{\overline{v}} \ker f'' \xrightarrow{\partial} \mathscr{N}'/Im\, f' \xrightarrow{\widetilde{u'}} \mathscr{N}/Im\, f \xrightarrow{\widetilde{v'}} \mathscr{N}''/Im\, f'' \to 0$$

is exact, where $\overline{u}$, $\overline{v}$ are restrictions of u, v, respectively, and $\widetilde{u'}$, $\widetilde{v'}$ are induced by u', v', respectively, and ∂ is the boundary morphism.

Proof. It is enough to show that ∂ is continuous. Let W be any open set in $\mathscr{N}'/\operatorname{Im} f'$. Then $\pi_1^{-1}(W)$ is open in $\mathscr{N}'$ where $\pi_1 : \mathscr{N}' \to \mathscr{N}'/\operatorname{Im} f'$ is the quotient map. But then $\pi_1^{-1}(W) = W_1 \cap \mathscr{N}'$ for some open set W_1 in $\mathscr{N}$ as $\mathscr{N}'$ has subspace topology. Since f is continuous and π is an open map, $\pi(f^{-1}(W_1))$ is open in $\mathscr{M}''$. So $\pi(f^{-1}(W_1)) \cap Ker\ f''$ is open in $Ker\ f''$. Now the proof will be complete if we show that $\partial^{-1}(W) = \pi(f^{-1}(W_1)) \cap Ker\ f''$. Let $m'' \in \partial^{-1}(W)$. Then $\partial(m'') = n' + \operatorname{Im} f'_x = \pi_{1x}(n') \in W$ for some $x \in X$, where $n' = u'(n') = f(m)$ with $\pi(m) = m''$. So $n' \in \pi_1^{-1}(W) = W_1 \cap \mathscr{N}' \subseteq W_1$. Hence $f(m) = n' \in W_1$ which implies that $m \in f^{-1}(W_1)$. So $m'' = \pi(m) \in \pi(f^{-1}(W_1)) \cap Ker\ f''$. On the other hand let $m'' \in \pi(f^{-1}(W_1)) \cap Ker\ f'$. Then $m'' = \pi(m)$ for some $m \in \mathscr{M}$ with $f(m) \in W_1$. By definition of ∂, $\partial(m'') = n' + \operatorname{Im} f'_x = \pi_{1x}(n')$ where $n' = u'(n') = f(m)$. So $n' \in W_1 \cap \mathscr{N}' = \pi_1^{-1}(W)$. Hence $\partial(m'') = \pi_1(n') \in W$ which means that $m'' \in \partial^{-1}(W)$. □

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