\textbf{\textgamma\text{- DERIVATIONS IN RINGS}}

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\textbf{Abstract.} We introduce the notion of $\gamma$-derivations in rings and obtain commutativity results in a prime ring $R$ admitting multiplicative $\gamma$-derivations. We show that the symmetry of $\gamma$ with various conditions on Lie products and Jordan products gives rise to commutativity of $R$. We obtain (i) a characterization of Galois field of any characteristic by using Lie product and $\gamma$-derivation, and (ii) a characterization of Galois field of characteristic 2 by using Jordan product and $\gamma$-derivation.

\section{Introduction}

A derivation $d$ on a ring $R$ is a linear map on $R$ which satisfies the Leibniz rule, that is, $d(xy) = xd(y) + d(x)y$ for all $x, y \in R$. Derivations on rings and nearrings have been widely studied in the literature. Posner [23] and Herstein [11, 12, 13] obtained some of the important early results on prime rings with derivations. Derivations and commutativity in prime and semiprime rings with different types of derivations have been widely investigated. Brešar [7] introduced the notion of generalized derivations. Argaç [2] generalized a well-known result of Posner [23] on commuting derivations to semiprime rings and obtained some sufficient conditions for a derivation to be commuting on a nonzero ideal of the ring. Recently, Gölbaşı and Öğurtsci [9] obtained certain sufficient conditions for a multiplicative semiderivation on a semiprime ring to be commuting on an ideal of the ring. Mamouni and Tamekkante [20] studied commutativity in prime rings admitting two generalized derivations. Birkenmeier, Heatherly and Lee [6] studied the interconnections between different types of prime ideals in nearrings. Bhavanari, Kuncham and Kedukodi [5] introduced the graph of a nearring with respect to an ideal of the nearring and studied the relation between 3-primeness of the ideal and ideal symmetry of the corresponding graph. Kedukodi, Kuncham and Bhavanari [15] introduced equiprime, 3-prime and $c$-prime fuzzy ideals of nearrings, and subsequently, Koppula, Kedukodi and Kuncham [16] related the ideas to decision making. Koppula, Kedukodi and Kuncham [17] studied the notion of perfect ideals of seminearings. Derivations in prime nearrings were first investigated by Bell and Mason [4] and several other results were obtained by Bell [3], Wang [24], Kamal and Shaalan [14], among others. Aishwarya, Kedukodi and Kuncham [1] obtained commutativity results in prime nearrings through permutation identities satisfied by certain subsets and gave a characterization of Galois field using permutation identities. It is well-known that Galois fields are extensively used in cryptography and coding theory (Lidl and Niederreiter [19], Mullen and Mummert [21], etc.). Different classes of codes based on Galois fields are used in various applications like error detection and correction, data transmission and storage, among many others. Gómez-Torrecillas, Lobillo, Navarro and Sánchez-Hernández [10] defined differential convolutional codes which are built from a derivation of the rational function field of a Galois field. In this paper, we introduce $\gamma$-derivation on a ring $R$ as a generalization of derivations and provide natural examples of $\gamma$-derivations. The definition of $\gamma$-derivation extends the notion of derivation studied in Zhu and Xiong [25, 26], Li and Pan [18]. We obtain commutativity results in prime rings using multiplicative $\gamma$-derivations satisfying conditions on Lie and Jordan products. These results generalize the commutativity results obtained by Bell [3], Kamal and Shaalan [14]. Fi-
nally, we obtain a characterization of Galois fields of any characteristic and a characterization of Galois fields of characteristic 2 by using \( \gamma \)-derivations.

We would like to point out that one of the consequences of the results on \( \gamma \)-derivations is that the symmetry of Pascal’s triangle arising due to the well-known discrete combinatorial formula \( ^nC_r = ^nC_{n-r} \), naturally induces the commutativity of multiplication in complex numbers.

## 2 Preliminaries

Let \( (R,+,\cdot) \) be a ring. For subsets \( A \) and \( B \) of \( R \), the product of the sets \( A \) and \( B \) is \( AB = \{ab | a \in A, b \in B\} \). If \( A = \{a\} \) or \( B = \{b\} \), we write \( AB \) simply as \( ab \) or \( Ab \) respectively. A non-empty subset \( K \) of \( R \) is called a subsemigroup of \( (R,\cdot) \) if \( KK \subseteq K \). A non-empty subset \( K \) of \( R \) is called a semigroup left ideal (resp. semigroup right ideal) of \( R \) if \( RK \subseteq K \) (resp. \( KR \subseteq K \)) and a semigroup ideal of \( R \) if \( RK \subseteq K \) and \( KR \subseteq K \). An element \( x \) in \( R \) is said to centralize a subset \( K \) of \( R \) if \( xk = kx \) for all \( k \in K \). The set \( \{x \in R | xr = rx \text{ for all } r \in R\} \) is called the multiplicative center of \( R \), and is denoted by \( Z(R) \). \( R \) is said to be a prime ring if for all \( a, b \in R \), \( ab = 0 \) implies that either \( a = 0 \) or \( b = 0 \). For \( a, b \in R \), \( [a,b] \) denotes the Lie product \( ab - ba \) and \( a \circ b \) denotes the Jordan product \( ab + ba \). For \( A, B \subseteq R \), \( [A,B] \) denotes the set \( \{[a,b] | a \in A, b \in B\} \) and \( A \circ B \) denotes the set \( \{a \circ b | a \in A, b \in B\} \). If \( A = \{a\} \) or \( B = \{b\} \), \( [A,B] \) is simply written as \( [a,B] \) or \( [A,b] \) respectively, and \( A \circ B \) is written as \( a \circ B \) or \( A \circ b \) respectively.

A function \( d : R \rightarrow R \) is called a \( \gamma \)-derivation if for all \( a,b \in R \),

\[
d(ab) = d(a)b + ad(b) - ad(1)b,
\]

where 1 is the unit of \( R \).

### Definition 2.1

Let \( \emptyset \neq K \subseteq R \). \( K \) is called a \{0\}-weak semigroup left ideal (\{0\}-weak semigroup right ideal) of \( R \) if there exists a nonzero subset \( I \) of \( K \) such that \( RI \subseteq K \) (resp. \( IR \subseteq K \)).

Note that every non-zero semigroup left ideal (resp. nonzero semigroup right ideal) of \( R \) is a \{0\}-weak semigroup left ideal of \( R \) (resp. \{0\}-weak semigroup right ideal of \( R \)).

### Definition 2.2

(Zhu and Xiong [26], Li and Pan [18]) A linear map \( d \) from a unital algebra \( A \) over a field \( F \) to an \( A \)-bimodule \( M \) is called a generalized derivation if for all \( a,b \in A \),

\[
d(ab) = d(a)b + ad(b) - ad(1)b,
\]

where 1 is the unit of \( A \).

### Proposition 2.3

(Bell [3]) Let \( R \) be a prime ring. If \( x \in Z(R) \) \( \setminus \{0\} \), then \( x \) is not a zero divisor.

### Proposition 2.4

(Bell [3]) Let \( K \) be a nonzero semigroup left ideal of a prime ring \( R \). If \( x \in R \) is such that \( x \) centralizes \( K \), then \( x \in Z(R) \).

### Proposition 2.5

(Bell [3]) Let \( R \) be a prime ring. If \( Z(R) \) contains a nonzero semigroup left ideal or a nonzero semigroup right ideal, then \( R \) is commutative.

## 3 \( \gamma \)-derivation

### Definition 3.1

Let \( R \) be a ring and let \( \gamma : R \times R \rightarrow R \). A function \( d : R \rightarrow R \) is called a multiplicative \( \gamma \)-derivation on \( R \) if for all \( x,y \in R \),

\[
d(xy) = x d(y) + x \gamma(x,y) y + d(x) y.
\]

We say that \( \gamma \) is symmetric if \( \gamma(x,y) = \gamma(y,x) \) for all \( x,y \in R \). It is clear that if \( \gamma(x,y) = c \) for all \( x,y \in R \) then \( \gamma \) is symmetric. When \( \gamma(x,y) = 0 \), we get \( d(xy) = x d(y) + d(x) y \), that is, \( d \) is a (usual) multiplicative derivation.

### Definition 3.2

Let \( R \) be a ring. A multiplicative \( \gamma \)-derivation \( d \) on \( R \) is called a \( \gamma \)-derivation on \( R \) if \( d \) is additive.

First, we give some examples to illustrate the concept and the idea involved.
Example 3.3. Let $F$ be a field and let $F[x]$ be the polynomial ring over $F$. Let $f(x)$ be an element of $F[x]$ with degree at least 2 and let $R = F[x]/\langle f(x) \rangle$, the quotient ring of $F[x]$ by the principal ideal generated by $f(x)$. Define $d : R \rightarrow R$ by

$$d(p(x) + \langle f(x) \rangle) = xp(x) + \langle f(x) \rangle$$

and let $\gamma : R \times R \rightarrow R$ be defined by

$$\gamma(p(x) + \langle f(x) \rangle, q(x) + \langle f(x) \rangle) = -x + \langle f(x) \rangle.$$  

For $a = p(x) + \langle f(x) \rangle$, $b = q(x) + \langle f(x) \rangle \in R$, we have

(i) $d(a) + d(b) = (xp(x) + \langle f(x) \rangle) + (xq(x) + \langle f(x) \rangle)$

$$= (xp(x) + xq(x)) + \langle f(x) \rangle$$

$$= x(p(x) + q(x)) + \langle f(x) \rangle$$

$$= d(a + b)$$

(ii) $ad(b) + a\gamma(a, b)b + d(a)b = (p(x) + \langle f(x) \rangle) (xq(x) + \langle f(x) \rangle)$

$$+ (p(x) + \langle f(x) \rangle) (-x + \langle f(x) \rangle) (q(x) + \langle f(x) \rangle)$$

$$+ (xp(x) + \langle f(x) \rangle)(q(x) + \langle f(x) \rangle)$$

$$= (xp(x)q(x) + \langle f(x) \rangle) - (xp(x)q(x) + \langle f(x) \rangle)$$

$$+ (xp(x)q(x) + \langle f(x) \rangle)$$

$$= xp(x)q(x) + \langle f(x) \rangle$$

$$= d(p(x)q(x) + \langle f(x) \rangle)$$

$$= d(ab).$$

By (i) and (ii), $d$ is a $\gamma$-derivation on $R$.

In particular, when $F$ is a Galois field and $R = F[x]/\langle x^n - 1 \rangle$ for some $n \geq 2$, the elements of $R$ can be seen as codewords of length $n$ over the field $F$. It is well-known that the ideals of $R$ exactly correspond to the cyclic codes. The function $d$ is the cyclic right shift operation on the set of codewords. The cyclic right shift operation is a $\gamma$-derivation on $R$.

Example 3.4. A generalized derivation $d$ (refer Definition 2.2) from a unital algebra $A$ over a field $F$ to an $A$-bimodule $M$ is a $\gamma$-derivation on $F$ when $A = M = F$, where $\gamma(x, y) = -d(1)$.

Example 3.5. Let $R$ be a ring and let $n$ be a natural number. Consider the ring $M_n(R) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in R, 1 \leq i, j \leq n \right\}$

of all matrices of order $n \times n$ over $R$. For $c \in R$, define $d_1, d_2 : M_n(R) \rightarrow M_n(R)$ by

$$d_1(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}) = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nn} \end{pmatrix}$$

and

$$d_2(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}) = \begin{pmatrix} a_{11}c & a_{12}c & \cdots & a_{1n}c \\ a_{21}c & a_{22}c & \cdots & a_{2n}c \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}c & a_{n2}c & \cdots & a_{nn}c \end{pmatrix}.$$
Let $\gamma : M_n(R) \times M_n(R) \to M_n(R)$ be defined by

$$
\gamma(A, B) = \begin{pmatrix}
-c & 0 & \ldots & 0 \\
0 & -c & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & -c
\end{pmatrix}_{n \times n}.
$$

Then $d_1$ and $d_2$ are multiplicative $\gamma$-derivations on $M_n(R)$.

**Example 3.6.** Consider the ring $M_2(\mathbb{R})$. For $\theta \in \mathbb{R}$, \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} is the standard rotation matrix where $\theta$ denotes the polar angle of rotation. Then $d : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by

$$
d(A) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} A
$$

is a multiplicative $\gamma$-derivation on $M_2(\mathbb{R})$, where $\gamma : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is defined by

$$
\gamma(A, B) = \begin{pmatrix} \cos(\pi + \theta) & -\sin(\pi + \theta) \\ \sin(\pi + \theta) & \cos(\pi + \theta) \end{pmatrix} = \begin{pmatrix} -\cos \theta & \sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix},
$$

the matrix corresponding to rotation by a polar angle $\pi + \theta$. Clearly, $d$ rotates the column vectors of $A$ by an angle $\theta$ in the anticlockwise direction.

**Example 3.7.** The identity map on a ring $R$ is a (usual) derivation only when the multiplication in $R$ is trivial. Such a severe restriction is not imposed by $\gamma$-derivations. For instance, if $R$ is any ring with 1 then the identity map is a $\gamma$-derivation on $R$; where $\gamma(x, y) = -1$.

**Example 3.8.** Let $R$ be a ring. Let $\gamma : R \times R \to R$ and let $f : R \to R$ be a $\gamma$-derivation on $R$. For an element $c \in Z(R)$, the functions $d_1 : R \to R$ defined by $d_1(x) = f(x)c$ and $d_2 : R \to R$ defined by $d_2(x) = c f(x)$ are a $\gamma_1$-derivation and a $\gamma_2$-derivation respectively on $R$, where $\gamma_1, \gamma_2 : R \times R \to R$ are defined by $\gamma_1(x, y) = \gamma(x, y)c$ and $\gamma_2(x, y) = c\gamma(x, y)$. The $\gamma_1$-derivation $d_1$ and the $\gamma_2$-derivation $d_2$ are induced by the $\gamma$-derivation $f$ on $R$.

In what follows, $\gamma : R \times R \to R$ and the image of $\gamma$ is denoted by

$$
\Gamma = \{\gamma(x, y) \mid x, y \in R\}.
$$

**Proposition 3.9.** Let $K$ be a subsemigroup of $(R, \cdot)$ and let $V$ be a non-empty subset of $R$ such that $KV \subseteq K$. If $d$ is a multiplicative $\gamma$-derivation on $R$ such that $d([V, K]) = \{0\}$ (resp. $d(V \circ K) = \{0\}$) and $[V, K] \Gamma V = \{0\}$ (resp. $(V \circ K) \Gamma V = \{0\}$), then $[v, K] K d(v) = \{0\} \forall v \in V$.

**Proof.** Let $k \in K$, $v \in V$. We have $kv \in K$ and hence

$$
d([v, kv]) = d(v(kv) - (kv)v) = d((vk - kv)v) = d([v, k]v) = 0
$$

(resp. $d(v \circ (kv)) = d(v(kv) + (kv)v) = d((vk + kv)v) = d((v \circ k)v) = 0$).

This gives

$$
[v, k] d(v) + [v, k] \gamma([v, k], v) v + d([v, k] v) v = 0
$$

(resp. $(v \circ k) d(v) + (v \circ k) \gamma(v \circ k, v) v + d(v \circ k) v) v = 0$).

Hence we get $[v, k] d(v) = 0$ (resp. $(v \circ k) d(v) = 0$). This implies that $vkd(v) = kvd(v)$ (resp. $vkd(v) = -kvd(v)$).

Now let $l \in K$. We have $lk \in K$ and hence $vlkd(v) = lkvd(v)$ (resp. $vlkd(v) = -lkvd(v)$). As $vkd(v) = kvd(v)$ (resp. $vkd(v) = -kvd(v)$), we get $vlkd(v) = lvkd(v) = 0$ (resp. $vlkd(v) = 0$). Therefore $vlkd(v) = lvkd(v)$. This gives $(vl - lv)kd(v) = [v, l] kd(v) = \{0\}$. Hence $[v, K] K d(v) = \{0\}$. □
Theorem 3.10. Let $S := \{c \in R \mid xc = yc \forall c \in Z(R), y \in R\}$. Then $S$ is a subgroup of $(R, +)$ containing $Z(R)$. If $R$ is prime and $Z(R) \neq \{0\}$ then $S = Z(R)$.

Proof. As $0 \in S$, the set $S$ is non-empty. Let $a, b \in S$. Let $x \in Z(R)$ and $y \in R$. We have $x(a - b)y = x(ay - by) = xy - yx = y(ax - bx) = y(a - b)x$, showing that $a - b \in S$. Hence $S$ is a subgroup of $(R, +)$. Now we will show that $Z(R) \subseteq S$. Let $b \in Z(R)$. Let $x \in Z(R)$ and $y \in R$. We have $x(by) = x(yb) = (yb)x$. Hence $b \in S$, which implies $Z(R) \subseteq S$. Suppose $R$ is prime and $Z(R) \neq \{0\}$. Let $a \in S$. Let $x \in Z(R) \setminus \{0\}$ and $y \in R$. We have $xay = yax$, which implies that $xay - yax = 0$.

This gives $ay - ya = 0$, that is, $ay = ya$. Hence $a \in Z(R)$, showing that $S \subseteq Z(R)$. Hence $S = Z(R)$.

The following result generalizes the well-known result by Wang [24]: If $x \in Z(R)$, then $d(x) \in Z(R)$.

Proposition 3.11. Let $\gamma$ be symmetric and let $\Gamma \subseteq S$. Let $d$ be a multiplicative $\gamma$-derivation on $R$. If $x \in Z(R)$, then $d(x) \in Z(R)$.

Proof. Let $y \in R$. We have $d(xy) = xd(y) + x \gamma(x, y)y + d(x)y$ and $d(yx) = d(y)x + y \gamma(y, x)x + yd(x)$. As $xy = yx$, we get $xd(y) + x \gamma(x, y)y + d(x)y = d(y)x + y \gamma(y, x)x + yd(x)$.

This gives $d(x)y = yd(x)$, and hence $d(x) \in Z(R)$.

4 $\gamma$ - derivation and commutativity of prime rings

In this section, $R$ denotes a prime ring unless specified otherwise.

Proposition 4.1. Let $K$ be a $\{0\}$-weak semigroup right (resp. left) ideal of $R$. If $x \in R$ is such that $Kx = \{0\}$ (resp. $xK = \{0\}$), then $x = 0$.

Proof. Suppose that $K$ is a $\{0\}$-weak semigroup right (resp. left) ideal of $R$. Let $I$ be a nonzero subset of $K$ such that $IR \subseteq K$ (resp. $RI \subseteq K$). Suppose $Kx = \{0\}$ (resp. $xK = \{0\}$). Then $(IR)x = \{0\}$ (resp. $x(RI) = \{0\}$), which implies that $(I \setminus \{0\})Rx = \{0\}$ (resp. $xR(I \setminus \{0\}) = \{0\}$). As $R$ is prime and $I$ is nonzero, we get $x = 0$.

The following result generalizes Proposition 4.2 of Kamal and Shaalan [14] for rings.

Proposition 4.2. Let $K$ be a nonzero semigroup left ideal of $R$ and let $V$ be a nonzero semigroup left ideal of $R$ such that $KV \subseteq V$. If $vk = -kv \forall v \in V, k \in K$, then $R$ is commutative and is of characteristic 2.

Proof. Let $v \in V, k, l \in K$. We have $kl \in K$ and hence $vl = (-k)v = (-k)(vl) = (-k)(vl) = (vk)l$, that is, $(vk)l = (lv)k$. This implies $(vk - kv)l = 0$. By Proposition 4.1, $vk = kv$. Now by Lemma 4.1 (iii) in Kamal and Shaalan [14], $R$ is commutative. We have $vk \in V$ and hence $-l(vk) = (vk)l = (vk)(kl) = -l(vk)l$ because $kl \in K$. As $lv \in V$, we have $-(vk)l = -l(vk)l = -l(v)k$, and hence $-l(vk)l = kv$. This gives $l(vk + lvk) = (lv + lv)k = (2lv)k = 0$. Hence $(2lv)K = \{0\}$. By Proposition 4.1, $2l = 2l = 0$, hence $(2lv) = \{0\}$. Again by Proposition 4.1, $2l = \{0\}$. Hence $2K = \{0\}$. By Proposition 2.8 in Kamal and Shaalan [14], $2R = \{0\}$, that is, $R$ is of characteristic 2.

Proposition 4.3. Let $K$ be a semigroup left (resp. right) ideal and a $\{0\}$-weak semigroup right (resp. left) ideal of $R$. If $x, y \in R$ are such that $xKy = \{0\}$, then $x = 0$ or $y = 0$.

Proof. Suppose $xKy = \{0\}$. As $RK \subseteq K$ (resp. $KR \subseteq K$), $xRKy = \{0\}$ (resp. $xK Ry = \{0\}$). As $R$ is prime, we get $x = 0$ or $Ky = \{0\}$ (resp. $xK = \{0\}$ or $y = 0$). By Proposition 4.1, $x = 0$ or $y = 0$. 

The following theorem extends Theorem 2.1 of Bell [3] and Theorem 3.4 of Kamal and Shaalan [14] for rings.

**Theorem 4.4.** Let $K$ be a nonzero semigroup left (resp. right) ideal of $R$. Let $\gamma$ be such that $[K, R \Gamma K] = \{0\}$ (resp. $[K, K \Gamma R] = \{0\}$). If $d$ is a multiplicative $\gamma$-derivation on $R$ such that $\{0\} \neq d(K) \subseteq Z(R)$, then $R$ is commutative.

**Proof.** Suppose $K$ is a nonzero semigroup left ideal of $R$ and $[K, R \Gamma K] = \{0\}$. Let $a \in K$ be such that $d(a) \neq 0$. Note that $d(a) \in Z(R)$. Let $k, l \in K$. Then $kl \in K$ and hence $d(kl) = ld(kl)$. This gives

$$k d(l) + k \gamma(k, l) l^2 + d(k) l^2 = l k d(l) + l k \gamma(k, l) l + l d(k) l.$$  

As $d(K) \subseteq Z(R)$ and $[K, R \Gamma K] = \{0\}$, we get

$$k l d(l) + k \gamma(k, l) l^2 + d(k) l^2 = l k d(l) + l k \gamma(k, l) l^2 + d(k) l^2.$$  

This gives $k l d(l) = l k d(l)$, that is, $(kl - lk) d(l) = 0$. Taking $l = a$, we get $(ka - ak) d(a) = 0$. Now by Proposition 2.3, $ka = ak$. Hence $a$ centralizes $K$.

As $ka \in K$, we have $d(ka) = l d(ka)$, that is,

$$k d(a) + l k \gamma(k, a) a l + d(k) a l = l k d(a) + l k \gamma(k, a) a + l d(k) a.$$  

This gives

$$k l d(a) + l k \gamma(k, a) a + d(k) l a = l k d(a) + l k \gamma(k, a) a + d(k) l a.$$  

Hence we get $k l d(a) = l k d(a)$, that is, $(kl - lk) d(a) = 0$. By Proposition 2.3, $kl = lk$, that is, $kl - lk = [k, l] = 0$. Therefore $[K, K] = \{0\}$.

Let $x \in R$. As $xa \in K$, we have $d(xa) = l d(xa)$, that is,

$$x d(a) + x \gamma(x, a) a l + d(x) a l = l x d(a) + l x \gamma(x, a) a + l d(x) a,$$  

which gives

$$x l d(a) + x \gamma(x, a) a l + l d(x) a = l x d(a) + x \gamma(x, a) a + l d(x) a$$  

because $K$ is a semigroup left ideal of $R$. Thus we have $x l d(a) = l x d(a)$, and hence $(x l - lx) d(a) = 0$. By Proposition 2.3, $xl = lx$. Hence $K \subseteq Z(R)$. Now by Proposition 2.5, $R$ is commutative.

The proof is analogous for the other case. \[\square\]

**Theorem 4.5.** Let $K$ be a nonzero semigroup ideal of $R$ and let $A$ be a nonzero semigroup left (resp. right) ideal of $R$. Let $\gamma$ be symmetric, $\Gamma \subseteq Z(R)$ and $[A, K] \Gamma = \{0\}$. If $d$ is a multiplicative $\gamma$-derivation on $R$ such that $d([A, K]) = \{0\}$ and $d(A) \neq \{0\}$, then $R$ is commutative.

**Proof.** As $[A, K] \Gamma = \{0\}$, we get $[A, K] \Gamma A = \{0\}$. Let $a \in A$ be such that $d(a) \neq 0$. By Proposition 3.9, we have $[a, K] kd(a) = \{0\}$. Now using Proposition 4.3, we get $[a, K] = \{0\}$, that is, $a$ centralizes $K$. By Proposition 2.4, we have $a \in Z(R)$. As $\gamma$ is symmetric, by Proposition 3.11 we get $d(a) \in Z(R)$. Let $y \in A$, $k \in K$. We have $ka \in K$ and hence $d([y, ka]) = 0$. This gives

$$d(yka - kay) = d(yka - kay) = d((yk - ky)a) = d([y, k] a) = 0.$$  

Therefore,

$$d([y, k] a) = [y, k] d(a) + [y, k] \gamma([y, k], a) a + d([y, k]) a = 0.$$  

This gives $[y, k] d(a) = 0$. Now by Proposition 2.3, we get $[y, k] = 0$. Hence $y$ centralizes $K$.

Now by Proposition 2.4, $y \in Z(R)$, and hence $A \subseteq Z(R)$. By Proposition 2.5, $R$ is commutative. \[\square\]
Example 4.6. Let \( n \) be a natural number. The function \( d \) on the ring of complex numbers \( \mathbb{C} \) defined by \( d(0) = 0 \) and \( d(re^{i\theta}) = r(\log r)^n e^{i\theta} \) for \( r \neq 0 \), is a multiplicative \( \gamma \)-derivation on \( \mathbb{C} \), where \( \gamma : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) is defined by

\[
\gamma(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \begin{cases} \sum_{k=0}^{n-1} nC_k (\log r_1)^{n-k} (\log r_2)^k & \text{if } n \geq 2, r_1 \neq 0 \text{ and } r_2 \neq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

For \( n \geq 2 \), the coefficients of powers of \( \log r_1, \log r_2 \) are binomial coefficients which can be computed as the interior part of the well-known Pascal’s triangle as shown in Figure 1. It follows easily from the symmetry of Pascal’s triangle that \( \gamma \) is symmetric. The ring \( \mathbb{C} \) is prime and all conditions of Theorem 4.5 hold with \( K = A = \mathbb{C} \). Using Theorem 4.5, we conclude that symmetry of \( \gamma \) induces commutativity of \( \mathbb{C} \) through the derivation \( d \).

\[
\begin{array}{ccccccccc}
1 & & & & & & & & 0th\,\, row \\
1 & 1 & & & & & & & 1st\,\, row \\
1 & 2 & 1 & & & & & & 2nd\,\, row \\
1 & 3 & 3 & 1 & & & & & 3rd\,\, row \\
1 & 4 & 6 & 4 & 1 & & & & \\
& & & & & & & &
\end{array}
\]

\begin{array}{ccccccccc}
\vdots & & & & & & & & \\
1 & nC_1 & nC_2 & \ldots & \ldots & nC_{n-1} & 1 & & \text{nth\,\, row }
\end{array}

Figure 1. Symmetry of \( \gamma \) (symmetry of Pascal’s triangle) induces commutativity of \( \mathbb{C} \)

Now we give a characterization of Galois fields in terms of Lie product and \( \gamma \)-derivation.

Theorem 4.7. Suppose

(1) \( R \) is finite;

(2) there exist a nonzero semigroup ideal \( K \) of \( R \), a nonzero semigroup left (resp. right) ideal \( A \) of \( R \), a function \( \gamma : R \times R \to R \) and a multiplicative \( \gamma \)-derivation \( d \) on \( R \) such that

(a) \( \gamma \) is symmetric, \( \Gamma \subseteq Z(R) \) and \( [A, K] \Gamma = \{0\} \);

(b) \( d([A, K]) = \{0\} \) and \( d(A) \neq \{0\} \).

Then \( R \) is a Galois field. Conversely, if \( R \) is a Galois field, then the conditions (1), (2a), (2b) hold in \( R \).

Proof. Let conditions (1), (2a), (2b) hold. By Theorem 4.5, \( R \) is a commutative ring. Let \( a, b \in R \) be such that \( ab = 0 \). Then for \( r \in R \), \( arb = abr = 0r = 0 \). This gives \( aRb = \{0\} \). As \( R \) is prime, we get either \( a = 0 \) or \( b = 0 \). Hence \( R \) is an integral domain. As \( R \) is finite, \( R \) is a Galois field.

Conversely, suppose that \( R \) is a Galois field with \( q = p^n \) elements. Then we have \( R = \{0, \alpha, \alpha^2, \ldots, \alpha^{q-1}\} \), where \( \alpha \) is a root of a primitive polynomial and \( \alpha^{q-1} = 1 \). Hence the condition (1) holds. It is clear that as \( R \) has no nonzero zero divisors, \( R \) is prime. Consider the map \( d : R \to R \) defined by \( d(0) = 0 \) and \( d(\alpha^k) = \alpha^{k-1} \) for \( 1 \leq k \leq q-1 \). Define \( \gamma : R \times R \to R \) by \( \gamma(x, y) = -\alpha^{-1} \). Let \( x, y \in R \). (i) Let \( x \neq 0 \) or \( y \neq 0 \). Then we have \( xd(y) + x\gamma(x, y)y + d(x)y = 0 + 0 + 0 = d(0) = d(xy) \). (ii) Let \( x \neq 0 \) and \( y \neq 0 \). Then \( x = \alpha^r \) and \( y = \alpha^s \) for some \( r, s \) where \( 1 \leq r, s \leq q-1 \). This gives \( xd(y) + x\gamma(x, y)y + d(x)y = \alpha^r\alpha^{s-1} + \alpha^s(-\alpha^{-1})\alpha^s + \alpha^{r-1}\alpha^s = \alpha^{r+s-1} - \alpha^{r+s-1} + \alpha^{r+s-1} = \alpha^{r+s} = d(\alpha^{r+s}) = d(\alpha^{s+r}) = d(\alpha^{s+r}) \). Hence \( d \) is a multiplicative \( \gamma \)-derivation on \( R \). Note that \( \gamma \) is symmetric and \( \Gamma \subseteq R = Z(R) \). By choosing \( K = A = R \), we have \( [A, K] \Gamma = \{0\} \Gamma = \{0\} \), \( d([A, K]) = d(\{0\}) = \{0\} \) and \( d(A) = d(R) \neq \{0\} \). Hence the conditions (2a), (2b) hold. \( \square \)
The following results extend Theorem 4.3 and Corollary 4.4 of Kamal and Shaalan [14] for rings.

**Proposition 4.8.** Let $K$ be a semigroup left ideal and a $\{0\}$-weak semigroup right ideal of $R$. Let $A$ be a non-empty subset of $R$ such that $KA \subseteq K$. If $d$ is a multiplicative $\gamma$-derivation on $R$ such that $d(A \circ K) = (A \circ K) \Gamma = \{0\}$ and $d(A) \neq \{0\}$, then $A \subseteq Z(R)$. Further, if $A$ is a nonzero semigroup left (resp. right) ideal of $R$, then $R$ is commutative.

**Proof.** As $(A \circ K) \Gamma = \{0\}$, we get $(A \circ K) \Gamma A = \{0\}$. Let $a \in A$ be such that $d(a) \neq 0$. By Proposition 3.9, we have $[a, K] K d(a) = \{0\}$. Then by Proposition 4.3, we have $[a, K] = \{0\}$, that is, $a$ centralizes $K$. By Proposition 2.4, $a \in Z(R)$. Let $y \in A, k \in K$. We have $ka \in K$ and hence $d(yoka) = 0$. This gives $d(yka + kay) = d(yka + kay) = d((yk + ky)a) = 0$. Thus we have
\[(y \circ k) d(a) + (y \circ k) \gamma(y \circ k, a) a + d(y \circ k) a = 0,
\]which gives
\[(y \circ k) d(a) = 0. \tag{1}\]

Let $l \in K$. Then we have $lk \in K$. Using Equation (1), we get $(y \circ lk) d(a) = 0$. This gives
\[ylk d(a) = -kly d(a) = (-l)(kly)d(a) = -l(-yk d(a)) = lky d(a).
\]

Therefore, we have
\[(ylk - kly) d(a) = (yl - ly) k d(a) = 0.
\]

Hence $(yl - ly) K d(a) = \{0\}$. As $d(a) \neq 0$, by Proposition 4.3, we get $yl - ly = 0$. Hence $y$ centralizes $K$. Now by Proposition 2.4, $y \in Z(R)$ and hence $A \subseteq Z(R)$. Suppose $A$ is a nonzero semigroup left (resp. right) ideal of $R$. By Proposition 2.5, $R$ is commutative. \hfill $\Box$

**Theorem 4.9.** Let $K$ be a nonzero semigroup left (resp. right) ideal of $R$ and let $\emptyset \neq A$ be a nonzero subset of $R$ such that $KA \subseteq K$. Let $\gamma$ be symmetric and $\Gamma \subseteq S$. If $d$ is a multiplicative $\gamma$-derivation on $R$ such that $d(A \circ K) = (A \circ K) \Gamma = \{0\}$ and $d(A) \neq \{0\}$, then $R$ is of characteristic 2. Further, if $A$ is a nonzero semigroup left (resp. right) ideal of $R$, then $R$ is commutative.

**Proof.** By Proposition 4.8, we have $A \subseteq Z(R)$. Let $a \in A$ be such that $d(a) \neq 0$. Let $y \in A \setminus \{0\}$ and $k \in K$. By Equation (1) in the proof of Proposition 4.8, we have $(y \circ k) d(a) = 0$. Also we have $a \in Z(R)$. Hence by Proposition 3.11, $d(a) \in Z(R)$. By Proposition 2.3, we get $y \circ k = 0$ because $d(a) \neq 0$. Hence $yk + ky = 0$. As $y \in Z(R)$, we get
\[ky + ky = (k + k)y = 0.
\]

Now Proposition 2.3 gives $k + k = 2k = 0$. Hence $2K = \{0\}$. By Proposition 2.8 in Kamal and Shaalan [14], we get $2R = \{0\}$. Hence $R$ is of characteristic 2. Suppose $A$ is a nonzero semigroup left (resp. right) ideal of $R$. As $A \subseteq Z(R)$ by Proposition 4.8, $R$ is commutative by Proposition 2.5. \hfill $\Box$

Now we give a characterization of Galois fields of characteristic 2 in terms of Jordan product and $\gamma$-derivation.

**Theorem 4.10.** Suppose

1. $R$ is finite;

2. there exist a nonzero semigroup left (resp. right) ideal $K$ of $R$, a nonzero semigroup left (resp. right) ideal $A$ of $R$, a function $\gamma : R \times R \rightarrow R$ and a multiplicative $\gamma$-derivation $d$ on $R$ such that

(a) $KA \subseteq K$;

(b) $\gamma$ is symmetric, $\Gamma \subseteq S$ and $(A \circ K) \Gamma = \{0\}$;
\[ d(A \circ K) = \{0\} \text{ and } d(A) \neq \{0\}. \]

Then \( R \) is a Galois field of characteristic 2. Conversely, if \( R \) is a Galois field of characteristic 2, then the conditions (1), (2a), (2b), (2c) hold in \( R \).

**Proof.** Let conditions (1), (2a), (2b), (2c) hold. By Theorem 4.9, \( R \) is a commutative ring of characteristic 2. Now as in the proof of Theorem 4.7, \( R \) is a Galois field. Conversely, suppose that \( R \) is a Galois field of characteristic 2 with \( q \) elements. Then \( q = 2^n \) for some \( n \geq 1 \). As \( R \) is of characteristic 2, for any \( a, b \in R \), the Jordan product of \( a \) and \( b \) coincides with the Lie product of \( a \) and \( b \). That is, \( a \circ b = ab + ba = ab - ba = [a, b] \). Now, the rest of the proof follows from Theorem 4.7.

**References**


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