Palestine Journal of Mathematics

Vol. 12(1)(2023) , 442–449

Generalized relative type (α,β) and generalized relative weak type (α,β) oriented some growth properties of composite entire and meromorphic functions

Tanmay Biswas and Chinmay Biswas

Communicated by Suheil Khoury

MSC 2020 Classifications: 30D35, 30D30, 30D20.

Keywords and phrases: Entire function, meromorphic function, growth, generalized relative order (α,β), generalized relative type (α,β), generalized relative weak type (α,β).

The authors are thankful to the Editor and referee for their valuable suggestions for the improvement of the paper.

Abstract The main aim of this paper is to prove some results related to the growth rates of composite entire and meromorphic functions on the basis of their generalized relative type (α,β) and generalized relative weak type (α,β), where α and β are continuous non-negative functions defined on $(-\infty,+\infty)$.

1 Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [9, 10, 14]. We also use the standard notations and definitions of the theory of entire functions which are available in [13] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, the Nevanlinna's characteristic function $T_f(r)$ (see [9, p.4]) plays the same role as $M_f(r)$. Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $T_f^{-1} : (T_f(0),\infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$. If f is non-constant then it has the following property:

Property (A) [2] : A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large values of r, $[M_f(r)]^2 \le M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [2].

Now let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x) = \alpha(x_0) \ge 0$ for $x \le x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and $\alpha \in L_2^0$, if $\alpha(\exp((1+o(1))x)) = (1+o(1))\alpha(\exp(x))$ as $x \to +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \le x \to +\infty$ for each $c \in (0,+\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1+o(1))\alpha(\exp(x))$ as $x_0 \le x \to +\infty$ for each $c \in (0,+\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$. Further we assume that throughout the present paper α_1, α_2, β, β_1, $\beta_2 \in L_1$ unless otherwise specifically stated.

The value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \ (\alpha \in L, \beta \in L)$$

introduced by Sheremeta [12], is called generalized order (α,β) of an entire function f. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different direction. For the purpose of further applications, Biswas et al. [4, 5] have rewritten the definition of the generalized order (α,β) of entire function in the following way after giving a minor modification to the original definition (e.g. see, [12]) which considerably extend the definition of φ-order of entire function introduced by Chyzhykov et al. [7]:

Definition 1.1. [4, 5] The generalized order (α,β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)} \text{ where } \alpha\in L_1.$$

If f is a meromorphic function, then

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ where } \alpha\in L_2.$$

Using the inequality $T_f(r)\le\log M_f(r)\le 3T_f(2r)$ $\{cf.$ [9]$\}$, for an entire function f, one may easily verify that

$$\begin{aligned}\rho_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)}=\limsup_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)}\\ \text{and } \lambda_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)}=\liminf_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)},\end{aligned}$$

when $\alpha\in L_2$.

Now in order to refine the growth scale namely the generalized order (α,β), Biswas et al. [5, 6] have introduced the definitions of another growth indicators, called generalized type (α,β) and generalized lower type (α,β) respectively of *a* meromorphic function which are as follows:

Definition 1.2. [5, 6] The generalized type (α,β) denoted by $\sigma_{(\alpha,\beta)}[f]$ and generalized lower type (α,β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]$ of meromorphic function f having finite positive generalized order (α,β) $(0<\rho_{(\alpha,\beta)}[f]<\infty)$ are defined as :

$$\begin{aligned}\sigma_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty}\frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}} \text{ and}\\ \overline{\sigma}_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty}\frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}},\ (\alpha\in L_2).\end{aligned}$$

If f is an entire function, then

$$\begin{aligned}\sigma_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty}\frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}} \text{ and}\\ \overline{\sigma}_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty}\frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}},\ (\alpha\in L_1).\end{aligned}$$

It is obvious that $0\le\overline{\sigma}_{(\alpha,\beta)}[f]\le\sigma_{(\alpha,\beta)}[f]\le\infty$.

Analogously, to determine the relative growth of two entire functions having same non-zero finite generalized lower order (α,β), Biswas et al. [5, 6] have introduced the definitions of generalized weak type (α,β) and generalized upper weak type (α,β) of a meromorphic function f of finite positive generalized lower order (α,β), $\lambda_{(\alpha,\beta)}[f]$ in the following way:

Definition 1.3. [5, 6] The generalized upper weak type (α,β) denoted by $\tau_{(\alpha,\beta)}[f]$ and generalized weak type (α,β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]$ of a meromorphic function f having finite positive generalized lower order (α,β) $(0<\lambda_{(\alpha,\beta)}[f]<\infty)$ are defined as:

$$\begin{aligned}\tau_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty}\frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and}\\ \overline{\tau}_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty}\frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}},(\alpha\in L_2).\end{aligned}$$

If f is an entire function, then

$$\tau_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty}\frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and}$$
$$\overline{\tau}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty}\frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}, (\alpha\in L_1).$$

It is obvious that $0\le\overline{\tau}_{(\alpha,\beta)}[f]\le\tau_{(\alpha,\beta)}[f]\le\infty$.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2, 11]) will come. Now in order to make some progress in the study of relative order, Biswas et al. [5] introduce the definitions of generalized relative order (α,β) and generalized relative lower order (α,β) of a meromorphic function with respect to another entire function in the following way:

Definition 1.4. [5] Let α, $\beta\in L_1$. The generalized relative order (α,β) and generalized relative lower order (α,β) of a meromorphic function f with respect to an entire function g denoted by $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:

$$\rho_{(\alpha,\beta)}[f]_g=\limsup_{r\to\infty}\frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g=\liminf_{r\to\infty}\frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

Now in order to refine the above growth scale, Biswas et al. [5] have introduced the definitions of other growth indicators, such as generalized relative type (α,β) and generalized relative lower type (α,β) of meromorphic function with respect to an entire function which are as follows:

Definition 1.5. [5] Let α, $\beta\in L_1$. The generalized relative type (α,β) denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized relative lower type (α,β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative order (α,β) are defined as:

$$\sigma_{(\alpha,\beta)}[f]_g=\limsup_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f]_g=\liminf_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Analogously, to determine the relative growth of a meromorphic function f having same non zero finite generalized relative lower order (α,β) with respect to an entire function g, Biswas et al. [5] have introduced the definitions of generalized relative upper weak type (α,β) denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type (α,β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of f with respect to g of finite positive generalized relative lower order (α,β) in the following way:

Definition 1.6. [5] Let α, $\beta\in L_1$. The generalized relative upper weak type (α,β) denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type (α,β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative lower order (α,β) are defined as:

$$\tau_{(\alpha,\beta)}[f]_g=\limsup_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f]_g=\liminf_{r\to\infty}\frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

In this paper we wish to prove some results related to the growth rates of composite entire and meromorphic functions on the basis of their generalized relative order (α,β), generalized relative type (α,β) and generalized relative weak type (α,β).

2 Main Results

First we present two lemmas which will be needed in the sequel.

Lemma 2.1. *[3] Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f(g)}(r) \leqslant \{1+o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

Lemma 2.2. *[8] Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right).$$

Now we present the main results of the paper.

Theorem 2.3. *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ and h satisfies the Property (A) where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then*

$$\limsup_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. Let us suppose that $\Delta > 2$ and $\delta \to 1+$ in Lemma 2.2. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1, Lemma 2.2 and the inequality $T_g(r) \leq \log M_g(r)$ {cf. [9] } for all sufficiently large values of r that

$$\begin{aligned}
T_h^{-1}(T_{f(g)}(r)) &\leqslant T_h^{-1}(\{1+o(1)\}T_f(M_g(r)))\\
i.e.,\ \alpha_1(T_h^{-1}(T_{f(g)}(r))) &\leqslant \alpha_1(\Delta(T_h^{-1}(T_f(M_g(r))))^{\delta})\\
i.e.,\ \alpha_1(T_h^{-1}(T_{f(g)}(r))) &\leqslant (1+o(1))\alpha_1(T_h^{-1}(T_f(M_g(r))))\\
i.e.,\ \alpha_1(T_h^{-1}(T_{f(g)}(r))) &\leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\beta_1(M_g(r))
\end{aligned}$$

$$i.e.,\ \alpha_1(T_h^{-1}(T_{f(g)}(r))) \leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)\exp(\alpha_2(M_g(r)))$$

$$i.e.,\ \alpha_1(T_h^{-1}(T_{f(g)}(r))) \leqslant$$

$$(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}. \tag{2.1}$$

Now from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]_h$, we obtain for all sufficiently large values of r that

$$\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}. \tag{2.2}$$

Therefore from (2.1) and (2.2), it follows for all sufficiently large values of r that

$$\frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq$$

$$\frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}}{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}}$$

$$i.e., \limsup_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Thus the theorem is established.□

Remark 2.4. In Theorem 2.3, if we replace the condition "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and other conditions remain same, then the conclusion of Theorem 2.3 remains valid with "$\lambda_{(\alpha_2,\beta_2)}[g]$" and "$\tau_{(\alpha_2,\beta_2)}[g]$" instead of "$\rho_{(\alpha_2,\beta_2)}[g]$" and "$\sigma_{(\alpha_2,\beta_2)}[g]$" respectively.

Remark 2.5. In Theorem 2.3, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ where k is an entire function" and other conditions remain the same, then Theorem 2.3 remains valid with

$$\text{“}\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))\text{” and “}\lambda_{(\alpha_2,\beta_2)}[g]_k\text{”}$$

instead of

$$\text{“}\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))\text{” and “}\lambda_{(\alpha_1,\beta_1)}[f]_h\text{” respectively.}$$

Remark 2.6. In Theorem 2.3, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ and $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ where k is an entire function" and other conditions remain the same, then Theorem 2.3 remains valid with "$\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))$", "$\lambda_{(\alpha_2,\beta_2)}[g]_k$" and "$\tau_{(\alpha_2,\beta_2)}[g]$" instead of
"$\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$", "$\lambda_{(\alpha_1,\beta_1)}[f]_h$" and "$\sigma_{(\alpha_2,\beta_2)}[g]$" respectively.

Using the notion of generalized lower type (α, β) we may state the following theorem without its proof because it can be carried out in the line of Theorem 2.3 .

Theorem 2.7. *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ and h satisfies the Property (A) where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g] \cdot \rho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Remark 2.8. In Theorem 2.7, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ where k is an entire function" and other conditions remain the same, then Theorem 2.7 remains valid with "$\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$" and "$\lambda_{(\alpha_2,\beta_2)}[g]_k$" instead of "$\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}$ and "$\lambda_{(\alpha_1,\beta_1)}[f]_h$" respectively.

Remark 2.9. In Theorem 2.7, if we replace the condition "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and other conditions remain the same, then the conclusion of Theorem 2.7 remains valid with

$$\text{“}\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))\text{” and “}\overline{\tau}_{(\alpha_2,\beta_2)}[g]\text{”}$$

instead of

$$\text{“}\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))\text{” and “}\overline{\sigma}_{(\alpha_2,\beta_2)}[g]\text{” respectively.}$$

Remark 2.10. In Theorem 2.7, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ where k is an entire function" and other conditions remain the same, then the conclusion of Theorem 2.7 remains valid with "$\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))$", "$\lambda_{(\alpha_2,\beta_2)}[g]_k$" and "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" instead of
"$\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$", "$\lambda_{(\alpha_1,\beta_1)}[f]_h$" and "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" respectively.

Now we state the following theorem without its proof as it can easily be carried out in the line in the line of Theorem 2.3.

Theorem 2.11. *Let f be a meromorphic function and g, h be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ or $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ and h satisfies the Property (A) where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \sigma_{(\alpha_2,\beta_2)}[g].$$

Remark 2.12. In Theorem 2.11, if we replace the condition "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and other conditions remain same, then Theorem 2.11 remains valid with

$$\text{“}\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))\text{” and “}\tau_{(\alpha_2,\beta_2)}[g]\text{”}$$

instead of

$$\text{“}\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))\text{” and “}\sigma_{(\alpha_2,\beta_2)}[g]\text{” respectively.}$$

Remark 2.13. In Theorem 2.11, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ or $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ where k is an entire function" and other conditions remain same, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.$$

Remark 2.14. In Theorem 2.11, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ or $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\rho_{(\alpha_2,\beta_2)}[g]_k > 0$ where k is an entire function" and other conditions remain same, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_2,\beta_2)}[g]_k}.$$

Remark 2.15. In Theorem 2.11, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$ or $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$ and $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$ and $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ where k is an entire function" and other conditions remain the same, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\tau_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.$$

Remark 2.16. In Remark 2.15, if we replace the conditions "$\lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g]_k > 0$" by "$\rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, $\rho_{(\alpha_2,\beta_2)}[g]_k > 0$" and other conditions remain the same, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\alpha_2(T_k^{-1}(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\tau_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]_h}{\rho_{(\alpha_2,\beta_2)}[g]_k}.$$

Theorem 2.17. *Let f be a meromorphic function and g, h be any two entire functions such that (i) $0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, (ii) $\rho_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$, (iii) $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$, (iv) $0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$ and h satisfies the Property (A) where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\exp(\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\sigma_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]}.$$

Proof. In view of condition (ii),we obtain from (2.1) for all sufficiently large values of r that

$$\alpha_1(T_h^{-1}(T_{f(g)}(r))) \leqslant$$

$$(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}. \tag{2.3}$$

Again from the definition of $\sigma_{(\alpha_1,\beta_1)}[f]_h$ we get for a sequence of values of r tending to infinity that

$$\exp(\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r)))))) \geq (\sigma_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}. \tag{2.4}$$

Now from (2.3) and (2.4), it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\exp(\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}}{(\sigma_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]_h}}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\exp(\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]_h\cdot\sigma_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]_h}.$$

□

Remark 2.18. In Theorem 2.17, if we replace the conditions "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$" respectively and other conditions remain same, then Theorem 2.17 remains valid with "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$" instead of "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\sigma_{(\alpha_1,\beta_1)}[f]_h$" respectively.

Remark 2.19. In Theorem 2.17, if we replace the conditions "$0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" and "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$" respectively and other conditions remain same, then Theorem 2.17 remains valid with "$\lambda_{(\alpha_1,\beta_1)}[f]_h$" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$" instead of "$\rho_{(\alpha_1,\beta_1)}[f]_h$" and "$\sigma_{(\alpha_1,\beta_1)}[f]_h$" respectively.

Remark 2.20. In Theorem 2.17, if we replace the condition "$0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$" and other conditions remain same, then Theorem 2.17 remains valid with "limit superior" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$" instead of "limit inferior" and "$\sigma_{(\alpha_1,\beta_1)}[f]_h$" respectively.

Now using the concept of generalized relative upper weak type (α,β), we may state the following theorem without its proof since it can be carried out in the line of Theorem 2.17.

Theorem 2.21. *Let f be a meromorphic function and g, h be any two entire functions such that (i) $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$, (ii) $\lambda_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$, (iii) $\tau_{(\alpha_2,\beta_2)}[g] < \infty$, (iv) $0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ and h satisfies the Property (A) where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\exp(\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]_h \cdot \tau_{(\alpha_2,\beta_2)}[g]}{\tau_{(\alpha_1,\beta_1)}[f]_h}.$$

Remark 2.22. In Theorem 2.21, if we replace the conditions "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$" respectively and other conditions remain same, then Theorem 2.21 remains valid with "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$" and "$\tau_{(\alpha_1,\beta_1)}[f]_h$" respectively.

Remark 2.23. In Theorem 2.21, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$" respectively and other conditions remain same, then Theorem 2.21 remains valid with "$\lambda_{(\alpha_1,\beta_1)}[f]_h$" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$" instead of "$\rho_{(\alpha_1,\beta_1)}[f]_h$" and "$\tau_{(\alpha_1,\beta_1)}[f]_h$" respectively.

Remark 2.24. In Theorem 2.21, if we replace the condition "$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$" by "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$" and other conditions remain same, then Theorem 2.21 remains valid with "limit superior" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$" instead of "limit inferior" and "$\tau_{(\alpha_1,\beta_1)}[f]_h$" respectively.

Remark 2.25. In Theorem 2.21, if we replace the conditions "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$" and "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$" and "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" respectively and other conditions remain same, then Theorem 2.21 remains valid with "$\sigma_{(\alpha_2,\beta_2)}[g]$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$".

Remark 2.26. In Theorem 2.21, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$","$\lambda_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ " by "$0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$", "$\rho_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f]_h < \infty$" respectively and other conditions remain same, then Theorem 2.21 remains valid with "$\sigma_{(\alpha_1,\beta_1)}[f]_h$" instead of "$\tau_{(\alpha_1,\beta_1)}[f]_h$".

Remark 2.27. In Theorem 2.21, if we replace the conditions "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$", "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ " by "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$", "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$" respectively and other conditions remain same, then Theorem 2.21 remains valid with "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$" and "$\tau_{(\alpha_1,\beta_1)}[f]_h$".

Remark 2.28. In Theorem 2.21, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$", "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$", "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ " by "$0 < \rho_{(\alpha_1,\beta_1)}[f]_h < \infty$", "$\rho_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h < \infty$" respectively and other conditions remain same, then Theorem 2.21 remains valid with "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$" instead of "$\tau_{(\alpha_2,\beta_2)}[g]$" and "$\tau_{(\alpha_1,\beta_1)}[f]_h$".

Remark 2.29. In Theorem 2.21, if we replace the conditions "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \lambda_{(\alpha_2,\beta_2)}[g]$", "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f]_h < \infty$ " by "$\lambda_{(\alpha_1,\beta_1)}[f]_h = \rho_{(\alpha_2,\beta_2)}[g]$", "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f]_h < \infty$" respectively and other conditions remain same, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\exp(\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h \cdot \sigma_{(\alpha_2,\beta_2)}[g]}{\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h}.$$

Remark 2.30. Under the same conditions of Remark 2.29, one can easily verify that

$$\limsup_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\exp(\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r))))))}\le\frac{\rho_{(\alpha_1,\beta_1)}[f]_h\cdot\sigma_{(\alpha_2,\beta_2)}[g]}{\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h}.$$

Remark 2.31. In Remark 2.29, if we replace the conditions "$0<\lambda_{(\alpha_1,\beta_1)}[f]_h\le\rho_{(\alpha_1,\beta_1)}[f]_h<\infty$", "$\lambda_{(\alpha_1,\beta_1)}[f]_h=\rho_{(\alpha_2,\beta_2)}[g]$", "$\sigma_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h<\infty$" by "$0<\lambda_{(\alpha_1,\beta_1)}[f]_h<\infty$", "$\rho_{(\alpha_1,\beta_1)}[f]_h=\lambda_{(\alpha_2,\beta_2)}[g]$", "$\tau_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h<\infty$" respectively and other conditions remain same, then conclusion of Remark 2.29 remains valid with "$\tau_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h$" instead of "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h$".

Remark 2.32. In Remark 2.29, if we replace the conditions "$0<\lambda_{(\alpha_1,\beta_1)}[f]_h\le\rho_{(\alpha_1,\beta_1)}[f]_h<\infty$", "$\lambda_{(\alpha_1,\beta_1)}[f]_h=\rho_{(\alpha_2,\beta_2)}[g]$", "$\sigma_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]_h<\infty$" by "$0<\rho_{(\alpha_1,\beta_1)}[f]_h<\infty$", "$\rho_{(\alpha_1,\beta_1)}[f]_h=\lambda_{(\alpha_2,\beta_2)}[g]$", "$\tau_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h<\infty$" respectively and other conditions remain same, then

$$\limsup_{r\to+\infty}\frac{\alpha_1(T_h^{-1}(T_{f(g)}(r)))}{\exp(\alpha_1(T_h^{-1}(T_f(\beta_1^{-1}(\beta_2(r))))))}\le\frac{\rho_{(\alpha_1,\beta_1)}[f]_h\cdot\tau_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]_h}.$$

References

[1] L. Bernal-Gonzaléz, *Crecimiento relativo de funciones enteras. Aportaciones al estudio de las funciones enteras con índice exponencial finito*, Doctoral Thesis, Universidad de Sevilla, Spain, (1984).

[2] L. Bernal, Orden relative de crecimiento de funciones enteras, *Collect. Math.*, **39**, 209–229 (1988).

[3] W. Bergweiler, On the Nevanlinna characteristic of a composite function, *Complex Variables Theory Appl.*, **10**, 225–236 (1988).

[4] T. Biswas and C. Biswas, Generalized (α,β) order based on some growth properties of wronskians, *Mat. Stud.*, **54**(1), 46–55 (2020).

[5] T. Biswas and C. Biswas, Some results on generalized relative order (α,β) and generalized relative type (α,β) of meromorphic function with respect to an entire function, *Ganita*, **70**(2), 239–252 (2020).

[6] T. Biswas and C. Biswas, On some growth properties of composite entire and meromorphic functions from the view point of their generalized type (α,β) and generalized weak type (α,β), *South East Asian J. Math. Math. Sci.*, **17**(1), 31–44 (2021).

[7] I. Chyzhykov and N. Semochko, Fast growing entire solutions of linear differential equations, *Math. Bull. Shevchenko Sci. Soc.*, **13**, 68–83 (2016).

[8] S. K. Datta, T. Biswas and C. Biswas, Measure of growth ratios of composite entire and meromorphic functions with a focus on relative order, *Int. J. Math. Sci. Eng. Appl.*, **8**(IV), 207–218 (2014).

[9] W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, (1964).

[10] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, De Gruyter, Berlin, (1993).

[11] B. K. Lahiri and D. Banerjee, Relative order of entire and meromorphic functions, *Proc. Natl. Acad. Sci. India, Sect. A*, **69(A)**(3), 339–354 (1999).

[12] M. N. Sheremeta, Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion, *Izv. Vyssh. Uchebn. Zaved Mat.*, **2**, 100–108 (1967)(in Russian).

[13] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company, New York, (1949).

[14] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, (1993).

Author information

Tanmay Biswas, Rajbari, Rabindrapally, R. N. Tagore Road, P.O. Krishnagar, P.S. Kotwali, Dist-Nadia, PIN-741101, West Bengal, India.
E-mail: `tanmaybiswas_math@rediffmail.com`

Chinmay Biswas, Department of Mathematics, Nabadwip Vidyasagar College, Nabadwip, Dist.- Nadia, PIN-741302,West Bengal, India.
E-mail: `chinmay.shib@gmail.com`

Received: August 23rd, 2021
Accepted: January 17th, 2022