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Symmetric identities for the generalized Hermite-based unified Apostol type polynomials

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Abstract In this paper, we introduce and investigate a new unification of the unified family of Hermite-based Apostol-Bernoulli, Euler and Genocchi polynomials and numbers. We obtain some summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. We give explicit relations for these polynomials and related to multiple power sums.

1 Introduction

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [2, 3] are defined as

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

It is easily seen from definition (1.1) that

$$H_n(2x,-1) = H_n(x)$$

and

$$H_n(x,-\frac{1}{2}) = He_n(x),$$

where $H_n(x)$ and $He_n(x)$ being ordinary Hermite polynomials. Also

$$H_n(x,0) = x^n.$$

The generating function for Hermite polynomial H_n(x,y) are given by [1, 5]:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.2}$$

Recently, Ozarslan [12] introduced the following unification of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Explicitly Ozarslan studied the following generating function:

$$f_{a,b}^{(\alpha)}(x;t,a,b) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}, \tag{1.3}$$

$$\left(\left|t + b\ln\left(\frac{\beta}{\alpha}\right)\right| < 2\pi, k \in \mathbb{N}_0; a,b \in \Re\backslash\{0\}; \alpha,\beta \in \mathbb{C}\right).$$

For $\alpha = 1$ in (1.3), we get

$$f_{a,b}(x;t,a,b) = \frac{2^{1-k}t^k}{\beta^b e^t - a^b} e^{xt} = \sum_{n=0}^{\infty} P_{n,\beta}(x;k,a,b)\frac{t^n}{n!}, \tag{1.4}$$

$$\left(\left|t+b\ln\left(\frac{\beta}{\alpha}\right)\right|<2\pi, k\in\mathbb{N}_0; a,b\in\Re\backslash\{0\}; \alpha,\beta\in\mathbb{C}\right).$$

From (1.3) and (1.4), we have

$$P_{n,\beta}^{(1)}(x;k,a,b)=P_{n,\beta}(x;k,a,b), (n\in\mathbb{N}).$$

Which is defined by Ozden and Simsek [14]. Now Ozden et al. [13] introduced many properties of these polynomials. We give some specific special cases:

1. By substituting $a=b=k=1$ and $\beta=\lambda$ into (1.3), one has the Apostol-Bernoulli polynomials $P_{n,\beta}^{(1)}(x;1,1,1)=B_n^{(\alpha)}(x;\lambda)$, which are defined by means of the following generating function

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, (\mid t+\log\lambda\mid<2\pi) \tag{1.5}$$

(see for details [7], [8], [9], [13], [14] see also the references cited in each of these earlier works).

For $\lambda=\alpha=1$ in (1.5), the result reduces to

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \mid t\mid<2\pi \tag{1.6}$$

where $B_n(x)$ denotes the classical Bernoulli polynomials (see from example [1]-[21]; see also the references cited in each of these earlier works).

2. If we substitute $b=\alpha=1$, $k=0$, $a=-1$ and $\beta=\lambda$ into (1.3), we have the Apostol-Euler polynomials $P_{n,\lambda}^{(1)}(x;0,-1,1)=E_n^1(x,\lambda)$

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, (\mid t+\log\lambda\mid<\pi) \tag{1.7}$$

(see for details [7], [8], [9], [13], [14] see also the references cited in each of these earlier works).

For $\lambda=1$ in (1.7), the result reduces to

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \mid t\mid<\pi \tag{1.8}$$

where $E_n(x)$ denotes the classical Euler polynomials (see from example [1], [7], [8], [9], [13], [14], [15], [16]; see also the references cited in each of these earlier works).

3. By substituting $b=\alpha=1$, $k=1$, $a=-1$ and $\beta=\lambda$ into (1.3), one has the Apostol-Genocchi polynomials $P_{n,\beta}^{(1)}(x;1,-1,1)=\frac{1}{2}G_n(x;\lambda)$, which is defined by means of the following generating function

$$\frac{2t}{\lambda e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x;\lambda)\frac{t^n}{n!}, (\mid t+\log\lambda\mid<\pi) \tag{1.9}$$

(see for details [7], [8], [9], [13], [14] see also the references cited in each of these earlier works)

4. By substituting $x=0$ in the generating function (1.3), we obtain the corresponding unification of the generating functions of Bernoulli, Euler and Genocchi numbers of higher order. Thus we have

$$P_{n,\beta}^{(\alpha)}(0;k,a,b)=P_{n,\beta}^{(\alpha)}(k,a,b), n\in\mathbb{N}.$$

Very recently, Pathan and Khan [15] introduced 2-variable Hermite-based Apostol type polynomials as follows:

Definition 1.1. The generalized Hermite-based Apostol type polynomials ${}_HP^{(\alpha)}_{n,\beta}(x, y; k, a, b, e)$ for nonnegative integer n are defined by

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HP^{(\alpha)}_{n,\beta}(x, y; k, a, b, e)\frac{t^n}{n!}, \tag{1.10}$$

$$(|t + b\ln(\frac{\beta}{\alpha})| < 2\pi, k \in \mathbb{N}_0; a, b \in \Re\backslash\{0\}; \alpha, \beta \in \mathbb{C}).$$

For the existence of the expansion, we need

(i) $| \ t \ |< 2\pi$ where $\alpha \in \mathbb{N}_0$, $k = 1$ and $\left(\frac{\beta}{a}\right)^b = 1$;$| \ t \ |< 2\pi$ when $\alpha \in \mathbb{N}_0$, $k = 2, 3, \cdots$ and $\left(\frac{\beta}{a}\right)^b = 1$; $| \ t \ |<| \ b\log\left(\frac{\beta}{a}\right) \ |$ when $\alpha \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $\left(\frac{\beta}{a}\right)^b \neq 1$ or $(\neq -1)$; $x, y \in \mathbb{R}$, $\beta \in \mathbb{C}/\{0\}$, $1^\alpha = 1$.

(ii) $| \ t \ |< 2\pi$ when $\left(\frac{\beta}{a}\right)^b = -1$; $| \ t \ |<| \ b\log\left(\frac{\beta}{a}\right) \ |$ when $\left(\frac{\beta}{a}\right)^b \neq -1$, $x, y \in \mathbb{R}$, $k = 0$, $\alpha, \beta \in \mathbb{C}$, $a, b, c \in \mathbb{C}$ $1^\alpha = 1$.

(iii)$| \ t \ |< 2\pi$ when $\alpha \varepsilon n_0$ and $\left(\frac{\beta}{a}\right)^b = -1$, $x, y \in \mathbb{R}$, $k \in \mathbb{N}$, $\beta \in \mathbb{C}$, $a, b, c \in \mathbb{C}/\{0\}$ $1^\alpha = 1$ where $w =| \ w \ | \ e^{i\theta}, -\pi \leq \theta < \pi$ and $\log(| \ w \ |) + i\theta$.

Table 1. Some special cases of the 2VHBATP ${}_HY^{(\alpha)}_n(x, y; k, a, b)$

S. No.	Values of the parameter	Relation between the 2VHBATP ${}_HP^{(\alpha)}_n(x, y; k, a, b)$ and its special case	Name of the resultant special polynomials	Generating functions and the resultant of special polynomials
I.	$k = a = b = 1, \beta = \lambda$	${}_HP^{(\alpha)}_n(x, y; 1, 1, \lambda)={}_HB^{(\alpha)}_n(x, y; \lambda)$	2-variable Hermite-based Apostal Bernoulli polynomial	$\left(\frac{t}{\lambda e^t-1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB^{(\alpha)}_n(x, y; \lambda)\frac{t^n}{n!}$
II.	$k + 1 = -a = b = 1, \beta = \lambda$	${}_HP^{(\alpha)}_n(x, y; 0, -1, 1, \lambda)={}_HE^{(\alpha)}_n(x, y; \lambda)$	2-variable Hermite-based Apostal Euler polynomial	$\left(\frac{2}{\lambda e^t+1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE^{(\alpha)}_n(x, y; \lambda)\frac{t^n}{n!}$
III.	$k = -2a = b = 1, 2\beta = \lambda$	${}_HP^{(\alpha)}_n(x, y; 1, -\frac{1}{2}, 1, \lambda)={}_HG^{(\alpha)}_n(x, y; \lambda)$	2-variable Hermite-based Apostal Genocchi polynomial	$\left(\frac{2t}{\lambda e^t+1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG^{(\alpha)}_n(x, y; \lambda)\frac{t^n}{n!}$

Note. In the case We note that for $\lambda = 1$, the result derived above for the 2VHbBP ${}_HB^{(\alpha)}_n(x, y; \lambda)$, 2VHbBEP ${}_HE^{(\alpha)}_n(x, y; \lambda)$ and 2VHbBGP ${}_HG^{(\alpha)}_n(x, y; \lambda)$ give the corresponding results for the 2-variable Hermite-based Bernoulli polynomials (2VHbBP) (of order α) ${}_HB^{(\alpha)}_n(x, y)$, 2-variable Hermite-based Euler polynomials (2VHbEP) (of order α) ${}_HE^{(\alpha)}_n(x, y)$ and 2-variable Hermite-based Genocchi polynomials (2VHbGP) (of order α) ${}_HG^{(\alpha)}_n(x, y)$. Again for $\alpha = 1$, we get the corresponding results for the 2-variable Hermite-based Bernoulli polynomials (2VHbBP) ${}_HB_n(x, y)$, 2-variable Hermite-based Euler polynomials (2VHbEP) ${}_HE_n(x, y)$ and 2-variable Hermite-based Genocchi polynomials (2VHbGP) ${}_HG_n(x, y)$.

Garg et al. [4, 20] introduced the following generalization of the Hurwitz-Lerch zeta function $\Phi(z, s; a)$:

$$\Phi^{(\rho,\sigma)}_{\mu,\nu}(z, s; a) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} z^n}{(\nu)_{\sigma n}(n + a)^s}, \tag{1.11}$$

$(m \in \mathbb{C}, a, \nu \in C \setminus \mathbb{Z}_0^-, p, \sigma \in \mathbb{R}, p < \sigma$ when $s, z \in \mathbb{C}(|z| = 1); p = \sigma$ and $R(s - m + \nu) > 0$, when $|z| = 1)$.

It is obvious that

$$\Phi_{\mu,1}^{(1,1)}(z,s;a) = \Phi_{\mu}^{*}(z,s;a) = \sum_{n=0}^{\infty} \frac{(\mu)_n z^n}{n!(n+a)^s}, \tag{1.12}$$

(see for details [4, 20]).

The multiple power sums are defined by Luo in [10, 11] as follows:

$$S_k^{(l)}(m,\lambda) = \sum_{\substack{0 \le \nu_1 \le \cdots \le \nu_m = l \\ \nu_1+\nu_2+\cdots+\nu_m = n}} \begin{pmatrix} l \\ \nu_1,\nu_2,\cdots,\nu_m \end{pmatrix} \lambda^{\nu_1+2\nu_2+\cdots+m\nu_m}(\nu_1+2\nu_2+\cdots+m\nu_m)^k. \tag{1.13}$$

From (1.13), we have

$$\left(\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}\right)^{(l)} = \lambda^{(-l)} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} S_k^{(l)(m,\lambda)} \right) \frac{t^n}{n!}. \tag{1.14}$$

For $l=1$, (1.14) reduces to

$$\frac{1-\lambda^m e^{mt}}{1-\lambda e^t} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} S_k^{(l)(m,\lambda)} \right) \frac{t^n}{n!}. \tag{1.15}$$

The generalized Stirling numbers of the second kinds $S(n,\nu,a,b,\beta)$ of order ν are defined in [17] as follows:

$$\sum_{n=0}^{\infty} S(n,\nu,a,b,\beta)\frac{t^n}{n!} = \frac{(\beta^b e^t - a^b)^\nu}{\nu!}. \tag{1.16}$$

Setting $\beta=\lambda$, $a=b=1$, (1.16) reduces to

$$\sum_{n=0}^{\infty} S(n,\nu,\lambda)\frac{t^n}{n!} = \frac{(\lambda e^t - 1)^\nu}{\nu!}. \tag{1.17}$$

In the last ten years many mathematicians studied the Apostol-type Bernoulli polynomials. Srivastava in [18] and Srivastava et al. in [20, 21] investigated and proved some relations and theorems for Bernoulli type polynomials and Apostol-Bernoulli-type polynomials. Luo in [10, 11] proved the multiplication theorems for the Apostol-Bernoulli and Apostol-Euler polynomials of higher-order and multiple alternating sums. Luo et al. [9] gave some symmetry relations between the Apostol-Bernoulli polynomials and Apostol-Euler polynomials. Ozarslan in [12] defined uniform form of the Apostol-Bernoulli, Euler and Genocchi polynomials $P_n^{(\alpha)}(x;ka,a,b)$ of order α. He gave the explicit representation of this unified family in terms of a Gaussian hypergeometric function. Also, he gave the recurrence relations and symmetry properties for the unified Apostol-type polynomials.

This paper is organized as follows. In Section 2, we give some explicit relation for the Hermite-based unified Apostol type polynomials. In section 3, we establish some implicit summation formulae for the Hermite-based unified Apostol type polynomials. In section 4, we prove the relation between Hurwitz-Lerch zeta function and the unified Apostol-type polynomials and give some symmetry relations for these Hermite-based unified Apostol-type polynomials.

2 A new class of Hermite-based Apostol-Bernoulli, Euler and Genocchi (ABEG) polynomials ${}_HP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$

In this section, we aim to obtain the explicit relations of the polynomials ${}_HP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$. By the motivation of the Pathan and Khan [15, 16] and Kurt [7, 8], we prove some relations for these polynomials and give the relations between the unified family of generalized Apostol-type polynomials and the Stirling numbers of the second kind $S(n,\nu,a,b,\beta)$ of order ν.

Theorem 2.1. Let $a,b>0$ and $a\neq b$. Then $x,y\in\mathbb{R}$ and $n\geq 0$, we have

$$_HP^{(\alpha)}_{n,\lambda}(x,y;1,1,1)={}_HB^{(\alpha)}_n(x,y;\lambda),{}_HP^{(\alpha)}_{n,\lambda}(x,y;0,1,-1)={}_HE^{(\alpha)}_n(x,y;\lambda),$$

$$_HP^{(\alpha)}_{n,\frac{\lambda}{2}}(x,y;1,-\frac{1}{2},1)={}_HG^{(\alpha)}_n(x,y;\lambda). \tag{2.1}$$

$$_HP^{(\alpha+\gamma)}_{n,\beta}(x+y,z+u;k,a,b)=\sum_{m=0}^{\infty}\binom{n}{m}{}_HP^{(\gamma)}_{m,\beta}(z,u;a,b){}_HP^{(\alpha)}_{n-m,\beta}(x,y;k,a,b). \tag{2.2}$$

$$_HP^{(\alpha)}_{n,\beta}(x+z,y;a,b)=\sum_{n=0}^{m}\binom{m}{n}P^{(\alpha)}_{n-m,\beta}(x;k,a,b)H_m(z,y). \tag{2.3}$$

Proof. The formula in (2.1) are obvious. Applying Definition (1.10), we have

$$\sum_{n=0}^{\infty}{}_HP^{(\alpha+\gamma)}_{n,\beta}(x+y,z+u;k,a,b)\frac{t^n}{n!}=\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}\sum_{m=0}^{\infty}{}_HP^{(\gamma)}_{m,\beta}(z,u;k,a,b)\frac{t^m}{m!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}_HP^{(\gamma)}_{m,\beta}(z,u;k,a,b){}_HP^{(\alpha)}_{n-m,\beta}(x,y;k,a,b)\frac{t^n}{(n-m)!}.$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.1). Again by Definition (1.10) of generalized polynomials, we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{(x+z)t+yt^2}=\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x+z,y;k,a,b)\frac{t^n}{n!} \tag{2.4}$$

which can be written as

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{xt}e^{zt+yt^2}=\sum_{n=0}^{\infty}P^{(\alpha)}_{n,\beta}(x;k,a,b)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(z,y)\frac{t^m}{m!}. \tag{2.5}$$

On replacing n by $n-m$ in the above equation and comparing the coefficients of $\frac{t^n}{n!}$, we get the desired result (2.3).

Theorem 2.2. The Hermite-based unified Apostol-type polynomials satisfy the following relation

$$\beta^b{}_HP^{(\alpha)}_{n,\beta}(x+1,y;k,a,b)-a^b{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)=2^{1-k}{}_HP^{(\alpha-1)}_{n-k,\beta}(x,y;k,a,b)\frac{n!}{(n-k)!}. \tag{2.6}$$

Proof. From (1.10), we have

$$\beta^b\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{(x+1)t+yt^2}-a^b\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{xt+yt^2}=2^{1-k}t^k\sum_{n=0}^{\infty}{}_HP^{(\alpha-1)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}\{\beta^b{}_HP^{(\alpha)}_{n,\beta}(x+1,y;k,a,b)-a^b{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\}\frac{t^n}{n!}=2^{1-k}\sum_{n=0}^{\infty}{}_HP^{(\alpha-1)}_{n,\beta}(x,y;k,a,b)\frac{t^{n+k}}{n!}$$

$$\sum_{n=0}^{\infty}\left(\beta^b{}_HP^{(\alpha)}_{n,\beta}(x+1,y;k,a,b)-a^b{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\right)\frac{t^n}{n!}=2^{1-k}\sum_{n=k}^{\infty}{}_HP^{(\alpha-1)}_{n-k,\beta}(x,y;k,a,b)\frac{t^n}{(n-k)!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, we arrive at the desired result (2.6).

Theorem 2.3. There is the following relation between the λ-Stirling numbers of second kinds and Hermite-based unified Apostol type polynomials ${}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)$:

$$a^{b\alpha}\alpha!\sum_{r=0}^{n}\binom{n}{r}{}_HP^{(\alpha)}_{n-r,\beta}(x,y;k,a,b)S\left(r,\alpha,\left(\frac{\beta}{a}\right)^{b}\right)=2^{(1-k)\alpha}H_{n-k\alpha}(x,y)\frac{n!}{(n-k\alpha)!}. \tag{2.7}$$

Proof. By using equation (1.10) and (1.17), we have

$$\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{xt+yt^2}$$

$$=\frac{2^{(1-k)\alpha}t^{k\alpha}}{a^{b\alpha}\left(\left(\frac{\beta}{a}\right)^{b}e^t-1\right)}e^{xt+yt^2}=\frac{2^{(1-k)\alpha}t^{k\alpha}e^{xt+yt^2}}{a^{b\alpha}\alpha!\sum\limits_{r=0}^{\infty}S\left(r,\alpha,\left(\frac{\beta}{a}\right)^{b}\right)\frac{t^r}{r!}}$$

$$\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}a^{b\alpha}\alpha!\sum_{r=0}^{\infty}S\left(r,\alpha,\left(\frac{\beta}{a}\right)^{b}\right)\frac{t^r}{r!}=2^{(1-k)\alpha}t^{k\alpha}\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}a^{b\alpha}\alpha!\sum_{r=0}^{n}\binom{n}{r}{}_HP^{(\alpha)}_{n-r,\beta}(x,y;k,a,b)S\left(r,\alpha,\left(\frac{\beta}{a}\right)^{b}\right)\frac{t^n}{n!}=2^{(1-k)\alpha}\sum_{n=0}^{\infty}H_{n-k\alpha}(x,y)\frac{t^n}{(n-k\alpha)!}$$

By comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result (2.7).

Theorem 2.4. There is the following relation between the λ-Stirling numbers of second kinds and Hermite-based unified Apostol type polynomials ${}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)$:

$${}_HP^{(\alpha-\gamma)}_{n-k\gamma,\beta}(x,y;k,a,b)=\frac{2^{(k-1)\gamma}\gamma!(n-k\gamma)!}{n!}\sum_{r=0}^{n}\binom{n}{r}{}_HP^{(\alpha)}_{n-r,\beta}(x,y;k,a,b)S\left(r,\gamma,a,b,\beta\right). \tag{2.8}$$

Proof. From (1.10) and (1.17), we have

$$\sum_{n=0}^{\infty}{}_HP^{(\alpha-\gamma)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha-\gamma}e^{xt+yt^2}$$

$$=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{xt+yt^2}\left(\frac{\beta^b e^t-a^b}{2^{1-k}t^k}\right)^{\gamma}$$

$$\sum_{n=0}^{\infty}{}_HP^{(\alpha-\gamma)}_{n,\beta}(x,y;k,a,b)\frac{t^{n+k\gamma}}{n!}=2^{(k-1)\gamma}\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}\sum_{r=0}^{\infty}S\left(r,\gamma,a,b,\beta\right)\frac{t^r}{r!}.$$

Using Cauchy product, comparing the coefficient of $\frac{t^n}{n!}$, we get the desired result (2.8).

3 Summation formulae for generalized Hermite-based Apostol type polynomials

The purpose of this section is to give some interesting generating functions, new results, and relations for the generalized Hermite-based unified Apostol type polynomials. We begin here some of these results in the following forms.

Theorem 3.1. The summation formulae for generalized Hermite-based unified Apostol type polynomials ${}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)$ holds true:

$${}_HP^{(\alpha)}_{n+m,\beta}(z,y;k,a,b)=\sum_{p,q=0}^{n,m}\binom{n}{p}\binom{m}{q}(z-x)^{p+q}{}_HP^{(\alpha)}_{n+m-p-q,\beta}(x,y;k,a,b). \tag{3.1}$$

Proof. We replace t by $t+u$ and rewrite the generating function (1.10) as

$$\left(\frac{2^{1-k}(t+u)^k}{\beta^b e^{(t+u)}-a^b}\right)^\alpha e^{y(t+u)^2}=e^{-x(t+u)}\sum_{k,l=0}^{\infty}{}_HP_{n+m,\beta}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.2}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$e^{(z-x)(t+u)}\sum_{n,m=0}^{\infty}{}_HP_{n+m}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!}=\sum_{n,m=0}^{\infty}{}_HP_{n+m,\beta}^{(\alpha)}(z,y;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{n,m=0}^{\infty}{}_HP_{n+m,\beta}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!}=\sum_{n,m=0}^{\infty}{}_HP_{n+m,\beta}^{(\alpha)}(z,y;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.4}$$

which on using formula [19,p.52(2)]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{3.5}$$

in the left hand side becomes

$$\sum_{p,q=0}^{\infty}\frac{(z-x)^{p+q}t^pu^q}{p!q!}\sum_{n,m=0}^{\infty}{}_HP_{n+m,\beta}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!}=\sum_{n,m=0}^{\infty}{}_HP_{n+m,\beta}^{(\alpha)}(z,y;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.6}$$

Now replacing n by $n-p$, m by $m-q$ and using the lemma [19,p.100(1)] in the left hand side of (3.6), we get

$$\sum_{n,m=0}^{\infty}\sum_{p,q=0}^{n,m}\frac{(z-x)^{p+q}}{p!q!}{}_HP_{n+m-p-q,\beta}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{(n-p)!}\frac{u^m}{(m-q)!}$$

$$=\sum_{n,m=0}^{\infty}{}_HP_{n+m,\beta}^{(\alpha)}(z,y;k,a,b)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.7}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the required result (3.1).

Remark 3.1. Taking $m=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formulae for generalized Hermite-based unified Apostol type polynomials ${}_HP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ holds true:

$${}_HP_{n,\beta}^{(\alpha)}(z,y;k,a,b)\ =\ \sum_{p=0}^{n}\binom{n}{p}(z-x)^p\,{}_HP_{n-p,\beta}^{(\alpha)}(x,y;k,a,b). \tag{3.8}$$

Remark 3.2. Replacing z by $z+x$ in (3.8), we obtain

$${}_HP_{n,\beta}^{(\alpha)}(z+x,y;k,a,b)\ =\ \sum_{p=0}^{n}\binom{n}{p}z^p\,{}_HP_{n-p,\beta}^{(\alpha)}(x,y;k,a,b). \tag{3.9}$$

Theorem 3.2. The following summation formulae for generalized Hermite-based unified Apostol type polynomials ${}_HP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ holds true:

$${}_HP_{n,\beta}^{(\alpha)}(x,y;k,a,b)=\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}P_{m,\beta}^{(\alpha)}(k,a,b)x^{n-2j-m}y^j\frac{n!}{m!j!(n-2j-m)!} \tag{3.10}$$

Proof. Applying the definition (1.10) to the term $\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^\alpha$ and expanding the exponential function e^{xt+yt^2} at $t=0$ yields

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^\alpha e^{xt+yt^2} = \left(\sum_{m=0}^{\infty} P_{m,\beta}^{(\alpha)}(k,a,b)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} y^j\frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{pmatrix} n \\ m\end{pmatrix} P_{m,\beta}^{(\alpha)}(k,a,b)x^{n-m}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty} y^j\frac{t^{2j}}{j!}\right)$$

Replacing n by $n-2j$, we have

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x,y;k,a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\begin{pmatrix} n-2j \\ m\end{pmatrix} P_{m,\beta}^{(\alpha)}(k,a,b)x^{n-m-2j}y^j\right)\frac{t^n}{(n-2j)!j!}. \tag{3.11}$$

Combining (3.11) and (1.10) and equating their coefficients of t^n produce the formula (3.10).

Theorem 3.3. The following summation formulae for generalized Hermite-based unified Apostol type polynomials ${}_HP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ holds true:

$${}_HP_{n,\beta}^{(\alpha)}(x+1,y;k,a,b) = \sum_{m=0}^{n}\begin{pmatrix} n \\ m\end{pmatrix} {}_HP_{m,\beta}^{(\alpha)}(x;k,a,b). \tag{3.12}$$

Proof. From (1.10), we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^\alpha e^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} {}_HP_{n,\beta}^{(\alpha)}(x+1,y;k,a,b)\frac{t^n}{n!}, \tag{3.13}$$

which can be written as

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^\alpha e^{xt+yt^2}e^t$$

$$= \left(\sum_{m=0}^{\infty} {}_HP_{m,\beta}^{(\alpha)}(x;k,a,b)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\begin{pmatrix} n \\ m\end{pmatrix} {}_HP_{m,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}. \tag{3.14}$$

Combining (3.13) and (3.14) and equating their coefficients of t^n leads to formula (3.12).

Theorem 3.4. The following summation formulae for generalized Hermite-based unified Apostol type polynomials ${}_HP_{n,\beta}^{(\alpha)}(x,y;k,a,b)$ holds true:

$${}_HP_{n,\beta}^{(\alpha)}(x,y;k,a,b) = \sum_{m=0}^{n}\begin{pmatrix} n \\ m\end{pmatrix} P_{n-m,\beta}^{(\alpha-1)}(k,a,b){}_HP_{m,\beta}^{(\alpha)}(x,y;k,a,b,e) \tag{3.15}$$

Proof. By the definition of generalized Hermite-based polynomials, we have

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^\alpha e^{xt+yt^2} = \left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)\sum_{n=0}^{\infty} {}_HP_n^{(\alpha)}(x,y;k,a,b)\frac{t^n}{n!}$$

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^\alpha e^{xt+yt^2} = \left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)\sum_{m=0}^{\infty} {}_HP_{m,\beta}^{(\alpha)}(x,y;k,a,b)\frac{t^m}{m!}.$$

Now replacing n by $n-m$ in the above equation and equating the coefficients of t^n leads to formula (3.15).

Theorem 3.5. The following summation formulae for generalized Hermite-based unified Apostol type polynomials ${}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)$ holds true:

$$ {}_HP^{(\alpha)}_{n,\beta}(x+1,y;k,a,b)=\sum_{m=0}^{n}\binom{n}{m}{}_HP^{(\alpha)}_{m,\beta}(x,y;k,a,b). \tag{3.16}$$

Proof. By the definition of generalized polynomials, we have

$$\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x+1,y;k,a,b)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}$$

$$=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{xt+yt^2}(e^t-1)$$

$$=\left(\sum_{m=0}^{\infty}{}_HP^{(\alpha)}_{m,\beta}(x,y;k,a,b)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)-\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}_HP^{(\alpha)}_{m,\beta}(x,y;k,a,b)\frac{t^n}{(n-m)!}-\sum_{n=0}^{\infty}{}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.16).

4 Some symmetry identities for Hermite-based unified generalized Apostol-type polynomials

In this section, we give general symmetry identities for the generalized unified Apostol-type polynomials $P^{(\alpha)}_{n,\beta}(x;k,a,b)$ and Hermite-based generalized unified Apostol type polynomials ${}_HP^{(\alpha)}_{n,\beta}(x,y;k,a,b)$ by applying the generating function (1.3) and (1.10). These results extend some known summation and identities studied by Ozarslan [12], Pathan and Khan [15, 16], Khan [5], Khan and Hiba [6], Kurt [7, 8]. Also we prove the relation between for the unified Apostol-type polynomials and Hurwitz-Lerch zeta function.

Theorem 4.1. The following symmetry relations for the Hermite-based unified Apostol-type polynomials hold true;

$$\sum_{m=0}^{c-1}\left(\frac{\beta}{a}\right)^{bm}\sum_{l=0}^{n}\binom{n}{l}{}_HP_{n-l,\beta}(dx,d^2z;k,a,b)c^{n-k-l}(dm)^l$$

$$=\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}\sum_{l=0}^{n}\binom{n}{l}{}_HP_{n-l,\beta}(cx,c^2z;k,a,b)d^{n-k-l}(cm)^l. \tag{4.1}$$

Proof. Let us consider

$$f(t)=\frac{2^{1-k}t^k}{\beta^b e^{dt}-a^b}e^{cdxt+c^2d^2yt^2}\frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^b e^{ct}-a^b} \tag{4.2}$$

$$=\frac{1}{d^k}\left(\frac{2^{1-k}t^k}{\beta^b e^{dt}-a^b}\right)e^{cdxt+c^2d^2yt^2}a^{b(d-1)}\left(\frac{1-(\frac{\beta}{a})^{bd}e^{dct}}{1-(\frac{\beta}{a})^{b}e^{ct}}\right)$$

$$=a^{b(d-1)}d^{(-k)}\sum_{n=0}^{\infty}{}_HP_{n,\beta}(cx,c^2z;k,a,b)\frac{(dt)^n}{n!}\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}e^{ctm}$$

$$= a^{b(d-1)}d^{(-k)}\sum_{n=0}^{\infty}{}_HP_{n,\beta}(cx,c^2z;k,a,b)\frac{(dt)^n}{n!}\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}\sum_{l=0}^{\infty}(cm)^l\frac{t^l}{l!}$$

$$f(t) = a^{b(d-1)}d^{(-k)}\sum_{n=0}^{\infty}\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}\left(\sum_{l=0}^{n}\binom{n}{l}{}_HP_{n-l,\beta}(cx,c^2z;k,a,b)d^{n-l}(cm)^l\right)\frac{t^n}{n!}. \tag{4.3}$$

On the similar lines, we can show that

$$f(t) = a^{b(d-1)}c^{(-k)}\sum_{n=0}^{\infty}\sum_{m=0}^{c-1}\left(\frac{\beta}{a}\right)^{bm}\left(\sum_{l=0}^{n}\binom{n}{l}{}_HP_{n-l,\beta}(dx,d^2z;k,a,b)c^{n-l}(dm)^l\right)\frac{t^n}{n!}. \tag{4.4}$$

On comparing the coefficients of $\frac{t^n}{n!}$ in (4.3) and (4.4), we arrive at the desired result (4.1).

Theorem 4.2. The Hermite-based unified Apostol-type polynomials satisfy the following relation:

$$c^k\sum_{l=0}^{n}\binom{n}{l}{}_HP_{n-l,\beta}(c^2y,c^2z;k,a,b)d^{n-l}c^l\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}(m+dx)^l$$

$$= d^k\sum_{l=0}^{n}\binom{n}{l}{}_HP_{n-l,\beta}(d^2y,d^2z;k,a,b)c^{n-l}d^l\sum_{m=0}^{c-1}\left(\frac{\beta}{a}\right)^{bm}(m+cx)^l. \tag{4.5}$$

Proof. Let us consider

$$f(t) = \frac{2^{1-k}t^k}{\beta^b e^{dt}-a^b}e^{cd(x+y)t+c^2d^2zt^2}\frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^b e^{ct}-a^b}$$

$$= \frac{a^{b(d-1)}}{d^k}\left(\frac{2^{1-k}(dt)^k}{\beta^b e^{dt}-a^b}\right)\left(\frac{1-(\frac{\beta}{a})^{bd}e^{cdt}}{1-(\frac{\beta}{a})^b e^{ct}}\right)e^{cd(x+y)t+c^2d^2zt^2}$$

$$= \frac{a^{b(d-1)}}{d^k}\sum_{n=0}^{\infty}{}_HP_{n,\beta}(c^2y,c^2z;k,a,b)\frac{d^nt^n}{n!}\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}e^{(m+dx)ct}$$

$$f(t) = \frac{a^{b(d-1)}}{d^k}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\left({}_HP_{n-l,\beta}(c^2y,c^2z;k,a,b)d^{n-l}c^l\sum_{m=0}^{d-1}\left(\frac{\beta}{a}\right)^{bm}(m+dx)^l\right)\frac{t^n}{n!}. \tag{4.6}$$

In a similar manner

$$f(t) = \frac{2^{1-k}t^k}{\beta^b e^{ct}-a^b}e^{cd(x+y)t+c^2d^2zt^2}\frac{\beta^{bd}e^{cdt}-a^{bd}}{\beta^b e^{dt}-a^b}$$

$$f(t) = \frac{a^{b(d-1)}}{d^k}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{s}\left({}_HP_{n-l,\beta}(d^2y,d^2z;k,a,b)c^{n-l}d^l\sum_{m=0}^{c-1}\left(\frac{\beta}{a}\right)^{bm}(m+cx)^l\right)\frac{t^n}{n!}. \tag{4.7}$$

By comparing the coefficients of $\frac{t^n}{n!}$ in (4.6) and (4.7), we get the result (4.5).

Theorem 4.3. For all $c,d,r\in\mathbb{N}$, $s,p\in\mathbb{N}_0$, we have the following symmetry relation between Hurwitz-Lerch zeta function and Hermite-based unified Apostol type polynomials:

$$\sum_{p=0}^{n-k\alpha}\binom{n-k\alpha}{p}d^{n-k\alpha-p}\sum_{s=0}^{p}\binom{p}{s}\sum_{s=0}^{r}\binom{r}{s}(-\alpha)^{r-s}S_s^{(\alpha)}\left(d,\left(\frac{\beta}{a}\right)^{\alpha}\right)c^p{}_HP_{p-r,\beta}\left(dy,d^2z;k,a,b\right)c^p$$

$$\times\Phi_a^*\left[\left(\frac{\beta}{a}\right)^b,p+k\alpha-n,cx\right]$$

$$= \sum_{p=0}^{n-k\alpha} \binom{n-k\alpha}{p} d^{n-k\alpha-p} \sum_{s=0}^{p} \binom{p}{s} \sum_{s=0}^{r} \binom{r}{s} (-\alpha)^{r-s} S_s^{(\alpha)} \left(c, \left(\frac{\beta}{a}\right)^b\right) d^p {}_HP_{p-r,\beta}\left(cx, c^2z; k, a, b\right)$$
$$\times \Phi_a^* \left(\left(\frac{\beta}{a}\right)^b, p+k\alpha-n, dy\right). \tag{4.8}$$

Proof. We now use

$$g(t) = \frac{(2^{(1-k)(\alpha+1)} t^{k(\alpha+1)})^{\alpha} (\beta^{bd} e^{cdt} - a^{bd}) e^{cd(x+y)t + c^2 d^2 z t^2}}{(\beta^b e^{ct} - a^b)^{\alpha+1} (\beta^b e^{dt} - a^b)^{\alpha+1}}$$

$$g(t) = \frac{t^{k\alpha} 2^{(1-k)\alpha} e^{cdxt}}{(\beta^b e^{dt} - a^b)^{\alpha+1}} \left(\frac{\beta^{bd} e^{cdt} - a^{bd}}{\beta^b e^{ct} - a^b}\right)^{\alpha} \frac{(ct)^k 2^{1-k}}{\beta^b e^{ct} - a^b} e^{cdyt + c^2 d^2 z t^2}$$

$$= \frac{2^{(1-k)\alpha} a^{bd\alpha - b\alpha - b} \beta^{-\alpha b} t^{k\alpha}}{c^k (-1)^{\alpha+1}} \sum_{m=0}^{\infty} \binom{m+\alpha}{m} \left(\frac{\beta^b}{a^b}\right)^m e^{dt(m+cx)} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} (-\alpha)^{n-s}$$
$$\times S_s^{(\alpha)} \left(d, \left(\frac{\beta}{a}\right)^b\right) c^r \frac{t^r}{r!} \sum_{n=0}^{\infty} {}_HP_{p-r,\beta}(dy, d^2z; k, a, b) c^n \frac{t^n}{n!}$$

$$= \sum_{n=k\alpha}^{\infty} \frac{n!}{(n-k\alpha)!} \frac{2^{(1-k)\alpha} a^{b(d\alpha-\alpha-1)} \beta^{-\alpha b}}{c^k (-1)^{\alpha+1}} \sum_{p=0}^{n-k\alpha} \binom{n-k\alpha}{p} d^{n-k\alpha-p} \sum_{s=0}^{p} \binom{p}{s}$$

$$\times \sum_{s=0}^{r} \binom{r}{s} (-\alpha)^{r-s} S_s^{(\alpha)} \left(d, \left(\frac{\beta}{a}\right)^{\alpha}\right) c^p {}_HP_{p-r,\beta}\left(dy, d^2z; k, a, b\right) c^p \Phi_a^* \left[\left(\frac{\beta}{a}\right)^b, p+k\alpha-n, cx\right] \frac{t^n}{n!}. \tag{4.9}$$

Using a similar plan, we get

$$g(t) = \frac{(2^{(1-k)(\alpha+1)} t^{k(\alpha+1)})^{\alpha} (\beta^{bd} e^{cdt} - a^{bd}) e^{cd(x+y)t + c^2 d^2 z t^2}}{(\beta^b e^{ct} - a^b)^{\alpha+1} (\beta^b e^{dt} - a^b)^{\alpha+1}}$$

$$g(t) = \sum_{n=k\alpha}^{\infty} \frac{n!}{(n-k\alpha)!} 2^{(1-k)\alpha} a^{b(d\alpha-\alpha-1)} \beta^{-\alpha b} \sum_{p=0}^{n-k\alpha} \binom{n-k\alpha}{p} d^{n-k\alpha-p} \sum_{s=0}^{p} \binom{p}{s}$$

$$\times \sum_{s=0}^{r} \binom{r}{s} (-\alpha)^{r-s} S_s^{(\alpha)} \left(c, \left(\frac{\beta}{a}\right)^b\right) d^p {}_HP_{p-r,\beta}\left(cx, c^2z; k, a, b\right) \Phi_a^* \left(\left(\frac{\beta}{a}\right)^b, p+k\alpha-n, dy\right) \frac{t^n}{n!}. \tag{4.10}$$

On comparing the coefficients of $\frac{t^n}{n!}$ in (4.9) and (4.10), we arrive at the desired result (4.8).

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