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Bicomplex two-parameter Mittag-Leffler function and properties with application to the fractional time wave equation

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Abstract. The aim of this paper is to define the bicomplex two-parameter Mittag-Leffler function, its region of convergence, and analyticity. Various properties, including recurrence relations, duplication formula, differential, and integral relations are established. Several interesting special cases of the bicomplex two-parameter Mittag -Leffler function have also been developed. Further, the bicomplex Laplace transform of two-parameter bicomplex Mittag-Leffler function has been evaluated. The bicomplex solution of the electromagnetic fractional time wave equation has been obtained for vacuum via the bicomplex Mittag-Leffler function.

1 Introduction

Bicomplex numbers are being studied for quite a long time, and a lot of work has been done in this area. Cockle [11, 12] introduced tessarines between 1848 and 1850, following which Segre [48] introduced bicomplex points as a natural completion of the complex projective straight line. During the last few years, researchers have aimed to establish different algebraic and geometric properties of bicomplex numbers and their applications (see, e.g., [9, 32, 43, 44, 46, 47]). In the recent developments, efforts have been done to extend the integral transforms [1, 2], holomorphic and meromorphic functions [9, 10], a number of bicomplex functions: like, Polygamma function [17], Hurwitz Zeta function [18], Gamma and Beta functions [19], Riemann Zeta function [43], bicomplex analysis and Hilbert space [25, 26, 27, 28, 29, 30] in the bicomplex variable from their complex counterpart.

Segre [48] defined the set of bicomplex numbers as:

Definition 1.1 (Bicomplex Number). In terms of real components, the set of bicomplex numbers is defined as

$$\mathbb{T} = \{\xi : \xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3 \mid x_0,\ x_1,\ x_2,\ x_3\ \in \mathbb{R}\}, \tag{1.1}$$

and, in terms of complex numbers, it can be written as

$$\mathbb{T} = \{\xi : \xi = z_1 + i_2 z_2 \mid z_1,\ z_2\ \in \mathbb{C}\}, \tag{1.2}$$

where $i_1,\ i_2$ and j are the imaginary units such that $i_1^2 = i_2^2 = -1$, $i_1 i_2 = i_2 i_1 = j$, $j^2 = 1$.

We shall use the notations, $x_0 = \text{Re}(\xi),\ x_1 = \text{Im}_{i_1}(\xi),\ x_2 = \text{Im}_{i_2}(\xi),\ x_3 = \text{Im}_j(\xi)$.

Segre studied the presence of zero divisors, which he called *nullifics*. He determined that the zero-divisors in bicomplex space constitute two ideals called infinite nullifics. The set

$$\mathbb{NC} = \mathbb{O}_2 = \{z_1 + z_2 i_2 \mid z_1^2 + z_2^2 = 0\} \tag{1.3}$$

of all the zero divisors of $\mathbb{T}$, is called the null-cone [45].

Two non-trivial idempotent elements in $\mathbb{T}$, denoted by e_1 and e_2 are defined as follows [40]: $e_1 = \dfrac{1+i_1 i_2}{2} = \dfrac{1+j}{2},\ e_2 = \dfrac{1-i_1 i_2}{2} = \dfrac{1-j}{2},\ e_1 + e_2 = 1,\ e_1.e_2 = 0$ and $e_1^2 = e_1,\ e_2^2 = e_2$.

Definition 1.2 (Idempotent Representation). Every element of $\mathbb{T}$ has a unique idempotent representation, defined by

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 = \xi_1 e_1 + \xi_2 e_2, \tag{1.4}$$

where $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$.

Projection mappings $P_1 : \mathbb{T} \to T_1 \subseteq \mathbb{C}$, $P_2 : \mathbb{T} \to T_2 \subseteq \mathbb{C}$ for a bicomplex number $\xi = z_1 + i_2 z_2$ are defined as [44]:

$$P_1(\xi) = P_1(z_1 + i_2 z_2) = P_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in T_1, \tag{1.5}$$

and

$$P_2(\xi) = P_2(z_1 + i_2 z_2) = P_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in T_2, \tag{1.6}$$

where

$$T_1 = \{\xi_1 = z_1 - i_1 z_2 \,|z_1, z_2 \in \mathbb{C}\} \text{ and } T_2 = \{\xi_2 = z_1 + i_1 z_2 \,|z_1, z_2 \in \mathbb{C}\}. \tag{1.7}$$

Let U be an open set, and $f : U \subseteq \mathbb{T} \to \mathbb{T}$ (see, e.g., [43, 47]) and $f(z_1 + i_2 z_2) = f_1(z_1, z_2) + i_2 f_2(z_1, z_2)$. Then f is $\mathbb{T}$-holomorphic iff f_1 and f_2 are holomorphic in U and

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad \text{on } U. \tag{1.8}$$

These equations in (1.8) are called the bicomplex Cauchy-Riemann equations.
Also,

$$f' = \frac{\partial f_1}{\partial z_1} + i_2 \frac{\partial f_2}{\partial z_1}. \tag{1.9}$$

In the Theorem 1.3, Riley [41] studied about the convergence of the bicomplex power series.

Theorem 1.3. *Let*

$$N(\xi) = \sqrt{\|\xi\|^2 + \sqrt{\|\xi\|^4 - |\xi|^4_{abs}}} = \max(|\xi_1|, |\xi_2|), \tag{1.10}$$

then $N(\xi)$ is a norm and if $\sum_{n=0}^{\infty} a_n \xi^n$ is a power series with component series $\sum_{n=0}^{\infty} b_n \xi_1^n$ and $\sum_{n=0}^{\infty} c_n \xi_2^n$, $a_n = b_n e_1 + c_n e_2$ both have same radius of convergence $R > 0$, then $\sum_{n=0}^{\infty} a_n \xi^n$ converges for $N(\xi) < R$ and diverges for $N(\xi) > R$. Here, $\|\xi\| = \frac{1}{\sqrt{2}}\sqrt{|\xi_1|^2 + |\xi_2|^2}$ and $|\xi|_{abs} = \sqrt{|\xi_1||\xi_2|}$, $\xi_1, \ \xi_2 \in \mathbb{C}$.

In the following theorem, Ringleb discussed the analyticity of a bicomplex function concerning its idempotent complex component functions (see, e.g., [41]). This theorem plays a crucial role in discussing the bicomplex functions' convergence.

Theorem 1.4 (Decomposition theorem of Ringleb [42]). *Let $f(\xi)$ be analytic in a region $U \subseteq \mathbb{T}$, and let $T_1 \subseteq \mathbb{C}$ and $T_2 \subseteq \mathbb{C}$ be the component regions of $\mathbb{T}$, in the ξ_1 and ξ_2 planes, respectively. Then there exists a unique pair of complex-valued analytic functions, $f_1(\xi_1)$ and $f_2(\xi_2)$, defined in $U_1 \subseteq T_1$ and $U_2 \subseteq T_2$, respectively, such that*

$$f(\xi) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2, \ \xi \in U. \tag{1.11}$$

Conversely, if $f_1(\xi_1)$ is any complex-valued analytic function in a region T_1 and $f_2(\xi_2)$ any complex-valued analytic function in a region T_2 then the bicomplex-valued function $f(\xi)$, defined by the equation (1.11), is an analytic function of the bicomplex variable ξ in the product-region $U = U_1 \times_e U_2$.

Definition 1.5 (j-Modulus). The j-modulus of $\xi \in \mathbb{T}$ is given by (see, e.g. [44])

$$|\xi|_j = |z_1 - i_1 z_2|e_1 + |z_1 + i_1 z_2|e_2 \in \mathbb{D}. \tag{1.12}$$

Definition 1.6 (Argument). The hyperbolic argument of $\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{T}$ is given by (see, e.g., [31])

$$\arg_j(\xi) = \arg(\xi_1)e_1 + \arg(\xi_1)e_2. \tag{1.13}$$

Definition 1.7. The idempotent representation of a hyperbolic number $w = x_1 + x_2 j \in \mathbb{D}$ can be represented as (see, e.g.[33])

$$w = w_1 e_1 + w_2 e_2, \quad w_1 = x_1 + x_2,\ w_2 = x_1 - x_2. \tag{1.14}$$

If $w_1, w_2 \geq 0$ then w is said to be a non-negative hyperbolic number and this set is represented by $\mathbb{D}^+$. When $w_1, w_2 > 0$ hyperbolic number w is said to be positive hyperbolic number. This concept enables the following partial order on hyperbolic numbers to be defined [33].

Definition 1.8. Let $u, v \in \mathbb{D}$ then partial order on $\mathbb{D}$ is defined as [33]

$$u \preccurlyeq v \text{ if } v - u \in \mathbb{D}^+ \text{ and } u \prec v \text{ if } v - u \in \mathbb{D}^+/\{0\}. \tag{1.15}$$

Definition 1.9 (Bicomplex Laplace Transform [24]). Let $f(x)$ be a bicomplex valued function of exponential order $K \in \mathbb{R}$. Then Laplace transform of $f(x)$ for $x \geq 0$, is defined as :

$$L[f(x);\xi] = \hat{f}(\xi) = \int_0^\infty f(x)e^{-\xi x}dx, \quad \xi = a_0 + i_1 a_1 + i_2 a_2 + j a_3 \in \mathbb{T}. \tag{1.16}$$

Here, $\hat{f}(\xi)$ exist and is convergent for all $\xi \in D$ where

$$D = \{\xi : H_\rho(\xi) \text{ represent a right half plane : } a_0 > K + |a_3|\}. \tag{1.17}$$

Definition 1.10 (Bicomplex Fourier Transform, [8]). Let $f(t)$ be a continuous function in $(-\infty, \infty)$ that satisfies the following estimates:

$$|f(t)| \leq C_1 \exp(-\sigma_1 t),\ t \geq 0, C_1 > 0,\ \sigma_1 > 0, \tag{1.18}$$

and

$$|f(t)| \leq C_2 \exp(-\sigma_2 t),\ t \leq 0, C_2 > 0,\ \sigma_2 > 0. \tag{1.19}$$

The Fourier transform of $f(t)$ is defined as

$$\bar{f}(\xi) = F(f(t)) = \int_{-\infty}^{\infty} \exp(i_1 \xi t) f(t) dt,\ \ \xi \in \mathbb{T}. \tag{1.20}$$

The Fourier transform $\bar{f}(\xi)$ exists and holomorphic for all $\xi = a_0 + a_1 i_1 + a_2 i_2 + a_3 i_1 i_2 \in \Omega$ where Ω is given by

$$\Omega = \left\{\xi \in \mathbb{T} : -\infty < a_0,\ \ a_3 < \infty;\ \ -\sigma_1 + |a_2| < a_1 < \sigma_2 - |a_2|;\ 0 \leq |a_2| < \frac{\sigma_1 + \sigma_2}{2}\right\}. \tag{1.21}$$

We would require the definition of the bicomplex gamma function defined by Goyal et al. [19]. In the Euler product form, the bicomplex gamma function is given by

$$\frac{1}{\Gamma\xi} = \xi e^{\gamma\xi} \prod_{n=1}^{\infty} \left(\left(1 + \frac{\xi}{n}\right) \exp\left(-\frac{\xi}{n}\right)\right), \tag{1.22}$$

where $\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 = \xi_1 e_1 + \xi_2 e_2$, provided that $z_1 \neq \frac{-(m+l)}{2}$, and $z_2 \neq i_1(\frac{l-m}{2})$ where $m,\ l \in \mathbb{N} \cup \{0\}$ and the Euler constant $\gamma(0 \leq \gamma \leq 1)$ is given by

$$\gamma = \lim_{n\to\infty}(H_n - \log n),\ \ H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{1.23}$$

Also,

$$\Gamma\xi = \Gamma\xi_1 e_1 + \Gamma\xi_2 e_2, \tag{1.24}$$

and in the integral form [19], for $p = p_1e_1 + p_2e_2,\ p_1, p_2 \in \mathbb{R}^+$,

$$\Gamma\xi = \int_\Gamma e^{-p}p^{\xi-1}dp = \left(\int_0^\infty e^{-p_1}p_1^{\xi_1-1}dp_1\right)e_1 + \left(\int_0^\infty e^{-p_2}p_2^{\xi_2-1}dp_2\right)e_2, \tag{1.25}$$

where $\Gamma = (\gamma_1, \gamma_2)$ and $\gamma_1 : 0$ to $\infty,\ \gamma_2 : 0$ to ∞.

The Mittag-Leffler (ML) function arrives intrinsically in the fractional analysis and fractional modeling. Recently many researchers have worked on various generalizations and the extensions of the Mittag- Leffler function [6, 7, 13, 15, 22]. Efforts have been made by authors to introduce the Mittag-Leffler function in bicomplex space along with its applications in fractional calculus and integral transform [4, 5]. The importance of ML function in applied science and engineering is continuously increasing. It is very useful in the area of fractional order differential and integral equations.

A two-parameter ML function is defined by Wiman [49] (see, also [22]) as follows:

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ z,\ \alpha, \beta \in \mathbb{C}. \tag{1.26}$$

For $\beta = 1$, equation (1.26) reduces to the one parameter (classical) ML function [38, 39] defined as

$$\mathbb{E}_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad \mathrm{Re}(\alpha) > 0,\ \ z \in \mathbb{C}. \tag{1.27}$$

For each $\mathrm{Re}(\alpha) > 0$, the ML function (1.26) has infinite radius of convergence and it is an entire function (see, e.g., [16]) of order ρ and type σ where

$$\rho = \lim\sup_{n\to\infty}\frac{n\log n}{\log\frac{1}{|a_n|}} = \frac{1}{\mathrm{Re}(\alpha)}, \tag{1.28}$$

and

$$\sigma = \frac{1}{e\rho}\lim\sup_{n\to\infty}(n|a_n|^{\frac{\rho}{n}}) = 1, \tag{1.29}$$

for the coefficients $a_n = \frac{1}{\Gamma(\alpha k+\beta)}$.

2 Bicomplex two-parameter Mittag-Leffler function

Here, we introduce the bicomplex two-parameter Mittag-Leffler function as

$$\mathbb{E}_{\alpha,\beta}(\xi) = \sum_{k=0}^{\infty}\frac{\xi^k}{\Gamma(\alpha k+\beta)}, \tag{2.1}$$

where $\xi, \alpha, \beta \in \mathbb{T},\ \xi = z_1 + i_2z_2$ and $|\,\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha),\ |\,\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta)$.
For $\beta = 1$, it reduces to the bicomplex one parameter ML function defined by Agarwal et al. [4]

$$\mathbb{E}_{\alpha}(\xi) = \sum_{k=0}^{\infty}\frac{\xi^k}{\Gamma(\alpha k+1)}, \tag{2.2}$$

for $|\,\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha)$.

Above definition of bicomplex two-parameter ML function is well justified by the following theorem:

Theorem 2.1. *Let* $\xi, \alpha, \beta \in \mathbb{T}, \xi = z_1 + i_2z_2 = \xi_1e_1 + \xi_2e_2,\ \alpha = \alpha_1e_1 + \alpha_2e_2,\ \beta = \beta_1e_1 + \beta_2e_2$. *Then*

$$\mathbb{E}_{\alpha,\beta}(\xi) = \sum_{k=0}^{\infty}\frac{\xi^k}{\Gamma(\alpha k+\beta)}, \tag{2.3}$$

is convergent for $|\,\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha),\ |\,\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta)$.

Proof. Consider the series

$$\sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k + \beta)}. \tag{2.4}$$

By using the idempotent representation for (2.4)

$$\sum_{k=0}^{\infty} \frac{\xi_1^k}{\Gamma(\alpha_1 k + \beta_1)} e_1 + \sum_{k=0}^{\infty} \frac{\xi_2^k}{\Gamma(\alpha_2 k + \beta_2)} e_2 = \mathbb{E}_{\alpha_1,\beta_1}(\xi_1)e_1 + \mathbb{E}_{\alpha_2,\beta_2}(\xi_2)e_2, \tag{2.5}$$

where $\xi = \xi_1 e_1 + \xi_2 e_2$, $\alpha = \alpha_1 e_1 + \alpha_2 e_2$ and $\beta = \beta_1 e_1 + \beta_2 e_2$.

Now, $\mathbb{E}_{\alpha_1,\beta_1}(\xi_1)$ and $\mathbb{E}_{\alpha_2,\beta_2}(\xi_2)$ are complex two-parameter ML function convergent for $\mathrm{Re}(\alpha_1) > 0$, $\mathrm{Re}(\beta_1) > 0$ and $\mathrm{Re}(\alpha_2) > 0$, $\mathrm{Re}(\beta_2) > 0$, respectively. Since $\mathbb{E}_{\alpha_1,\beta_1}(\xi_1)$ and $\mathbb{E}_{\alpha_2,\beta_2}(\xi_2)$ are convergent in T_1 and T_2, respectively, by Ringleb decomposition Theorem 1.4, (2.4) is also convergent in $\mathbb{T}$ and is denoted by $\mathbb{E}_{\alpha,\beta}(\xi)$.

Further, let

$$\begin{aligned} \alpha &= p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3 \\ &= (p_0 + i_1 p_1) + i_2(p_2 + i_1 p_3) \\ &= \alpha_1 e_1 + \alpha_2 e_2, \end{aligned} \tag{2.6}$$

where $\alpha_1 = (p_0 + p_3) + i_1(p_1 - p_2)$ and $\alpha_2 = (p_0 - p_3) + i_1(p_1 + p_2)$.

Since $\mathrm{Re}(\alpha_1) > 0$ and $\mathrm{Re}(\alpha_2) > 0$,

$$\begin{aligned} &\Rightarrow \quad p_0 + p_3 > 0 \text{ and } p_0 - p_3 > 0 \\ &\Rightarrow \quad |p_3| < p_0 \\ &\Rightarrow \quad |\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha). \end{aligned} \tag{2.7}$$

Similarly, let

$$\begin{aligned} \beta &= q_0 + i_1 q_1 + i_2 q_2 + i_1 i_2 q_3 \\ &= (q_0 + i_1 q_1) + i_2(q_2 + i_1 q_3) \\ &= \beta_1 e_1 + \beta_2 e_2, \end{aligned} \tag{2.8}$$

where $\beta_1 = (q_0 + q_3) + i_1(q_1 - q_2)$ and $\beta_2 = (q_0 - q_3) + i_1(q_1 + q_2)$.

Since, $\mathrm{Re}(\beta_1) > 0$ and $\mathrm{Re}(\beta_2) > 0$,

$$\begin{aligned} &\Rightarrow \quad q_0 + q_3 > 0 \text{ and } q_0 - q_3 > 0 \\ &\Rightarrow \quad |q_3| < q_0 \\ &\Rightarrow \quad |\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta). \end{aligned} \tag{2.9}$$

This completes the proof. □

By substituting the value of the bicomplex gamma function defined by equation (1.22) in the equation (2.5), we get the following representation for the Mittag-Leffler function:

Theorem 2.2. *Let* $\xi, \alpha, \beta \in \mathbb{T}$ *where* $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$, $\alpha = \alpha_1 e_1 + \alpha_2 e_2$, $\beta = \beta_1 e_1 + \beta_2 e_2$ *with* $|\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha)$, $|\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta)$. *Then*

$$\mathbb{E}_{\alpha,\beta}(\xi) = \sum_{k=0}^{\infty} \xi^k (\alpha k + \beta) e^{\gamma(\alpha k + \beta)} \prod_{n=1}^{\infty} \left(\left(1 + \frac{(\alpha k + \beta)}{n}\right) \exp\left(-\frac{(\alpha k + \beta)}{n}\right) \right). \tag{2.10}$$

Remark 2.3. Also, using the integral form of the gamma function, the bicomplex two-parameter ML function can be represented as

$$\mathbb{E}_{\alpha,\beta}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\int_{\Gamma} e^{-p} p^{\alpha k + \beta - 1} dp}, \tag{2.11}$$

where $\Gamma = (\gamma_1, \gamma_2)$ is as defined in (1.22).

Theorem 2.4. *The bicomplex two-parameter ML function satisfies the bicomplex Cauchy - Riemann equations in* $\mathbb{T}$.

Proof. By the result (2.5), we have

$$\begin{aligned}
\mathbb{E}_{\alpha,\beta}(\xi) &= \mathbb{E}_{\alpha_1,\beta_1}(\xi_1)e_1 + \mathbb{E}_{\alpha_2,\beta_2}(\xi_2)e_2 \\
&= \mathbb{E}_{\alpha_1,\beta_1}(z_1 - i_1z_2)e_1 + \mathbb{E}_{\alpha_2,\beta_2}(z_1 + i_1z_2)e_2 \\
&= \mathbb{E}_{\alpha_1,\beta_1}(z_1 - i_1z_2)\left(\frac{1+i_1i_2}{2}\right) + \mathbb{E}_{\alpha_2,\beta_2}(z_1 + i_1z_2)\left(\frac{1-i_1i_2}{2}\right) \\
&= \left(\frac{1}{2}\left(\mathbb{E}_{\alpha_1,\beta_1}(z_1 - i_1z_2) + \mathbb{E}_{\alpha_2,\beta_2}(z_1 + i_1z_2)\right)\right) \\
&\quad + i_2\left(\frac{i_1}{2}\left(\mathbb{E}_{\alpha_1,\beta_1}(z_1 - i_1z_2) - \mathbb{E}_{\alpha_2,\beta_2}(z_1 + i_1z_2)\right)\right) \\
&= f_1(z_1,z_2) + i_2f_2(z_1,z_2),
\end{aligned} \tag{2.12}$$

where,

$$f_1(z_1,z_2) = \frac{1}{2}\left(\mathbb{E}_{\alpha_1,\beta_1}(z_1 - i_1z_2) + \mathbb{E}_{\alpha_2,\beta_2}(z_1 + i_1z_2)\right),$$
$$f_2(z_1,z_2) = \frac{i_1}{2}\left(\mathbb{E}_{\alpha_1,\beta_1}(z_1 - i_1z_2) - \mathbb{E}_{\alpha_2,\beta_2}(z_1 + i_1z_2)\right).$$

Now,

$$\begin{aligned}
\frac{\partial f_1}{\partial z_1} &= \frac{1}{2}\left(\mathbb{E}'_{\alpha_1,\beta_1}(z_1 - i_1z_2) + \mathbb{E}'_{\alpha_2,\beta_2}(z_1 + i_1z_2)\right), \\
\frac{\partial f_1}{\partial z_2} &= \frac{-i_1}{2}\left(\mathbb{E}'_{\alpha_1,\beta_1}(z_1 - i_1z_2) - \mathbb{E}'_{\alpha_2,\beta_2}(z_1 + i_1z_2)\right), \\
\frac{\partial f_2}{\partial z_1} &= \frac{i_1}{2}\left(\mathbb{E}'_{\alpha_1,\beta_1}(z_1 - i_1z_2) - \mathbb{E}'_{\alpha_2,\beta_2}(z_1 + i_1z_2)\right), \\
\frac{\partial f_2}{\partial z_2} &= \frac{1}{2}\left(\mathbb{E}'_{\alpha_1,\beta_1}(z_1 - i_1z_2) + \mathbb{E}'_{\alpha_2,\beta_2}(z_1 + i_1z_2)\right).
\end{aligned}$$

From the above equations, it can be easily shown that

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}. \tag{2.13}$$

Hence, the bicomplex Cauchy-Riemann equations are satisfied by the bicomplex two-parameter ML function in $\mathbb{T}$. □

Theorem 2.5. *The bicomplex ML function* $\mathbb{E}_{\alpha,\beta}(\xi)$, $|\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha)$ *is an entire function in the bicomplex domain.*

Proof. Let $\sum_{n=0}^{\infty} a_n\xi^n$ represents a bicomplex power series where $a_n,\ \xi \in \mathbb{T},\ a_n = b_ne_1 + c_ne_2$, $\xi = \xi_1e_1 + \xi_2e_2$. Then by Ringleb decomposition Theorem 1.4, the series

$$\sum_{n=0}^{\infty} a_n\xi^n = \left(\sum_{n=0}^{\infty} b_n\xi_1^n\right)e_1 + \left(\sum_{n=0}^{\infty} c_n\xi_2^n\right)e_2, \tag{2.14}$$

converges iff $\sum_{n=0}^{\infty} b_n\xi_1^n$ and $\sum_{n=0}^{\infty} c_n\xi_2^n$ converge in the complex domains (see, e.g., [41]). Now, from equation (2.5), the idempotent components $\mathbb{E}_{\alpha_1,\beta_1}(\xi_1)$ and $\mathbb{E}_{\alpha_2,\beta_2}(\xi_2)$ are complex ML functions with infinite radius of convergence (say R) for $\mathrm{Re}(\alpha_1) > 0,\ \ \mathrm{Re}(\alpha_2) > 0$, respectively [15, p.18]. Thus, $\mathbb{E}_{\alpha_1,\beta_1}(\xi_1)$ and $\mathbb{E}_{\alpha_2,\beta_2}(\xi_2)$ are convergent respectively for

$$|\xi_1| < R,\ |\xi_2| < R. \tag{2.15}$$

Equation (1.10), thus implies

$$N(\xi) = \sqrt{\|\xi\|^2 + \sqrt{\|\xi\|^4 - |\xi|^4_{abs}}} = \max(|\xi_1|, |\xi_2|) < R. \tag{2.16}$$

Hence by Theorem 1.3, $\mathbb{E}_{\alpha,\beta}(\xi)$ converges in the bicomplex domain and has infinite radius of convergence [41]. As ML function is entire function in the complex domain, the bicomplex ML function is also an entire function (Riley [41, p.141]). □

Theorem 2.6 (Order and Type). *The bicomplex ML function* $\mathbb{E}_{\alpha,\beta}(\xi),\ \xi,\ \alpha,\ \beta \in \mathbb{T},\ \alpha = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3,\ |\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha),\ |\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta)$, *is an entire function of order* $\rho = \dfrac{p_0 - p_3 j}{(p_0^2 - p_3^2)}$ *and type* $\sigma = 1$.

Proof. From equation (2.7), the order is given by

$$\begin{aligned}
\rho &= \lim_{n\to\infty}\sup \frac{n\log n}{\log\frac{1}{|a_n|}} \\
&= \lim_{n\to\infty}\sup \frac{n\log n}{\log\Gamma(\alpha n+\beta)} \\
&= \left(\lim_{n\to\infty}\sup \frac{n\log n}{\log\Gamma(\alpha_1 n+\beta_1)}\right) e_1 + \left(\lim_{n\to\infty}\sup \frac{n\log n}{\log\Gamma(\alpha_2 n+\beta_2)}\right) e_2 \\
&= \left(\frac{1}{\mathrm{Re}(\alpha_1)}\right) e_1 + \left(\frac{1}{\mathrm{Re}(\alpha_2)}\right) e_2 \qquad \text{[using equation (1.28)]} \\
&= \left(\frac{1}{p_0+p_3}\right) e_1 + \left(\frac{1}{p_0-p_3}\right) e_2 \\
&= \frac{p_0 - p_3 j}{(p_0^2 - p_3^2)}.
\end{aligned} \tag{2.17}$$

Since $|p_3| < p_0,\ p_3 \neq p_0$.

The type of the bicomplex ML function $\mathbb{E}_{\alpha,\beta}(\xi)$ is given by

$$\begin{aligned}
\sigma &= \frac{1}{e\rho}\lim_{n\to\infty}\sup\,(n|a_n|^{\frac{\rho}{n}}) \\
&= \frac{1}{e\rho}\lim_{n\to\infty}\sup\,\left(n\left|\frac{1}{\Gamma(\alpha n+\beta)}\right|_j^{\frac{\rho}{n}}\right) \\
&= \left(\frac{1}{e\rho}\lim_{n\to\infty}\sup\left(n\left|\frac{1}{\Gamma(\alpha_1 n+\beta_1)}\right|^{\frac{\rho}{n}}\right)\right) e_1 + \left(\frac{1}{e\rho}\lim_{n\to\infty}\sup\left(n\left|\frac{1}{\Gamma(\alpha_2 n+\beta_2)}\right|^{\frac{\rho}{n}}\right)\right) e_2 \\
&= 1\cdot e_1 + 1\cdot e_2 \qquad \text{[using equation (1.29)]} \\
&= 1.
\end{aligned} \tag{2.18}$$

□

2.1 Some Special Cases

Let $\xi \in \mathbb{T}$ where $\xi = z_1 + i_2 z_2$, then for specific values of the parameters $\alpha,\ \beta$, we obtain various bicomplex functions as special cases. To mention, a few are:

(i) $\mathbb{E}_{1,1}(\xi) = e^{\xi}$,

(ii) $\mathbb{E}_{1,2}(\xi) = \dfrac{e^{\xi}-1}{\xi}$,

(iii) $\mathbb{E}_{2,1}(\xi) = \cosh\sqrt{\xi}$,

(iv) $\mathbb{E}_{2,1}(-\xi^2) = \cos\xi$,

(v) $\mathbb{E}_{2,2}(\xi) = \dfrac{\sinh\sqrt{\xi}}{\sqrt{\xi}}$,

(vi) $\mathbb{E}_{2,2}(-\xi^2) = \dfrac{\sin\xi}{\xi}$,

(vii) $\mathbb{E}_{2,3}(\xi) = \dfrac{\cosh\sqrt{\xi}-1}{\sqrt{\xi}}$,

(viii) $\mathbb{E}_{1,3}(\xi) = \dfrac{e^{\xi}-1-\xi}{\xi^2}$.

Theorem 2.7. *Let* $\xi,\ \alpha,\ \beta \in \mathbb{T}$. *Then*

(i) $\mathbb{E}_{\alpha,0}(\xi) = \xi\mathbb{E}_{\alpha,\alpha}(\xi), \quad |\operatorname{Im_j}(\alpha)| < \operatorname{Re}(\alpha)$,

(ii) $\mathbb{E}_{0,\beta}(\xi) = \dfrac{1}{\Gamma\beta}\dfrac{1}{1-\xi}, \quad \|\xi\| < 1,\ |\operatorname{Im_j}(\beta)| < \operatorname{Re}(\beta)$.

Proof. (i) Since

$$\lim_{\xi\to 0}\frac{1}{\Gamma\xi} = \lim_{\xi_1\to 0}\frac{1}{\Gamma\xi_1}e_1 + \lim_{\xi_2\to 0}\frac{1}{\Gamma\xi_2}e_2 = 0\cdot e_1 + 0\cdot e_2 = 0, \tag{2.19}$$

and from (2.3), we get for $\beta = 0$

$$\begin{aligned}\mathbb{E}_{\alpha,0}(\xi) &= \sum_{k=1}^{\infty}\frac{\xi^k}{\Gamma(\alpha k)}\\ &= \sum_{k=0}^{\infty}\frac{\xi^{k+1}}{\Gamma(\alpha k+\alpha)}\\ &= \xi\sum_{k=0}^{\infty}\frac{\xi^k}{\Gamma(\alpha k+\alpha)}\\ &= \xi\mathbb{E}_{\alpha,\alpha}(\xi).\end{aligned} \tag{2.20}$$

(ii) Again, from (2.3) and (2.19), we get for $\alpha = 0$

$$\begin{aligned}\mathbb{E}_{0,\beta}(\xi) &= \sum_{k=1}^{\infty}\frac{\xi^k}{\Gamma(\beta)}\\ &= \frac{1}{\Gamma\beta}\sum_{k=0}^{\infty}\xi^k\\ &= \frac{1}{\Gamma\beta}\frac{1}{1-\xi}, \quad \|\xi\| < 1.\end{aligned} \tag{2.21}$$

□

Recurrence Relations

Complex two-parameter ML function satisfies following recurrence relations (see, e.g., [20, 22])

$$\mathbb{E}_{\alpha,\beta}(z) = \frac{1}{\Gamma\beta} + z\mathbb{E}_{\alpha,\beta+\alpha}(z), \tag{2.22}$$

$$\mathbb{E}_{\alpha,\beta}(z) = \beta\mathbb{E}_{\alpha,\beta+1}(z) + \alpha z\frac{d}{dz}\mathbb{E}_{\alpha,\beta+1}(z), \tag{2.23}$$

$$\mathbb{E}_{\alpha,r+1}(z) = r(r+2)\mathbb{E}_{\alpha,r+3}(z) + \alpha z(\alpha+2(r+1))\mathbb{E}'_{\alpha,r+3}(z) + z^2\mathbb{E}''_{\alpha,r+3}(z) + \mathbb{E}_{\alpha,r+2}(z), \tag{2.24}$$

and

$$3\mathbb{E}_{1,4}(z) + 5z\mathbb{E}'_{1,4} + z^2\mathbb{E}''_{1,4} = \mathbb{E}_{1,2} - \mathbb{E}_{1,3}, \tag{2.25}$$

where $\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0$.

We establish here the recurrence relations for bicomplex ML function.

Theorem 2.8 (Recurrence Relations). *Let* $\xi,\ \alpha,\ \beta \in \mathbb{T}$ *where* $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$, $\alpha = \alpha_1 e_1 + \alpha_2 e_2,\ \beta = \beta_1 e_1 + \beta_2 e_2$ *with* $|\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha),\ |\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta)$. *Then*

(i) $\mathbb{E}_{\alpha,\beta}(\xi) = \frac{1}{\Gamma\beta} + \xi\mathbb{E}_{\alpha,\beta+\alpha}(\xi)$,

(ii) $\mathbb{E}_{\alpha,\beta}(\xi) = \beta\mathbb{E}_{\alpha,\beta+1}(\xi) + \alpha\xi\dfrac{d}{d\xi}\mathbb{E}_{\alpha,\beta+1}(\xi)$,

(iii) $\mathbb{E}_{\alpha,r+1}(\xi) = r(r+2)\mathbb{E}_{\alpha,r+3}(\xi) + \alpha\xi(\alpha+2(r+1))\mathbb{E}'_{\alpha,r+3}(\xi) + \xi^2\mathbb{E}''_{\alpha,r+3}(\xi) + \mathbb{E}_{\alpha,r+2}(\xi)$,

(iv) $3\mathbb{E}_{1,4}(\xi) + 5z\mathbb{E}'_{1,4} + \xi^2\mathbb{E}''_{1,4} = \mathbb{E}_{1,2} - \mathbb{E}_{1,3}$.

Proof. (i) By using the results (2.5) and (2.22), we have

$$\begin{aligned}\mathbb{E}_{\alpha,\beta}(\xi) &= \mathbb{E}_{\alpha_1,\beta_1}(\xi_1)e_1 + \mathbb{E}_{\alpha_2,\beta_2}(\xi_2)e_2\\ &= \left(\frac{1}{\Gamma\beta_1} + \xi_1\mathbb{E}_{\alpha_1,\beta_1+\alpha_1}(\xi_1)\right)e_1 + \left(\frac{1}{\Gamma\beta_2} + \xi_2\mathbb{E}_{\alpha_2,\beta_2+\alpha_2}(\xi_2)\right)e_2\\ &= \frac{1}{\Gamma\beta} + \xi\mathbb{E}_{\alpha,\beta+\alpha}(\xi).\end{aligned} \tag{2.26}$$

(ii) By using the results (2.5) and (2.23), we have

$$\begin{aligned}\mathbb{E}_{\alpha,\beta}(\xi) =& \mathbb{E}_{\alpha_1,\beta_1}(\xi_1)e_1 + \mathbb{E}_{\alpha_2,\beta_2}(\xi_2)e_2\\ =& \left(\beta_1\mathbb{E}_{\alpha_1,\beta_1+1}(\xi_1) + \alpha_1\xi_1\frac{d}{d\xi_1}\mathbb{E}_{\alpha_1,\beta_1+1}(\xi_1)\right)e_1\\ &+ \left(\beta_2\mathbb{E}_{\alpha_2,\beta_2+1}(\xi_2) + \alpha_2\xi_2\frac{d}{d\xi_2}\mathbb{E}_{\alpha_2,\beta_2+1}(\xi_1 2\right)e_2\\ =& \beta\mathbb{E}_{\alpha,\beta+1}(\xi) + \alpha\xi\frac{d}{d\xi}\mathbb{E}_{\alpha,\beta+1}(\xi).\end{aligned} \tag{2.27}$$

(iii) By using the results (2.5) and (2.24), we have

$$\begin{aligned}\mathbb{E}_{\alpha,r+1}(\xi) =& \mathbb{E}_{\alpha,r+1}(\xi_1)e_1 + \mathbb{E}_{\alpha,r+1}(\xi_2)e_2\\ =& \left(r(r+2)\mathbb{E}_{\alpha,r+3}(\xi_1) + \alpha\xi_1(\alpha+2(r+1))\mathbb{E}'_{\alpha,r+3}(\xi_1) + \xi_1^2\mathbb{E}''_{\alpha,r+3}(\xi_1) + \mathbb{E}_{\alpha,r+2}(\xi_1)\right)e_1\\ &+ \left(r(r+2)\mathbb{E}_{\alpha,r+3}(\xi_2) + \alpha\xi_2(\alpha+2(r+1))\mathbb{E}'_{\alpha,r+3}(\xi_2) + \xi_2^2\mathbb{E}''_{\alpha,r+3}(\xi_2) + \mathbb{E}_{\alpha,r+2}(\xi_2)\right)e_2\\ =& r(r+2)\mathbb{E}_{\alpha,r+3}(\xi) + \alpha\xi(\alpha+2(r+1))\mathbb{E}'_{\alpha,r+3}(\xi) + \xi^2\mathbb{E}''_{\alpha,r+3}(\xi) + \mathbb{E}_{\alpha,r+2}(\xi).\end{aligned} \tag{2.28}$$

(iv) By substituting $\alpha = 1$ and $r = 3$ in (iii), the desired result is obtained. □

Duplication Formula
Duplication formula for the complex two-parameter ML function are given by [16, p.7]

$$\mathbb{E}_{\alpha,\beta}(z) + \mathbb{E}_{\alpha,\beta}(-z) = 2\mathbb{E}_{2\alpha,\beta}(z^2), \tag{2.29}$$

and

$$\mathbb{E}_{\alpha,\beta}(z) - \mathbb{E}_{\alpha,\beta}(-z) = 2z\mathbb{E}_{2\alpha,\alpha+\beta}(z^2), \tag{2.30}$$

where $\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0$.

We establish here, the duplication formula for the bicomplex ML function.

Theorem 2.9 (Duplication Formula). *Let* $\xi, \alpha, \beta \in \mathbb{T}$ *with* $|\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha)$, $|\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta)$. *Then*

(i) $\mathbb{E}_{\alpha,\beta}(\xi) + \mathbb{E}_{\alpha,\beta}(-\xi) = 2\mathbb{E}_{2\alpha,\beta}(\xi^2)$,

(ii) $\mathbb{E}_{\alpha,\beta}(\xi) - \mathbb{E}_{\alpha,\beta}(-\xi) = 2\xi\mathbb{E}_{2\alpha,\alpha+\beta}(\xi^2)$.

Proof. (i) By using the results (2.3) and (2.29), we have

$$
\begin{aligned}
\mathbb{E}_{\alpha,\beta}(\xi) + \mathbb{E}_{\alpha,\beta}(-\xi) &= \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k+\beta)} + \sum_{k=0}^{\infty} \frac{(-\xi)^k}{\Gamma(\alpha k+\beta)} \\
&= \sum_{k=0}^{\infty} \frac{\xi^k + (-\xi)^k}{\Gamma(\alpha k+\beta)} \\
&= \sum_{k=0}^{\infty} \frac{\xi^k(1+(-1)^k)}{\Gamma(\alpha k+\beta)} \\
&= 2 + 2\frac{\xi^2}{\Gamma(2\alpha+\beta)} + 2\frac{\xi^4}{\Gamma(4\alpha+\beta)} + 2\frac{\xi^6}{\Gamma(6\alpha+\beta)} + \dots \\
&= 2\left(1 + \frac{(\xi^2)^1}{\Gamma((2\alpha)+\beta)} + \frac{(\xi^2)^2}{\Gamma(2(2\alpha)+\beta)} + \frac{(\xi^2)^2}{\Gamma(3(2\alpha)+\beta)} + \dots\right) \\
&= 2\sum_{k=0}^{\infty} \frac{(\xi^2)^k}{\Gamma((2\alpha)k+\beta)} \\
&= 2\mathbb{E}_{2\alpha,\beta}(\xi^2).
\end{aligned}
\tag{2.31}
$$

(ii) By using the results (2.3) and (2.30), we have

$$
\begin{aligned}
\mathbb{E}_{\alpha,\beta}(\xi) - \mathbb{E}_{\alpha,\beta}(-\xi) &= \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k+\beta)} - \sum_{k=0}^{\infty} \frac{(-\xi)^k}{\Gamma(\alpha k+\beta)} \\
&= \sum_{k=0}^{\infty} \frac{\xi^k - (-\xi)^k}{\Gamma(\alpha k+\beta)} \\
&= \sum_{k=0}^{\infty} \frac{\xi^k(1-(-1)^k)}{\Gamma(\alpha k+\beta)} \\
&= 2\frac{\xi^1}{\Gamma(\alpha+\beta)} + 2\frac{\xi^3}{\Gamma(3\alpha+\beta)} + 2\frac{\xi^5}{\Gamma(5\alpha+\beta)} + 2\frac{\xi^7}{\Gamma(7\alpha+\beta)} + \dots \\
&= 2\xi\left(\frac{1}{\Gamma(0+\alpha+\beta)} + \frac{(\xi^2)^1}{\Gamma((2\alpha)+(\alpha+\beta))} + \frac{(\xi^2)^2}{\Gamma(2(2\alpha)(\alpha+\beta))} + \dots\right) \\
&= 2\xi\sum_{k=0}^{\infty} \frac{(\xi^2)^k}{\Gamma((2\alpha)k+(\alpha+\beta))} \\
&= 2\xi\mathbb{E}_{2\alpha,\alpha+\beta}(\xi^2).
\end{aligned}
\tag{2.32}
$$

□

Differential Relation

Complex two-parameter ML function satisfies the following differential relation (see, e.g., [15, p.58])

$$
\left(\frac{d}{dz}\right)^m \left(z^{\beta-1}\mathbb{E}_{\alpha,\beta}(z^\alpha)\right) = z^{\beta-m-1}\mathbb{E}_{\alpha,\beta-m}(z^\alpha), \quad (m \geq 1),
\tag{2.33}
$$

where $\alpha > 0,\ \mathrm{Re}(\beta) > m$.

We establish here the differential relation for bicomplex ML function.

Theorem 2.10 (Differential Relation). *Let* $\xi,\ \beta \in \mathbb{T}$ *where* $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2,\ \beta = \beta_1 e_1 + \beta_2 e_2,\ \mathrm{Re}(\beta) > m + |\,\mathrm{Im_j}(\beta)|,\ \alpha > 0$. *Then*

$$\left(\frac{d}{d\xi}\right)^m \left(\xi^{\beta-1}\mathbb{E}_{\alpha,\beta}(\xi^\alpha)\right) = \xi^{\beta-m-1}\mathbb{E}_{\alpha,\beta-m}(\xi^\alpha), \quad m \geq 1. \tag{2.34}$$

Proof. By using the results (2.3) and (2.33), we have

$$\begin{aligned}\left(\frac{d}{d\xi}\right)^m \left(\xi^{\beta-1}\mathbb{E}_{\alpha,\beta}(\xi^\alpha)\right) &= \left(\frac{d}{d\xi_1}\right)^m \left(\xi_1^{\beta_1-1}\mathbb{E}_{\alpha,\beta_1}(\xi_1)^\alpha\right) e_1 + \left(\frac{d}{d\xi_2}\right)^m \left(\xi_2^{\beta_2-1}\mathbb{E}_{\alpha,\beta_2}(\xi_2)^\alpha\right) e_2\\ &= \left(\xi_1^{\beta_1-m-1}\mathbb{E}_{\alpha,\beta_1-m}(\xi_1^\alpha)\right) e_1 + \left(\xi_2^{\beta_2-m-1}\mathbb{E}_{\alpha,\beta_2-m}(\xi_2^\alpha)\right) e_2\\ &= \xi^{\beta-m-1}\mathbb{E}_{\alpha,\beta-m}(\xi^\alpha), \quad m \geq 1.\end{aligned} \tag{2.35}$$

□

Integral Relations

Integral relations for complex two-parameter ML function are given by (see, e.g., [15, p.61])

$$z^\beta\mathbb{E}_{1,\beta+1}(\lambda z) = \frac{1}{\Gamma\beta}\int_0^z (z-t)^{\beta-1}e^{\lambda t}dt, \tag{2.36}$$

$$z^\beta\mathbb{E}_{2,\beta+1}(\lambda z^2) = \frac{1}{\Gamma\beta}\int_0^z (z-t)^{\beta-1}\cosh\sqrt{\lambda}t dt, \tag{2.37}$$

$$z^{\beta+1}\mathbb{E}_{2,\beta+2}(\lambda z^2) = \frac{1}{\Gamma\beta}\int_0^z (z-t)^{\beta-1}\frac{\sinh\sqrt{\lambda}t}{\sqrt{\lambda}}dt, \tag{2.38}$$

where $\beta > 0,\ \lambda \in \mathbb{C}$.

We establish here the integral relation for bicomplex ML function.

Theorem 2.11 (Integral Relations). *Let* $\xi,\ \lambda \in \mathbb{T},\ \lambda \notin \mathbb{NC}$ *where* $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2,\ \beta > 0$. *Then*

(i) $\xi^\beta\mathbb{E}_{1,\beta+1}(\lambda\xi) = \frac{1}{\Gamma\beta}\int_0^\xi(\xi-t)^{\beta-1}e^{\lambda t}dt,$

(ii) $\xi^\beta\mathbb{E}_{2,\beta+1}(\lambda\xi^2) = \frac{1}{\Gamma\beta}\int_0^\xi(\xi-t)^{\beta-1}\cosh\sqrt{\lambda}t dt,$

(iii) $\xi^{\beta+1}\mathbb{E}_{2,\beta+2}(\lambda\xi^2) = \frac{1}{\Gamma\beta}\int_0^\xi(\xi-t)^{\beta-1}\dfrac{\sinh\sqrt{\lambda}t}{\sqrt{\lambda}}dt.$

Proof. (i) By using the results (2.5) and (2.36), we have

$$\begin{aligned}\xi^\beta\mathbb{E}_{1,\beta+1}(\lambda\xi) &= \left(\xi_1^\beta\mathbb{E}_{1,\beta+1}(\lambda_1\xi_1)\right) e_1 + \left(\xi_2^\beta\mathbb{E}_{1,\beta+1}(\lambda_2\xi_2)\right) e_2\\ &= \left(\frac{1}{\Gamma\beta}\int_0^{\xi_1}(\xi_1-t)^{\beta-1}e^{\lambda_1 t}dt\right) e_1 + \left(\frac{1}{\Gamma\beta}\int_0^{\xi_2}(\xi_2-t)^{\beta-1}e^{\lambda_2 t}dt\right) e_2\\ &= \frac{1}{\Gamma\beta}\int_0^\xi(\xi-t)^{\beta-1}e^{\lambda t}\, dt.\end{aligned} \tag{2.39}$$

(ii) By using the results (2.5) and (2.37), we have

$$\begin{aligned}\xi^{\beta}\mathbb{E}_{2,\beta+1}(\lambda\xi^2) &= \left(\xi_1^{\beta}\mathbb{E}_{2,\beta+1}(\lambda_1\xi_1^2)\right)e_1 + \left(\xi_2^{\beta}\mathbb{E}_{2,\beta+1}(\lambda_2\xi_2^2)\right)e_2\\ &= \left(\frac{1}{\Gamma\beta}\int_0^{\xi_1}(\xi_1-t)^{\beta-1}\cosh\sqrt{\lambda_1}tdt\right)e_1 + \left(\frac{1}{\Gamma\beta}\int_0^{\xi_2}(\xi_2-t)^{\beta-1}\cosh\sqrt{\lambda_2}tdt\right)e_2\\ &= \frac{1}{\Gamma\beta}\int_0^{\xi}(\xi-t)^{\beta-1}\cosh\sqrt{\lambda}t\,dt.\end{aligned} \tag{2.40}$$

(iii) By using the results (2.5) and (2.38), we have

$$\begin{aligned}\xi^{\beta+1}\mathbb{E}_{2,\beta+2}(\lambda\xi^2) &= \left(\xi_1^{\beta+1}\mathbb{E}_{2,\beta+2}(\lambda_1\xi_1^2)\right)e_1 + \left(\xi_2^{\beta+1}\mathbb{E}_{2,\beta+2}(\lambda_2\xi_2^2)\right)e_2\\ &= \left(\frac{1}{\Gamma\beta}\int_0^{\xi_1}(\xi_1-t)^{\beta-1}\frac{\sinh\sqrt{\lambda_1}t}{\sqrt{\lambda_1}}dt\right)e_1 + \left(\frac{1}{\Gamma\beta}\int_0^{\xi_2}(\xi_2-t)^{\beta-1}\frac{\sinh\sqrt{\lambda_2}t}{\sqrt{\lambda_2}}dt\right)e_2\\ &= \frac{1}{\Gamma\beta}\int_0^{\xi}(\xi-t)\}^{\beta-1}\frac{\sinh\sqrt{\lambda}t}{\sqrt{\lambda}}dt.\end{aligned} \tag{2.41}$$

□

Theorem 2.12. *Let* $\xi,\ \alpha,\ \beta \in \mathbb{T}$ *where* $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2,\ \alpha = \alpha_1 e_1 + \alpha_2 e_2,\ \beta = \beta_1 e_1 + \beta_2 e_2$ *with* $|\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha),\ |\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta)$ *and* $r \in \mathbb{N}$. *Then*

$$\xi^r\mathbb{E}_{\alpha,\beta+r\alpha}(\xi) = \mathbb{E}_{\alpha,\beta} - \sum_{n=0}^{r-1}\frac{\xi^n}{\Gamma(\alpha n+\beta)}. \tag{2.42}$$

Proof. We have

$$\begin{aligned}\mathbb{E}_{\alpha,\beta} - \sum_{n=0}^{r-1}\frac{\xi^n}{\Gamma(\alpha n+\beta)} &= \sum_{n=r}^{\infty}\frac{\xi^n}{\Gamma(\alpha n+\beta)}\\ &= \sum_{k=o}^{\infty}\frac{\xi^{k+r}}{\Gamma(\alpha(k+r)+\beta)} \quad [\text{put } n = k+r]\\ &= \xi^r\sum_{k=o}^{\infty}\frac{\xi^k}{\Gamma(\alpha k+\alpha r+\beta)}\\ &= \xi^r\mathbb{E}_{\alpha,\beta+r\alpha}(\xi).\end{aligned} \tag{2.43}$$

□

We obtain the following special cases of the Theorem 2.12 for $r = 2, 3$ and 4, respectively where $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$, $\alpha = \alpha_1 e_1 + \alpha_2 e_2,\ \beta = \beta_1 e_1 + \beta_2 e_2$ with $|\mathrm{Im_j}(\alpha)| < \mathrm{Re}(\alpha)$ and $|\mathrm{Im_j}(\beta)| < \mathrm{Re}(\beta)$,

(i) $\xi^2\mathbb{E}_{\alpha,\beta+2\alpha}(\xi) = \mathbb{E}_{\alpha,\beta}(\xi) - \frac{1}{\Gamma\beta} - \frac{\xi}{\Gamma(\beta+\alpha)}$,

(ii) $\xi^3\mathbb{E}_{\alpha,\beta+3\alpha}(\xi) = \mathbb{E}_{\alpha,\beta}(\xi) - \frac{1}{\Gamma\beta} - \frac{\xi}{\Gamma(\beta+\alpha)} - \frac{\xi^2}{\Gamma(\beta+2\alpha)}$,

(iii) $\xi^4\mathbb{E}_{\alpha,\beta+4\alpha}(\xi) = \mathbb{E}_{\alpha,\beta}(\xi) - \frac{1}{\Gamma\beta} - \frac{\xi}{\Gamma(\beta+\alpha)} - \frac{\xi^2}{\Gamma(\beta+2\alpha)} - \frac{\xi^3}{\Gamma(\beta+3\alpha)}$.

2.2 Bicomplex Laplace transform of the two-parameter Mittag-Leffler function and Caputo fractional derivative

Here, we define the bicomplex Laplace transform of the bicomplex two-parameter ML function and Caputo fractional derivative (CFD), that would be required in the sequel.

Bicomplex Laplace transform of the two-parameter ML function

The Laplace transform (LT) of the two-parameter ML function in complex space is given by (see, e.g., [15, 16])

$$L[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha});s]=\int_0^{\infty}e^{-st}E_{\alpha,\beta}(\lambda t^{\alpha})t^{\beta-1}dt=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \tag{2.44}$$

where $\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(s)>0,\ |\lambda s^{-\alpha}|<1,\ \alpha,\ \beta,\ \lambda,\ s\in\mathbb{C}$.

Theorem 2.13. *Let $s,\ \alpha,\ \beta,\ \lambda\in\mathbb{T}$ where $s=s_1e_1+s_2e_2$. Bicomplex Laplace transform of the two-parameter ML function is given by*

$$L[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha});s]=\int_0^{\infty}e^{-st}E_{\alpha,\beta}(\lambda t^{\alpha})t^{\beta-1}dt=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \tag{2.45}$$

where $|\mathrm{Im_j}(\alpha)|<\mathrm{Re}(\alpha),\ |\mathrm{Im_j}(\beta)|<\mathrm{Re}(\beta),\ |\mathrm{Im_j}(s)|<\mathrm{Re}(s),\ |\lambda s^{-\alpha}|_j\prec 1$.

Proof. Writing the bicomplex Laplace transform in idempotent components we have

$$\begin{aligned}L[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha});s]&=L[t^{\beta_1-1}E_{\alpha_1,\beta_1}(\lambda_1t^{\alpha_1});s_1]\,e_1+L[t^{\beta_2-1}E_{\alpha_2,\beta_2}(\lambda_2t^{\alpha_2});s_2]\,e_2\\&=\int_0^{\infty}e^{-s_1t}E_{\alpha_1,\beta_1}(\lambda_1t^{\alpha_1})t^{\beta_1-1}dt\,e_1+\int_0^{\infty}e^{-s_2t}E_{\alpha_2,\beta_2}(\lambda_2t^{\alpha_2})t^{\beta_2-1}dt\,e_2\\&=\frac{s_1^{\alpha_1-\beta_1}}{s_1^{\alpha_1}-\lambda_1}e_1+\frac{s_2^{\alpha_2-\beta_2}}{s_2^{\alpha_2}-\lambda_2}e_2,\quad |\lambda_1s_1^{-\alpha_1}|<1,\ |\lambda_2s_2^{-\alpha_2}|<1\\&=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda},\end{aligned} \tag{2.46}$$

where

$$\begin{aligned}|\lambda s^{-\alpha}|_j&=|\lambda_1s_1^{-\alpha_1}|e_1+|\lambda_2s_2^{-\alpha_2}|e_2\qquad[\text{ from equation (1.12) }]\\&\prec 1\cdot e_1+1\cdot e_2=1.\end{aligned} \tag{2.47}$$

□

Bicomplex Laplace transform of Caputo fractional derivative

For a function $f\in AC[a,b]$ the Caputo fractional derivative of order μ is defined as [23]

$${}^{C}D^{\mu}f(t)=\frac{1}{\Gamma(n-\mu)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\mu+1-n}}d\tau,\ n-1<\mu\le n,\ n\in\mathbb{N},\ t>0. \tag{2.48}$$

Let $s=s_1e_1+s_2e_2=a_0+a_1i_1+a_2i_2+a_3i_1i_2\in\mathbb{T}$ and $f(t)$ be a real-valued function of exponential order $K\in\mathbb{R}$. Then the bicomplex Laplace transform of Caputo fractional derivative of order μ is given by

$$L({}^{C}D^{\mu}f(t);s)=s^{\mu}F(s)-\sum_{k=o}^{n-1}s^{\mu-k-1}f^{(k)}(0),\ n-1<\mu\le n\in\mathbb{N}, \tag{2.49}$$

where $F(s)$ is bicomplex Laplace transform of $f(t)$ (see, e.g [24]). $F(s)$ is convergent for all $s\in D$,

$$D=\{s:H_{\rho}(s)\text{ represent a right half plane : }\ a_0>K+|a_3|\}. \tag{2.50}$$

Since the bicomplex Laplace transform of $f(t)$ convergent for all $s\in D$, bicomplex Laplace transform of Caputo fractional derivative is also convergent for all $s\in D$.

3 Bicomplex Solution For Fractional Electromagnetic Wave Equation in Vaccum

Motivated by the work of Gómez [14], we use the following result for replacing the ordinary derivative in electromagnetic wave equation by the fractional derivative

$$\frac{\partial}{\partial t} \to \frac{1}{\sigma^{1-\alpha}} \frac{\partial^\alpha}{\partial t^\alpha},\ 0 < \alpha \leq 1, \tag{3.1}$$

where σ has the dimension of time. When $\alpha = 1$, the above expression reduces to the ordinary derivative.

The Maxwell's equations in vacuum for electromagnetic field are (see, e.g., [3])

$$\begin{aligned} \bigtriangledown \times \mathsf{E} &= -\mu_0 \frac{\partial \mathsf{H}}{\partial t}, & (3.2)\\ \bigtriangledown \times \mathsf{H} &= \epsilon_0 \frac{\partial \mathsf{E}}{\partial t}, & (3.3)\\ \bigtriangledown . \mathsf{E} &= 0, & (3.4)\\ \bigtriangledown . \mathsf{H} &= 0, & (3.5)\\ & & (3.6) \end{aligned}$$

where complex valued vectors $\mathsf{E},\ \mathsf{H}$ represent the intensities of electric and magnetic fields, respectively. Here μ_0 is permeability and ϵ_0 is the permittivity of free space.

The bicomplex vector field F is (see, e.g., [3]) defined by

$$\mathsf{F} = \sqrt{\epsilon_0}\, \mathsf{E} + i_2 \sqrt{\mu_0}\, \mathsf{H}, \tag{3.7}$$

and the bicomplex Maxwell's vector equations are given as

$$\begin{aligned} \bigtriangledown \times \mathsf{F} &= i_2 \frac{\partial \mathsf{F}}{\partial t}, \qquad \frac{1}{c} = \sqrt{\mu_0 \epsilon_0}, & (3.8)\\ \bigtriangledown . \mathsf{F} &= 0. & (3.9) \end{aligned}$$

Assume that the time derivative is fractional and the space derivative is ordinary, using Caputo fractional derivative and the result (3.1) we get the bicomplex fractional Maxwell's vector equations as follows

$$\begin{aligned} \bigtriangledown \times \mathsf{F} &= i_2 \frac{1}{c\sigma^{1-\alpha}} \frac{\partial^\alpha \mathsf{F}}{\partial t^\alpha}, \qquad \frac{1}{c} = \sqrt{\mu_0 \epsilon_0}, \quad 0 < \alpha \leq 1 & (3.10)\\ \bigtriangledown . \mathsf{F} &= 0. & (3.11) \end{aligned}$$

Assuming that the wave is traveling in x-direction, i.e., a vanishing x-component, then above equations (3.10) and (3.11) reduce to the following system of bicomplex differential equations:

$$\begin{aligned} -\frac{\partial F_z}{\partial x} &= i_2 \frac{1}{c\sigma^{1-\alpha}} \frac{\partial^\alpha F_y}{\partial t^\alpha}, & (3.12)\\ \frac{\partial F_y}{\partial x} &= i_2 \frac{1}{c\sigma^{1-\alpha}} \frac{\partial^\alpha F_z}{\partial t^\alpha}, & (3.13)\\ \frac{\partial F_z}{\partial y} &- \frac{\partial F_y}{\partial z} = 0, & (3.14)\\ \frac{\partial F_y}{\partial y} &+ \frac{\partial F_z}{\partial z} = 0. & (3.15) \end{aligned}$$

Putting $Q_z = i_2 F_z$ in equations (3.12) and (3.13), respectively, we obtain

$$\begin{aligned} \frac{\partial Q_z}{\partial x} &= \frac{1}{c\sigma^{1-\alpha}} \frac{\partial^\alpha F_y}{\partial t^\alpha}, & (3.16)\\ \frac{\partial F_y}{\partial x} &= \frac{1}{c\sigma^{1-\alpha}} \frac{\partial^\alpha Q_z}{\partial t^\alpha}. & (3.17) \end{aligned}$$

Differentiating (3.16) and (3.17) and using (3.17) and (3.16), respectively therein, we get

$$\frac{\partial^2}{\partial x^2}F_y(x,t) = \frac{1}{c^2\sigma^{2-2\alpha}}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}F_y(x,t), \tag{3.18}$$

$$\frac{\partial^2}{\partial x^2}Q_z(x,t) = \frac{1}{c^2\sigma^{2-2\alpha}}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}Q_z(x,t). \tag{3.19}$$

where $0 < \alpha \leq 1$ and initial conditions are: $F_y(x,0) = A_1h_1(x)$ and $F_y^{(1)}(x,0) = B_1g_1(x)$. Taking the bicomplex Fourier transform of (3.18) with respect to x, we get

$$\frac{d^{2\alpha}}{dt^{2\alpha}}\bar{F}_y(\xi,t) + c^2\sigma^{2-2\alpha}\xi^2\bar{F}_y(\xi,t) = 0. \tag{3.20}$$

where $\bar{F}_y(\xi,t)$ is the Fourier transform of $F_y(x,t)$.
Fourier transform of initial conditions are given by $\bar{F}_y(\xi,0) = A_1\bar{h}_1(\xi)$ and $\bar{F}_y^{(1)}(\xi,0) = B_1\bar{g}_1(\xi)$. By taking the bicomplex Laplace transform of (3.20) with respect to t, we get

$$s^{2\alpha}\hat{\bar{F}}_y(\xi,s) - \sum_{k=0}^{1} s^{2\alpha-k-1}\bar{F}_y^{(k)}(\xi,0) + c^2\sigma^{2-2\alpha}\xi^2\hat{\bar{F}}_y(\xi,s) = 0.$$

$$\Rightarrow \left(s^{2\alpha} + c^2\sigma^{2-2\alpha}\xi^2\right)\hat{\bar{F}}_y(\xi,s) - \sum_{k=0}^{1} s^{2\alpha-k-1}\bar{F}_y^{(k)}(\xi,0) = 0.$$

$$\Rightarrow \left(s^{2\alpha} + c^2\sigma^{2-2\alpha}\xi^2\right)\hat{\bar{F}}_y(\xi,s) - s^{2\alpha-1}\bar{F}_y(\xi,0) - s^{2\alpha-2}\bar{F}_y^{(1)}(\xi,0) = 0.$$

$$\Rightarrow \left(s^{2\alpha} + c^2\sigma^{2-2\alpha}\xi^2\right)\hat{\bar{F}}_y(\xi,s) = A_1s^{2\alpha-1}\bar{h}_1(\xi) + B_1s^{2\alpha-2}\bar{g}_1(\xi).$$

$$\Rightarrow \hat{\bar{F}}_y(\xi,s) = A_1\left(\frac{s^{2\alpha-1}}{s^{2\alpha} + c^2\sigma^{2-2\alpha}\xi^2}\right)\bar{h}_1(\xi) + B_1\left(\frac{s^{2\alpha-2}}{s^{2\alpha} + c^2\sigma^{2-2\alpha}\xi^2}\right)\bar{g}_1(\xi). \tag{3.21}$$

Taking the bicomplex inverse Laplace transform of (3.21) with respect to s with the help of equation (2.46), we get

$$\bar{F}_y(\xi,t) = A_1\,\mathbb{E}_{2\alpha,1}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{h}_1(\xi) + B_1t\,\mathbb{E}_{2\alpha,2}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{g}_1(\xi). \tag{3.22}$$

Since $\mathbb{E}_{\alpha,1}(\xi) = \mathbb{E}_{\alpha}(\xi)$, hence from equation (3.22), we get

$$\bar{F}_y(\xi,t) = A_1\,\mathbb{E}_{2\alpha}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{h}_1(\xi) + B_1t\,\mathbb{E}_{2\alpha,2}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{g}_1(\xi). \tag{3.23}$$

Taking the bicomplex inverse Fourier transform, (3.23) gives

$$F_y(x,t) = \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi x}\bar{F}_y(\xi,t)d\xi, \tag{3.24}$$

where $\Gamma = (\gamma_1,\gamma_2)$ is a closed contour in the bicomplex space, γ_1 and γ_2 are closed contours in the complex space along the horizontal lines $\{-p < \mathrm{Im}_j(P_1,\xi) < q\}$ and $\{-p < \mathrm{Im}_j(P_2,\xi) < q\}$, respectively. Thus,

$$\begin{aligned}
F_y(x,t) =& \frac{1}{2\pi}\int_\Gamma e^{-i_1\xi x}\left(A_1\mathbb{E}_{2\alpha}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{h}_1(\xi) + B_1t\mathbb{E}_{2\alpha,2}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{g}_1(\xi)\right)d\xi \\
=& \frac{1}{2\pi}A_1\int_\Gamma \left(e^{-i_1\xi x}\mathbb{E}_{2\alpha}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{h}_1(\xi)\right)d\xi \\
&+ \frac{1}{2\pi}B_1\int_\Gamma \left(e^{-i_1\xi x}t\mathbb{E}_{2\alpha,2}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{g}_1(\xi)\right)d\xi \\
=& \frac{1}{2\pi}A_1\int_\Gamma \left(e^{-i_1\xi x}\mathbb{E}_{2\alpha}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{h}_1(\xi)\right)d\xi \\
&+ \frac{1}{2\pi}B_1t\int_\Gamma \left(e^{-i_1\xi x}\mathbb{E}_{2\alpha,2}(-c^2\sigma^{2-2\alpha}\xi^2t^{2\alpha})\bar{g}_1(\xi)\right)d\xi.
\end{aligned} \tag{3.25}$$

Using the result [35, p.23, Eq. (A.8)] for $0 < \alpha < 1$

$$F^{-1}\left[\mathbb{E}_{2\alpha}(-c^2\sigma^{2-2\alpha}\xi^2 t^{2\alpha});\xi\right] = F^{-1}\left[\mathbb{E}_{2\alpha}(-k^2 t^{2\alpha});\xi\right] = \frac{1}{2}\mathbb{M}_{\alpha}(|x|,t) = \frac{1}{2}t^{-\alpha}\mathbb{M}_{\alpha}\left(\frac{|x|}{t^{\alpha}}\right), \tag{3.26}$$

where $k^2 = c^2\sigma^{2-2\alpha}\xi^2$ and $\mathbb{M}_{\alpha}(|x|,t)$ is M-Wright function, details can be found in [34, 35]. Further using the result [21, P.6, Eq. (25)], for $\alpha > 0$

$$F^{-1}(\mathbb{E}_{2\alpha,2}(-c^2\sigma^{2-2\alpha}\xi^2 t^{2\alpha});\xi) = \frac{1}{2|x|}H^{2,1}_{3,3}\left[\frac{|x|}{(c^2\sigma^{2-2\alpha})^{1/2}t^{\alpha}}\middle|\begin{matrix}(1,\frac{1}{2}),(2,\alpha),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right]. \tag{3.27}$$

Here, $H^{p,q}_{m.n}(z)$ is the H-function. Background material on H-function can be found in many books (see, e.g., [36, 37]).

$$\begin{aligned} F_y(x,t) =& \frac{A_1}{2}t^{-\alpha}\mathbb{M}_{\alpha}\left(\frac{|x|}{t^{\alpha}}\right) * h_1(x) \\ &+ B_1\frac{t}{2|x|}H^{2,1}_{3,3}\left[\frac{|x|}{(c^2\sigma^{2-2\alpha})^{1/2}t^{\alpha}}\middle|\begin{matrix}(1,\frac{1}{2}),(2,\alpha),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right] * g_1(x), \end{aligned} \tag{3.28}$$

where $*$ denotes the convolution of two functions.
Similarly,

$$F_z(x,t) = -i_2 Q_z(x,t), \tag{3.29}$$

and hence,

$$\begin{aligned} F_z(x,t) =& -i_2\frac{A_2}{2}t^{-\alpha}\mathbb{M}_{\alpha}\left(\frac{|x|}{t^{\alpha}}\right) * h_1(x) \\ &- i_2 B_2\frac{t}{2|x|}H^{2,1}_{3,3}\left[\frac{|x|}{(c^2\sigma^{2-2\alpha})^{1/2}t^{\alpha}}\middle|\begin{matrix}(1,\frac{1}{2}),(2,\alpha),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right] * g_1(x). \end{aligned} \tag{3.30}$$

Therefore, wave traveling in x-direction with vector field F_x is

$$\boldsymbol{F_x} = F_y(x,t)\hat{y} + F_z(x,t)\hat{z}. \tag{3.31}$$

$$\begin{aligned} \boldsymbol{F_x} \quad =& \left(\frac{A_1}{2}t^{-\alpha}\mathbb{M}_{\alpha}\left(\frac{|x|}{t^{\alpha}}\right) * h_1(x) + B_1\frac{t}{2|x|}H^{2,1}_{3,3}\left[\frac{|x|}{(c^2\sigma^{2-2\alpha})^{1/2}t^{\alpha}}\middle|\begin{matrix}(1,\frac{1}{2}),(2,\alpha),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right] * g_1(x)\right)\hat{y} \\ &-i_2\left(\frac{A_2}{2}t^{-\alpha}\mathbb{M}_{\alpha}\left(\frac{|x|}{t^{\alpha}}\right) * h_1(x) + B_2\frac{t}{2|x|}H^{2,1}_{3,3}\left[\frac{|x|}{(c^2\sigma^{2-2\alpha})^{1/2}t^{\alpha}}\middle|\begin{matrix}(1,\frac{1}{2}),(2,\alpha),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right] * g_1(x)\right)\hat{z}. \end{aligned}$$

Similarly, for the wave traveling in y-direction with vector field F_y and initial conditions

$$F_x(y,0) = C_1 h_2(y),\ Q_z(y,0) = C_2 h_2(y) \text{ and } F_x^{(1)}(y,0) = D_1 g_2(y),\ Q_z^{(1)}(y,0) = D_2 g_2(y),$$

we get

$$F_z(y,t) = \frac{C_1}{2}t^{-\alpha}\mathbb{M}_{\alpha}\left(\frac{|y|}{t^{\alpha}}\right)*h_2(y)+D_1\frac{t}{2|y|}H^{2,1}_{3,3}\left[\frac{|y|}{(c^2\sigma^{2-2\alpha})^{1/2}t^{\alpha}}\middle|\begin{matrix}(1,\frac{1}{2})(2,\alpha)(1,\frac{1}{2})\\(1,\frac{1}{2})(1,1)(1,\frac{1}{2})\end{matrix}\right]*g_2(y), \tag{3.32}$$

and

$$\begin{aligned} F_x(y,t) =& -i_2\frac{C_2}{2}t^{-\alpha}\mathbb{M}_{\alpha}\left(\frac{|y|}{t^{\alpha}}\right) * h_2(y) \\ &- i_2 D_2\frac{t}{2|y|}H^{2,1}_{3,3}\left[\frac{|y|}{(c^2\sigma^{2-2\alpha})^{1/2}t^{\alpha}}\middle|\begin{matrix}(1,\frac{1}{2}),(2,\alpha),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right] * g_2(y). \end{aligned} \tag{3.33}$$

Therefore, wave traveling in y-direction with vector field $\boldsymbol{F_y}$ is

$$\boldsymbol{F_y} = F_z(y,t)\hat{z} + F_x(y,t)\hat{x}. \tag{3.34}$$

Since (3.34) is the solution of bicomplex Maxwell's equations, it satisfies the equations (3.10) and (3.11). So we obtain the values of C_2 and D_2 in terms of C_1 and D_1.

To find the wave traveling in z-direction with initial conditions

$$F_x(z,0) = R_1 h_3(z),\ Q_y(z,0) = R_2 h_3(z) \text{ and } F_x^{(1)}(z,0) = S_1 g_3(z),\ Q_y^{(1)}(z,0) = S_2 g_3(z),$$

proceeding in a similar way, we get

$$\begin{aligned} F_x(z,t) =& \frac{R_1}{2} t^{-\alpha}\mathbb{M}_\alpha\left(\frac{|z|}{t^\alpha}\right) * h_3(z) \\ &+ S_1 \frac{t}{2|z|} H_{3,3}^{2,1}\left[\frac{|z|}{(c^2\sigma^{2-2\alpha})^{1/2}t^\alpha}\middle|\begin{matrix}(1,\frac{1}{2}),(2,\alpha),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right] * g_3(z), \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} F_y(z,t) =& -i_2 R_2 \frac{1}{2} t^{-\alpha}\mathbb{M}_\alpha\left(\frac{|z|}{t^\alpha}\right) * h_3(z) \\ &- i_2 S_2 t \frac{1}{2|z|} H_{3,3}^{2,1}\left[\frac{|z|}{(c^2\sigma^{2-2\alpha})^{1/2}t^\alpha}\middle|\begin{matrix}(1,\frac{1}{2}),(2,\alpha),(1,\frac{1}{2})\\(1,\frac{1}{2}),(1,1),(1,\frac{1}{2})\end{matrix}\right] * g_3(z). \end{aligned} \tag{3.36}$$

Therefore, the wave traveling in z-direction with vector field $\boldsymbol{F_z}$ is

$$\boldsymbol{F_z} = F_x(z,t)\hat{x} + F_y(z,t)\hat{y}. \tag{3.37}$$

Since (3.37) is the solution of bicomplex Maxwell's equations, it satisfies the equations (3.10) and (3.11). So we obtain the values of R_2 and S_2 in terms of R_1 and S_1.

Now, by applying the superposition principle on equations (3.31), (3.34), and (3.37), we obtain the solution of equations (3.10) and (3.11) as

$$\mathsf{F} = (F_x(y,t) + F_x(z,t))\,\hat{x} + (F_y(x,t) + F_y(z,t))\,\hat{y} + (F_z(x,t) + F_z(y,t))\,\hat{z}. \tag{3.38}$$

4 Conclusion

In this paper, the bicomplex two-parameter ML function has been defined, which is an extension of the complex two-parameter ML function. Various properties, including recurrence relations, duplication formula, differential and integral relations are established. Here, bicomplex Laplace transform of Caputo fractional derivative and two-parameter bicomplex Mittag-Leffler function have been evaluated. The electromagnetic fractional time wave equation for vacuum is solved using bicomplex analysis. The results obtained are interesting and found to involve two-parameter bicomplex Mittag-Leffler function and H-function. The bicomplex analysis is emerging as a great tool for solving problems of mathematical physics.

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