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SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH SOME EXISTENCE AND STABILITY RESULTS

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Abstract. The current study discusses boundary value problems of nonlinear fractional differential equations with Riemann−Caputo derivatives. Some existence and uniqueness results are proposed by using the Banach contraction principle and Krasnosel'skii fixed point theorems. Furthermore, some sufficient conditions are given to estimate the existence of integrable solution in L_p spaces. Also, we make use the Ulam-Hyers stability for the given problem to derive the desired results. Finally, two illustrative examples are studied to support the obtained results.

1 Introduction

Fractional differential equations have both important and active roles in mathematical modeling and processing in different fields as physics, engineering and biological phenomena etc. Fractional calculus applications and examples can be found in the books ([14],[23],[26]). Nowadays, much attention has concentrated on the study of existence and uniqueness of solutions for nonlinear fractional differential equations with initial and boundary conditions by using techniques of fixed-point theorems, see ([1],[7]-[11],[16],[19]-[21],[28],[29]).
Afshari et al. [2] studied the existence and uniqueness of positive solutions for some nonlinear fractional differential equations with boundary value problems. Mohamed [17] obtained new existence results of solution for boundary value problems of fractional order involving two Caputo's fractional derivatives. The investigation is based on Hölder's inequality together with Banach contraction principle and Schaefer's fixed point theorem. Rain et al. [25] established some criteria of existence for the boundary value problems involving Caputo fractional derivatives, by using Banach's and Schaefer's fixed point theorems. It is worth remarking that Sousa et al. [27] investigated the existence and uniqueness of mild and strong solutions of fractional semilinear evolution equations by means of the Banach fixed point theorem and the Gronwall inequality. Hyers–Ulam stability has been one of the most effective research topics in differential equations, and obtained a series of results, see ([3],[5],[12],[13]).
In [6], the authors discussed the Ulam-Hyers stability of linear and nonlinear nabla fractional Caputo difference equations on finite intervals. Muniyappan and Rajan [18] proved Hyers-Ulam and Hyers-Ulam-Rassias stability for the fractional differential eqaution wih boundary condition in the sense of Caputo fractional derivative of order $\alpha \in (0, 1)$. Rabha [24] examined Ulam stability for the Cauchy differential equation of fractional order in the unit disk.
Wherefore, motivated by the above works, we investigate the existence and uniqueness of solution for the fractional differential equation with boundary conditions of the form

$$^{R}D^{\beta}\, {}^{C}D^{\alpha}y(t) = h(t, y), \quad 1 < \alpha \le 2, \quad 0 < \beta \le 1, \tag{1.1}$$

$$y(a) = \gamma_1, \quad y(T) = \gamma_2, \quad y^{'}(a) = y^{'}(T), \qquad t \in I = [a, T]. \tag{1.2}$$

Where D^{β} and D^{α} are the Riemann-Liouville and Caputo fractional derivatives of order β and α, respectively and $h : I \times \mathbb{R} \to \mathbb{R}$, be a continuous function, where a, T, γ_1, γ_2 are constants. The aims are estimated to establish the existence of solution for the problem (1.1)–(1.2) by using Banach contraction principle and Krasnosel'skii fixed point teorem in $C(I, \mathbb{R})$.

Then we investigate the p-integrable solution in $L^p[a,T]$. Finally, Ulam-Hyers stability result for the problem (1.1)−(1.2) is presented and some examples are given to explain the results.

2 Preliminaries

Let us give some necessary basic definitions, lemmas and theorems which are useful throughout this paper, for references see ([4], [14], [15], [22], [23]).

Definition 2.1. For a continuous function $y : (0,\infty) \to R$ the Riemann-Liouville fractional integral of order α is defined as

$$ {}_a^tI^{\alpha} y(t) \;=\; \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)ds, $$

where $\Gamma(.)$ is Gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative of order α for a function $y(t)$ is defined by

$$ {}_a^tD^{\alpha} y(t) \;=\; \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} y(s)ds, $$

where $\alpha > 0$, $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of α.

Definition 2.3. The Caputo fractional derivative of order α for a function $y(t)$ is defined by

$$ {}_a^tD^{\alpha} y(t) \;=\; \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} y^{(n)}(s)ds, $$

where $\alpha > 0$, $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer of α.

Lemma 2.4. *If $\alpha > 0$; n is the smallest integer $n > \alpha$; $y(t)$ is in $L(a,b)$ and ${}_a^tI^{1-\alpha}y$ exists and absolutely continuous on $[a,b]$, then ${}_a^{a^+}I^{i-\alpha}y = k_i$ exists for $i = 1,2,...,n$; ${}_a^tI^{\alpha}y$ exists a.e on $[a,b]$, is in $L(a,b)$ and*

$$ {}_a^tI^{\alpha}\, {}_a^sD^{\alpha} y(t) \;=\; y(t) - \sum_{i=1}^{n} \frac{k_i(t-a)^{\alpha-i}}{\Gamma(\alpha-i+1)} \quad a.e. \quad on \quad a \le t \le b $$

Furthermore, the inequality holds everywhere on $(a,b]$, if in additional, $y(t)$ is continuous on $(a,b]$.

Lemma 2.5. *Let $\alpha > 0$, then the Caputo fractional differential equation ${}_a^tD^{\alpha}y(t) = 0$ has the solution*

$$ y(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + ... + c_{n-1}(t-a)^{n-1}, $$

where $c_i \in R, i = 0,1,2,...,n-1$ $(n = [\alpha]+1)$.

Lemma 2.6. *Let $\alpha > 0$, then*

$$ {}_a^tI^{\alpha}\, {}_a^sD^{\alpha} y(t) = y(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + ... + c_{n-1}(t-a)^{n-1}, $$

for some $c_i \in R, i = 0,1,2,...,n-1$ $(n = [\alpha]+1)$.

Theorem 2.7. *(Arzela-Ascoli Theorem)*
If X is compact and $f \subseteq C(X)$, then f totally is bounded if and only if f is bounded and equicontinuous.

Theorem 2.8. *(Krasnosel'skii fixed point teorem)*
Let M be a closed-convex bounded nonempty subset of a Banach space X. Let A and B be two operators such that
(i) $Ax + By \in M$, whenever $x, y \in M$,
(ii) A is compact and continuous,
(iii) B is a contraction mapping.
Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.9. *(Bochner theorem)*
A measurable function $h : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is Bochner integrable if $\|h\|$ is Lebesgue integrable.

Lemma 2.10. *For any $y(t) \in C(I)$, $1 < \alpha \le 2$ and $0 < \beta \le 1$, then the boundary value problem (1.1)–(1.2) has a solution:*

$$\begin{aligned} y(t) =& \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} h(s,y(s))ds \\ &- \frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \\ &+ \gamma_1 + (\frac{t-a}{T-a})(\gamma_2-\gamma_1) + \frac{t-a}{\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \\ &- \frac{t-a}{(T-a)\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-1} h(s,y(s))ds. \end{aligned} \tag{2.1}$$

Proof. Using Lemma (2.4) and Lemma (2.6), the solution of problem (1.1)–(1.2) can be written as:

$$y(t) = \ {}_a^t I^{\alpha+\beta} h + c_0 \frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + c_1 + c_2(t-a). \tag{2.2}$$

Applying the boundary conditions (1.2), we obtain

$$\begin{aligned} c_0 &= \frac{-1}{(T-a)^{\alpha+\beta-2}} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds, \quad c_1 = \gamma_1, \\ c_2 &= \frac{1}{(T-a)} \Big[\gamma_2 - \gamma_1 - \frac{1}{\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-1} h(s,y(s))ds \\ &\quad + \frac{(T-a)}{\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \Big]. \end{aligned}$$

Substituting the values of c_0, c_1 and c_2 in (2.2), yields the solution (2.1). □

3 Main Results (Existence of solution in $C(I, \mathbb{R})$ space)

We now present the existence and uniqueness of solutions to the problem (1.1)–(1.2). Considering the space $C(I, \mathbb{R})$ be Banach space of all continuous functions from I into $\mathbb{R}$, with the norm:

$$\|y\| = sup\{|y(t)| : t \in I\}.$$

Actually, the results are based on Banach contraction principle and Krasnosel'skii fixed point teorem. To prove the main results, we introduce the following assumptions:
(**H1**) There exists a constant $K > 0$ such that $|h(t,y)| \le K$, for each $t \in I$ and all $y \in \mathbb{R}$.
(**H2**) There exists a constant $L > 0$ such that $|h(t,y_1) - h(t,y_2)| \le L|y_1 - y_2|$, for each $t \in I$ and all $y_1, y_2 \in \mathbb{R}$.

Theorem 3.1. *Assume that the hypotheses **(H1)** and **(H2)** hold. If*

$$\frac{2LT^{\alpha+\beta}(2\alpha+2\beta-1)}{\Gamma(\alpha+\beta+1)(\alpha+\beta-1)} < 1.$$

Then the boundary value problem (1.1)–(1.2) has a unique solution.

Proof. Consider the operator $F : C(I, \mathbb{R}) \longrightarrow C(I, \mathbb{R})$ defined by

$$\begin{aligned}(Fy)(t) =&\frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-s)^{\alpha+\beta-1}h(s,y(s))ds\\&-\frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}}\int_a^T (T-s)^{\alpha+\beta-2}h(s,y(s))ds\\&+\gamma_1+(\frac{t-a}{T-a})(\gamma_2-\gamma_1)+\frac{t-a}{\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-2}h(s,y(s))ds\\&-\frac{t-a}{(T-a)\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-1}h(s,y(s))ds.\end{aligned}$$

Let $B_r = \{y \in C : \|y\| \leq r\}$, we show that $FB_r \subset B_r$, for $y \in B_r$, we have

$$\begin{aligned}||(Fy)(t)|| \leq& \frac{1}{\Gamma(\alpha+\beta)}\sup_{t\in I}\left\{\int_a^t (t-s)^{\alpha+\beta-1}|h(s,y(s))|ds\right\}\\&+\frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}}\sup_{t\in I}\left\{\int_a^T (T-s)^{\alpha+\beta-2}|h(s,y(s))|ds\right\}+|\gamma_1|\\&+\frac{(t-a)}{(T-a)}|\gamma_2-\gamma_1|+\frac{t-a}{\Gamma(\alpha+\beta)}\sup_{t\in I}\left\{\int_a^T (T-s)^{\alpha+\beta-2}|h(s,y(s))|ds\right\}\\&+\frac{(t-a)}{(T-a)\Gamma(\alpha+\beta)}\sup_{t\in I}\left\{\int_a^T (T-s)^{\alpha+\beta-1}|h(s,y(s))|ds\right\},\\||Fy|| \leq& KT^{\alpha+\beta}\left(\frac{2}{\Gamma(\alpha+\beta+1)}+\frac{2}{(\alpha+\beta-1)\Gamma(\alpha+\beta)}\right)+2|\gamma_1|+|\gamma_2|.\end{aligned}$$

Now, take $x, y \in C$ and for each $t \in I$, we obtain

$$\begin{aligned}&||(Fy_1)(t)-(Fy_2)(t)|| \leq \frac{1}{\Gamma(\alpha+\beta)}\sup_{t\in I}\left\{\int_a^t (t-s)^{\alpha+\beta-1}|h(s,y_1(s))-h(s,y_2(s))|ds\right\}\\&+\frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}}\sup_{t\in I}\left\{\int_a^T (T-s)^{\alpha+\beta-2}|h(s,y_1(s))-h(s,y_2(s))|ds\right\}\\&+\frac{(t-a)}{\Gamma(\alpha+\beta)}\sup_{t\in I}\left\{\int_a^T (T-s)^{\alpha+\beta-2}|h(s,y_1(s))-h(s,y_2(s))|ds\right\}\\&+\frac{(t-a)}{(T-a)\Gamma(\alpha+\beta)}\sup_{t\in I}\left\{\int_a^T (T-s)^{\alpha+\beta-1}|h(s,y_1(s))-h(s,y_2(s))|ds\right\},\\&||Fy_1-Fy_2|| \leq Lw||y_1-y_2||.\end{aligned}$$

where

$$w = \frac{2T^{\alpha+\beta}(2\alpha+2\beta-1)}{\Gamma(\alpha+\beta+1)(\alpha+\beta-1)}.$$

If $Lw < 1$, then the operator F is a contraction mapping. Hence, it follows from Banach's fixed point theorem that the problem (1.1)–(1.2) has a unique solution on $[a, T]$. □

Theorem 3.2. *Assume that **(H1)** holds with*

$$|h(t,y(t))| \leq \varphi(t), \quad \textit{where} \quad \varphi(t) \in L_1(I).$$

If

$$\frac{LT^{\alpha+\beta}(3\alpha+3\beta-1)}{\Gamma(\alpha+\beta+1)(\alpha+\beta-1)} < 1.$$

Then the boundary value problem (1.1)–(1.2) has at least one solution.

Proof. Let A and B be two operators defined as follows:

$$(Ay)(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} h(s, y(s))ds,$$

$$(By)(t) = -\frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}} \int_a^T (T-s)^{\alpha+\beta-2} h(s, y(s))ds + \gamma_1 + (\frac{t-a}{T-a})(\gamma_2 - \gamma_1)$$
$$+ \frac{t-a}{\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-2} h(s, y(s))ds - \frac{t-a}{(T-a)\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-1} h(s, y(s))ds.$$

It is observed that if $x, y \in B_r$, then $Ax + By \in B_r$, we have

$$||Ax + By|| \leq w \; ||\varphi(t)||_{L_1} + 2|\gamma_1| + |\gamma_2|.$$

Now, we prove that By is contraction mapping,

$$||By_1 - By_2|| \leq \frac{LT^{\alpha+\beta}(3\alpha + 3\beta - 1)}{\Gamma(\alpha+\beta+1)(\alpha+\beta-1)} ||y_1 - y_2||.$$

The operator B is a contraction because of $\frac{LT^{\alpha+\beta}(3\alpha + 3\beta - 1)}{\Gamma(\alpha+\beta+1)(\alpha+\beta-1)} < 1$. As we all know, A is a continuous result from the continuity of h. Moreover,

$$||Ay(t)|| \leq \frac{||\varphi(t)||_{L_1} T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},$$

which implies that A is uniformly bounded on B_r . Now we verify that A is completely continuous. Let $t_1, t_2 \in I, t_1 < t_2$, and $y \in B_r$, we get

$$||Ay(t_2) - Ay(t_1)|| \leq \frac{K}{\Gamma(\alpha+\beta+1)} \Big[(t_1 - a)^{\alpha+\beta} + 2(t_2 - t_1)^{\alpha+\beta} + (t_2 - a)^{\alpha+\beta} \Big].$$

The operator A is compact due to Arzela-Ascoli theorem and three conditions are satisfied by application of Krasnoselskii's fixed point theorem. Hence, boundary value problem (1.1)–(1.2) has at least one solution on $[a, T]$. □

4 Uniqueness of p-integrable solutions in $L^p[a, T]$ spaces

We study the existence and uniqueness theorem in $L^p[a, T]$ spaces for the boundary value problem (1.1)–(1.2), where $L^p(I, \mathbb{R})$ is Banach space for all Lebesgue measurable functions are defined $f : I \longrightarrow \mathbb{R}$, with the norm:

$$||f||_{L^p(I,\mathbb{R})} = \left(\int_I |f(t)|^p dt \right)^{\frac{1}{p}} < \infty, \;\; 1 \leq p < \infty.$$

The Result is based on Banach contraction principle, as follow

Theorem 4.1. *Assume that the hypotheses **(H1)** and **(H2)** hold. If*

$$\frac{2L(T-a)^{(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \Bigg[\left(\frac{p-1}{p(\alpha+\beta)-1} \right)^{p-1} \left(\frac{1}{p(\alpha+\beta)} + \frac{2^{2p}}{p+1} \right)$$
$$+ 2^p \left(\frac{p-1}{p(\alpha+\beta-1)-1} \right)^{p-1} \left(\frac{1}{p(\alpha+\beta-1)+1} + \frac{2^p}{p+1} \right) \Bigg]^{\frac{1}{p}} < 1.$$

Then the boundary value problem (1.1)–(1.2) has a unique solution on [a,T].

Proof. Define the operator F on $L^p[a,T]$ as

$$\begin{aligned} Fy(t) =& \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} h(s,y(s))ds \\ & - \frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \\ & + \gamma_1 + (\frac{t-a}{T-a})(\gamma_2-\gamma_1) + \frac{t-a}{\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \\ & - \frac{t-a}{(T-a)\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-1} h(s,y(s))ds, \end{aligned}$$

we have to prove that F maps every function $y \in L^p[a,T]$ into a function which belongs to $L^p[a,T]$. It is clear that $Fy(t)$ is continuous on I and it is measurable. For $t \in I$, we have

$$\begin{aligned} |Fy(t)|^p =& \Bigg| \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} h(s,y(s))ds \\ & - \frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \\ & + \gamma_1 + (\frac{t-a}{T-a})(\gamma_2-\gamma_1) + \frac{t-a}{\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \\ & - \frac{t-a}{(T-a)\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-1} h(s,y(s))ds \Bigg|^p, \end{aligned} \tag{4.1}$$

for all $t \in I$, the equation (4.1) becomes:

$$\begin{aligned} |Fy(t)|^p \leq & 2^p \left| \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} h(s,y(s))ds \right|^p \\ + & 2^{2p} \left| \frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds + \gamma_1 + (\frac{t-a}{T-a})(\gamma_2-\gamma_1) \right|^p \\ + & 2^{3p} \left| \frac{t-a}{\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \right|^p \\ + & 2^{3p} \left| \frac{t-a}{(T-a)\Gamma(\alpha+\beta)} \int_a^T (T-s)^{\alpha+\beta-1} h(s,y(s))ds \right|^p. \end{aligned} \tag{4.2}$$

Now, we have to show that $(Fy(t))^p$ is Lebsegue integrabl, by Hölder's inequality, from equation (4.2) we obtain

$$\left[\frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} h(s,y(s))ds \right]^p \leq \frac{(t-a)^{p(\alpha+\beta)-1}}{\left(\frac{p(\alpha+\beta)-1}{p-1} \right)^{p-1}} \int_a^t (h(s,y(s)))^p ds.$$

Thus, $\left[\frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1} h(s,y(s))ds \right]^p$ is Lebesgue integrable with respect to $s \in [a,t]$, for all $t \in I$, by Lemma(2.9) the term $(t-s)^{\alpha+\beta-1} h(s,y(s))$ is Bochner integrable with respect to $s \in [a,t]$ for all $t \in I$. Since

$$\left[\int_a^T (T-s)^{\alpha+\beta-2} h(s,y(s))ds \right]^p \leq \frac{(T-a)^{p(\alpha+\beta-1)-1}}{\left(\frac{p(\alpha+\beta-1)-1}{p-1} \right)^{p-1}} \int_a^T (h(s,y(s)))^p ds,$$

and

$$\left[\int_a^T (T-s)^{\alpha+\beta-1}h(s,y(s))ds\right]^p \leq \frac{(T-a)^{p(\alpha+\beta)-1}}{\left(\frac{p(\alpha+\beta)-1}{p-1}\right)^{p-1}}\int_a^T (h(s,y(s)))^p ds.$$

Therefore, $\left[\int_a^T (T-s)^{\alpha+\beta-2}h(s,y(s))ds\right]^p$ and $\left[\int_a^T (T-s)^{\alpha+\beta-1}h(s,y(s))ds\right]^p$ are Lebesgue integrable with respect to $s \in I$, by Lemma (2.9) the term $(T-s)^{\alpha+\beta-1}h(s,y(s))$ and $(T-s)^{\alpha+\beta-2}h(s,y(s))$ are Bochner integrable with respect to $s \in I$ for all $t \in I$. Hence $(Fy(t))^p$ is Lebesgue integrable and therefore F maps $L^p[a,T]$ into itself.
Now, to show that F is a contraction mapping, letting $y_1, y_2 \in L^p[a,T]$, we have

$$\begin{aligned}\int_a^T |F(y_1(t)) - F(y_2(t))|^p dt &\leq \int_a^T \Bigg(\frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-s)^{\alpha+\beta-1}\Big|h(s,y_1(s)) - h(s,y_2(s))\Big|ds \\ &+ \frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}}\int_a^T (T-s)^{\alpha+\beta-2}\Big|h(s,y_1(s)) - h(s,y_2(s))\Big|ds \\ &+ \frac{(t-a)}{\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-2}\Big|h(s,y_1(s)) - h(s,y_2(s))\Big|ds \\ &+ \frac{(t-a)}{(T-a)\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-1}\Big|h(s,y_1(s)) - h(s,y_2(s))\Big|ds\Bigg)^p dt.\end{aligned} \tag{4.3}$$

Since $h(t,y)$ satisfies Lipschitz condition **(H2)**, therefore from inequality (4.3), we get

$$\begin{aligned}\|F(y_1(t)) - F(y_2(t))\|_p^p &\leq \frac{2^pL^p}{(\Gamma(\alpha+\beta))^p}\int_a^T \left(\int_a^t (t-s)^{\alpha+\beta-1}|y_1(s)-y_2(s)|\,ds\right)^p dt \\ &+ \frac{2^{2p}L^p\ (T-a)^{-(\alpha+\beta-2)p}}{(\Gamma(\alpha+\beta))^p}\int_a^T (t-a)^{p(\alpha+\beta-1)}\left(\int_a^T (T-s)^{\alpha+\beta-2}\ |y_1(s)-y_2(s)|\,ds\right)^p dt \\ &+ \frac{2^{3p}L^p}{(\Gamma(\alpha+\beta))^p}\int_a^T (t-a)^p\left(\int_a^T (T-s)^{\alpha+\beta-2}\ |y_1(s)-y_2(s)|\,ds\right)^p dt \\ &+ \frac{2^{3p}L^p}{((T-a)\Gamma(\alpha+\beta))^p}\int_a^T (t-a)^p\left(\int_a^T (T-s)^{\alpha+\beta-1}\ |y_1(s)-y_2(s)|\,ds\right)^p dt,\end{aligned} \tag{4.4}$$

by Hölder's inequality, equation (4.4) becomes

$$\begin{aligned}\|F(y_1(t)) - F(y_2(t))\|_p^p &\leq \frac{2^pL^p}{(\Gamma(\alpha+\beta))^p}\left(\frac{p-1}{p(\alpha+\beta)-1}\right)^{p-1}\int_a^T (t-a)^{p(\alpha+\beta)-1}\int_a^t |y_1-y_2|^p\,dsdt \\ &+ \frac{2^{2p}L^p(T-a)^{p(\alpha+\beta)}}{(\Gamma(\alpha+\beta))^p\ (p(\alpha+\beta-1)+1)}\left(\frac{p-1}{p(\alpha+\beta-1)-1}\right)^{p-1}\left(\int_a^T |y_1-y_2|^p\,ds\right) \\ &+ \frac{2^{3p}L^p(T-a)^{p(\alpha+\beta)}}{(\Gamma(\alpha+\beta))^p\ (p+1)}\left(\frac{p-1}{p(\alpha+\beta-1)-1}\right)^{p-1}\left(\int_a^T |y_1-y_2|^p\,ds\right) dt \\ &+ \frac{2^{3p}L^p(T-a)^{p(\alpha+\beta)}}{(\Gamma(\alpha+\beta))^p\ (p+1)}\left(\frac{p-1}{p(\alpha+\beta)-1}\right)^{p-1}\left(\int_a^T |y_1-y_2|^p\,ds\right),\end{aligned}$$

$$\|Fy_1 - Fy_2\|_p \leq \zeta\,\|y_1 - y_2\|_p\,,$$

where

$$\zeta = \frac{2L(T-a)^{(\alpha+\beta)}}{\Gamma(\alpha+\beta)}\Bigg[\left(\frac{p-1}{p(\alpha+\beta)-1}\right)^{p-1}\left(\frac{1}{p(\alpha+\beta)}+\frac{2^{2p}}{p+1}\right) + 2^p\left(\frac{p-1}{p(\alpha+\beta-1)-1}\right)^{p-1}\left(\frac{1}{p(\alpha+\beta-1)+1}+\frac{2^p}{p+1}\right)\Bigg]^{\frac{1}{p}}.$$

If $\zeta < 1$, then F is a contraction mapping and has a unique fixed point say $y(t) \in L^p[a,T]$ which is the unique solution of the boundary value problem (1.1)–(1.2). □

5 Hyers–Ulam stability

In this section, we study the Hyers–Ulam stability of the boundary value problem

$$\begin{aligned} &{}^{R}D^{\beta}\,{}^{C}D^{\alpha}z(t) = h(t,z(t)), \quad t \in I, \\ &y(a) = \gamma_1, \quad y(T) = \gamma_2, \quad y'(a) = y'(T). \end{aligned} \tag{5.1}$$

Definition 5.1. The boundary value problem (5.1) is Hyers–Ulam stable, if there exists a real constant $c_h > 0$ such that for any $\epsilon > 0$, and for every solution $y \in C(I,\mathbb{R})$ of the inequality

$$\left|{}^{R}D^{\beta}\,{}^{C}D^{\alpha}y(t) - h(t,y(t))\right| \le \epsilon, \quad \forall\, t \in I. \tag{5.2}$$

there exists a solution $z \in C(I,\mathbb{R})$ of equation (5.1) with

$$|y(t) - z(t)| \le c_h\epsilon, \quad \forall\, t \in I.$$

Theorem 5.2. *Assume that $h : I \times \mathbb{R} \to \mathbb{R}$, be a continuous function and (H2) hold. Then the solution of the boundary value problem (5.1) is Hyers–Ulam stable.*

Proof. Let $y \in C(I,\mathbb{R})$ be a solution of the inequality (5.2). Denote by z as the unique solution of problem (5.1), we have

$$\begin{aligned} z(t) =&\frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-s)^{\alpha+\beta-1}h(s,z(s))ds \\ &- \frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}}\int_a^T (T-s)^{\alpha+\beta-2}h(s,z(s))ds \\ &+ \gamma_1 + \frac{(t-a)}{(T-a)}(\gamma_2-\gamma_1) + \frac{t-a}{\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-2}h(s,z(s))ds \\ &- \frac{(t-a)}{(T-a)\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-1}h(s,z(s))ds, \end{aligned}$$

by differential inequality (5.2), we get

$$\begin{aligned} &\Bigg|y(t) - \frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-s)^{\alpha+\beta-1}h(s,y(s))ds \\ &- \frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}}\int_a^T (T-s)^{\alpha+\beta-2}h(s,y(s))ds \\ &+ \gamma_1 + \frac{(t-a)}{(T-a)}(\gamma_2-\gamma_1) + \frac{t-a}{\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-2}h(s,y(s))ds \\ &- \frac{(t-a)}{(T-a)\Gamma(\alpha+\beta)}\int_a^T (T-s)^{\alpha+\beta-1}h(s,y(s))ds\Bigg| \le \frac{\epsilon(t-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \end{aligned}$$

for all $t \in I$, from above it follows that:

$$\begin{aligned}|y(t)-z(t)| \leq & \frac{\epsilon(t-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\left(\frac{1}{\Gamma(\alpha+\beta)} \int_a^t(t-s)^{\alpha+\beta-1}\right. \\ & +\frac{(t-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)(T-a)^{\alpha+\beta-2}} \int_a^T(T-s)^{\alpha+\beta-2}+\frac{t-a}{\Gamma(\alpha+\beta)} \int_a^T(T-s)^{\alpha+\beta-2} \\ & \left.+\frac{(t-a)}{(T-a) \Gamma(\alpha+\beta)} \int_a^T(T-s)^{\alpha+\beta-1}\right)\Big|h(s, y(s))-h(s, z(s))\Big| ds,\end{aligned} \tag{5.3}$$

for all $t \in I$, the equation (5.3) becomes:

$$|y-z| \leq \frac{\epsilon(T)^{\alpha+\beta}}{\left(1-\left(\frac{2LT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2LT^{\alpha+\beta}}{(\alpha+\beta-1)\Gamma(\alpha+\beta)}\right)\right)\Gamma(\alpha+\beta+1)}.$$

Thus, the equation (5.1) is Hyers–Ulam stable. □

6 An Examples

In this section, we present some examples to illustrate our results.

Example 6.1. Consider the following boundary value problem:

$$\begin{aligned} {}^{R}D^{1/2}({}^{C}D^{3/2}y)(t) &= \frac{cos^3\pi t}{(t+13)}\left(\frac{y^2+|y|}{|y|+1}+e^{-t}\right), \quad t\in[0,1] \\ y(0) &= 8, \quad y(1)=10, \quad y^{'}(0)=y^{'}(1). \end{aligned} \tag{6.1}$$

Here $\alpha=3/2,\ \ \beta=1/2,\ a=0,\ T=1$, and

$$h(t,y)=\frac{cos^3\pi t}{(t+13)}\left(\frac{y^2+|y|}{|y|+1}+e^{-t}\right),$$

by Lipschitz condition, we have

$$\Big|h(t,y_1)-h(t,y_2)\Big| \leq \frac{cos^3\pi t}{(t+13)}|y_1-y_2|,$$

where $L=0.076923076$. To estimate the contraction mapping, apply Theorem (3.1) to get $w=0.2307692307<1$. Thus the problem (6.1) has a unique solution.
Now, by Theorem (4.1), let $p=2$, we get $\zeta=0.4492670576$, it follows that the problem (6.1) has a unique solution.

Example 6.2. Consider the following boundary value problem:

$$\begin{aligned} {}^{R}D^{0.3}\,{}^{C}D^{1.7}y(t) &= \frac{\sqrt{t}}{(t+27)}(sint+tan^{-1}y), \quad t\in[1,2] \\ y(1) &= 5, \quad y(2)=1, \quad y^{'}(1)=y^{'}(2) \end{aligned} \tag{6.2}$$

Here $\alpha=1.7,\ \ \beta=0.3,\ a=1,\ T=2$, and

$$h(t,y)=\frac{\sqrt{t}}{(t+27)}(sint+tan^{-1}y),$$

by using $(H2)$, the Lipschitz constant is $L=0.048765984$. Moreover, we have
$w=0.5851908<1$. Thus by Theorem (3.1) the problem (6.2) has a unique solution.
Now, from Theorem (4.1) when $p=2$, it is clear that $\zeta=0.2848163553$. Then the problem (6.2) has a unique p- integrable solutin in $L^2[1,2]$. Thus according to Theorem (5.2) the problem (6.2) is Ulam-Hyers stable.

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