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SCALABILITY AND K-FRAMES

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Abstract Recent studies on K-frames show that parseval K-frames can be used to manage data loss in signal communication. So the construction of parseval K-frames is desirable and scaling is the easiest way for this construction. In this paper we deal with K-frames which can be scaled to parseval K-frames and tight K-frames and we term such K-frames as scalable K-frames and A-scalable K-frames respectively. We prove some of the results related to scalable K-frames. Also we give characterization result for scalable K-frames.

1 Introduction

Frames in Hilbert spaces were introduced as a generalization of orthonormal bases, by R. J. Duffin and A. C. Schaffer in 1956. Frames have their own advantages compared to bases. The main advantage is the redundancy of frames. Frames span the whole Hilbert space, but the representation of an element using frames need not be unique. This flexibility makes frames an important tool in different areas of research, in theory and in application. Frame theory plays an important role in signal processing, sampling theory, coding and communications and so on. We refer [7] for an introduction to frame theory.

For different applications in theory and application, some special kinds of frames have been introduced. One such frame is K-frame. The concept of K-frames was introduced by L. Gavruta [6], to study atomic systems with respect to bounded linear operators. K-frames are more general than classical frames. Although the span limit of K-frames is restricted to range of K, this generality of K-frames makes K-frames practically important.

G. Kuttyniok, K. A. Okoudjou, F. Philipp, and E. K. Tuley in [4] introduced Scalable frames. Scalable frames have a wide range of applications. Recent studies on K-frames shows that K-frames can be used to deal with the problem of data loss in signal communication. Parseval K-frames and tight K-frames are mainly used for this purpose. So we are interested in constructions which modify a given K-frame into a parseval K-frame or a tight K-frame. The easiest way to get a K-frame from a given K-frame is by scaling the vectors. So it is desirable to have a characterization of K-frames which can be scaled to parseval K-frames or tight K-frames. We term such K-frames as scalable K-frames and A-scalable K-frames respectively.

Some basic definitions and results related to frames and K-frames are contained in section 2. In section 3 we have proved some lead off results on scalable K-frames. Section 4 contains our main result which characterizes scalable K-frames. Throughout this paper, H represent a complex separable Hilbert space, $B(H)$, the space of all linear bounded operators on H. For $K \in B(H)$, we denote $R(K)$ the range of K and $D(K)$ the domain of K. Also, N denote the finite or countable index set.

2 Preliminaries

In this section we give some basic definitions and results about frames and K-frames. For several generalizations and applications in frame theory, refer [1, 2, 3, 9, 10].

Definition 2.1. [7] For a separable Hilbert space H, a sequence $\{f_n\}_{n\in N} \subset H$ is said to be a frame for H if there exist $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n\in N} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$$

for all $f \in H$. A, B are called lower and upper frame bounds. If $A = B$ then we call it a A-tight frame and if $A = B = 1$ we call it a parseval frame.

Definition 2.2. [6]Let $K \in B(H)$. We say that $\{f_n\}_{n\in N} \subset H$ is a K-frame for H if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{n\in N} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$$

for all $f \in H$. A, B are called lower and upper K-frame bounds. If

$$\sum_{n\in N} |\langle f, f_n\rangle|^2 = A\|K^*f\|^2$$

holds then we call it a A-tight K-frame. If

$$\sum_{n\in N} |\langle f, f_n\rangle|^2 = \|K^*f\|^2$$

holds then we call it a parseval K-frame.

Let $\{f_n\}_{n\in N}$ be any of the two sequences defined above. Then we can define two operators.

$$T_F : H \to l^2(N)$$

defined by

$$T_F(f) = \{\langle f, f_n\rangle\}_{n\in N}$$

denote the associated analysis operator. Its adjoint

$$T_F^* : l^2(N) \to H$$

defined by

$$T_F^*(\{c_n\}_{n\in N}) = \sum_{n\in N} c_n f_n$$

is called the synthesis operator. From the properties of T_F, it follows that the frame operator

$$S_F : H \to H$$

defined by

$$S_F f = T_F^* T_F = \sum_{n\in N} \langle f, f_n\rangle f_n$$

for all $f \in H$ is a bounded, positive and self-adjoint operator in H. If $\{f_n\}_{n\in N}$ is a frame, then S_F is invertible also. But in the case of K-frames S_F is not invertible on H. But if K has closed range, then S_F is invertible on $R(K)$.

Theorem 2.3. *[10] $\{f_n\}_{n\in N}$ is a parseval K-frame for H if and only if $S_F = KK^*$.*

Definition 2.4. [8] A sequence $\{a_n\}_{n\in N}$ is said to be semi-normalized if there exist $a, b > 0$ such that $a \leq a_n \leq b$ for all n.

Definition 2.5. [3] A sequence $\{a_n\}_{n\in N}$ is said to be positively confined if $0 < \inf_n a_n \leq \sup_n a_n < +\infty$.

Definition 2.6. [4] Diagonal operator D_a in $l^2(N)$ corresponding to a sequence $a = \{a_n\}_{n\in N} \subset \mathbb{K}$ is defined by

$$D_a\{v_n\}_{n\in N} = \{a_n v_n\}_{n\in N}.$$

D_a (possibly unbounded) is a self-adjoint operator.

Theorem 2.7. *[6] Let $L_1 \in B(H_1, H)$ and $L_2 \in B(H_2, H)$. Then following statements are equivalent:*
(i)$R(L_1) \subset R(L_2)$.
(ii)$L_1L_1^* \leq \lambda^2 L_2L_2^*$ for some $\lambda \geq 0$.
(iii)there exists a bounded operator $X \in L(H_1, H_2)$ so that $L_1 = L_2X$.

This theorem is called Douglas' Majorization theorem.

3 Some results about scalability of K-frames

Definition 3.1. A K-frame $\{f_n\}_{n\in N}$ for H is said to be a scalable K-frame for H if there exist non-negative scalars $\{a_n\}_{n\in N}$ such that $\{a_nf_n\}_{n\in N}$ is a parseval K-frame for H.

The following example from [5] is a scalable K-frame.

Example 3.2. Let $H = \mathbb{C}^3$ and N denote the index set $\{1, 2, 3\}$. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for H. Let $K \in B(\mathbb{C}^3)$ be defined by

$$Ke_1 = e_1, Ke_2 = e_1, Ke_3 = e_2.$$

Then

$$\{f_n\}_{n\in N} = \{Ke_1, Ke_2, Ke_3\} = \{e_1, e_1, e_2\}$$

is a K-frame for H. Take $\{a_n\}_{n\in N} = \{1, 1, 1\}$. Then $\{a_nf_n\}_{n\in N}$ is a parseval K-frame for H and hence $\{f_n\}_{n\in N}$ is a scalable K-frame for H.

Definition 3.3. A K-frame $\{f_n\}_{n\in N}$ for H is said to be a A-scalable K-frame for H if there exist non-negative scalars $\{a_n\}_{n\in N}$ such that $\{a_nf_n\}_{n\in N}$ is a A-tight K-frame for H.

All K-frames are not scalable. Also, all scalings of vectors of a K-frame need not result in a parseval K-frame. But, some particular scalings of a K-frame sometimes give a new K-frame. The following two results help us to identify two such scalings.

Theorem 3.4. *Let $\{f_n\}_{n\in N}$ be a K-frame for H with K-frame bounds A, B and $\{a_n\}_{n\in N}$ be a semi-normalized sequence. Then $\{a_nf_n\}_{n\in N}$ is also a K-frame.*

Proof. Suppose $\{f_n\}_{n\in N}$ is a $K-$frame for H. Therefore there exist $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{n\in N} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$$

for all $x \in H$. Since $\{a_n\}_{n\in N}$ is a semi-normalized sequence, there exist $a, b > 0$ such that

$$\sum_{n\in N} |\langle f, a_nf_n\rangle|^2 = \sum_{n\in N} a_n^2|\langle f, f_n\rangle|^2 \leq b^2B\|f\|^2$$

and

$$\sum_{n\in N} |\langle f, a_nf_n\rangle|^2 \geq a^2A\|K^*f\|^2$$

where A and B are optimal K-frame bounds for$\{f_n\}_{n\in N}$. Hence $\{a_nf_n\}_{n\in N}$ is a K-frame with bounds $A^{'} = Aa^2$ and $B^{'} = Bb^2$ □

Remark 3.5. Let $\{a_n\}_{n\in N}$ be a real sequence. In the above result if $\{g_n\}_{n\in N}$ is a K-dual of $\{f_n\}_{n\in N}$, then $\{a_n^{-1}g_n\}_{n\in N}$ is a K-dual of $\{a_nf_n\}_{n\in N}$, which is evident from the following fact.

$$\sum_{n\in N} \langle f, a_nf_n\rangle a_n^{-1}g_n = \sum_{n\in N} \langle f, f_n\rangle g_n = f$$

for all $f \in H$.

Theorem 3.6. *Let $\{f_n\}_{n\in N}$ be a K-frame for H with K-frame bounds A, B and $\{a_n\}_{n\in N}$ be a positively confined sequence. Then $\{a_nf_n\}_{n\in N}$ is also a K-frame for H.*

Proof. We have,

$$\sum_{n\in N} |\langle f, a_nf_n\rangle|^2 = \sum_{n\in N} a_n^2|\langle f, a_nf_n\rangle|^2$$

and

$$(\inf_n a_n)^2 \sum_{n\in N} |\langle f, f_n\rangle|^2 \leq \sum_{n\in N} a_n^2|\langle f, f_n\rangle|^2 \leq (\sup_n a_n)^2 \sum_{n\in N} |\langle f, f_n\rangle|^2$$

$$A(\inf_n a_n)^2\|K^*f\|^2 \leq \sum_{n\in N} |\langle f, a_nf_n\rangle|^2 \leq B(\sup_n a_n)^2\|f\|^2$$

for all $f \in H$. Hence $\{a_nf_n\}_{n\in N}$ is a K-frame for H. □

Remark 3.7. In the case of finite frames, if $\{f_n\}_{n\in N}$ is a K-frame for M-dimensional Hilbert space H^M, then given any non-negative sequence $\{a_n\}_{n\in N}$ of scalars, we have $\{a_n f_n\}_{n\in N}$ is also a K-frame for H^M.

Theorem 3.8. *Let $T \in B(H)$. Then $\{f_n\}_{n\in N}$ is a scalable K-frame for $R(T^*)$ if and only if $\{Tf_n\}_{n\in N}$ is a scalable TK-frame for H.*

Proof. Suppose $\{f_n\}_{n\in N}$ is a scalable K-frame for $R(T^*)$. This implies that there exist $\{a_n\}_{n\in N}$ such that

$$\sum_{n\in N} |\langle g, a_n f_n\rangle|^2 = \|K^* g\|^2$$

for all $g \in R(T^*)$,where $g = T^* f$ for some $f \in H$.This implies,

$$\sum_{n\in N} |\langle T^* f, a_n f_n\rangle|^2 = \|K^*(T^* f)\|^2$$

for all $f \in H$, which implies

$$\sum_{n\in N} |\langle f, a_n T f_n\rangle|^2 = \|(TK)^* f\|^2$$

for all $f \in H$.Hence $\{Tf_n\}_{n\in N}$ is a scalable TK-frame for H.
Conversely suppose that $\{Tf_n\}_{n\in N}$ is a scalable TK-frame for H. This implies

$$\sum_{n\in N} |\langle f, a_n T f_n\rangle|^2 = \|(TK)^* f\|^2$$

for all $f \in H$. This implies

$$\sum_{n\in N} |\langle T^* f, a_n f_n\rangle|^2 = \|K^* T^* f\|^2$$

for all $f \in H$. That is

$$\sum_{n\in N} |\langle g, a_n f_n\rangle|^2 = \|K^* g\|^2$$

for all $g \in R(T^*)$. Hence $\{f_n\}_{n\in N}$ is a scalable K-frame for $R(T^*)$. □

Remark 3.9. (1)If $T \in B(H)$ and $\{f_n\}_{n\in N}$ is a scalable K-frame for H, then $\{Tf_n\}_{n\in N}$ is a scalable TK-frame for H.
(2)Let $\{f_n\}_{n\in N}$ be a scalable K-frame for H. Then,

$$\sum_{n\in N} |\langle f, a_n K f_n\rangle|^2 = \sum_{n\in N} |\langle K^* f, a_n f_n\rangle|^2 = \|K^*(K^* f)\|^2 = \|(K^2)^* f\|^2$$

for all $f \in H$. This implies that $\{a_n K f_n\}_{n\in N}$ is a parseval K^2-frame for H. In general, $\{a_n K^s f_n\}_{n\in N}$ is a parseval K^{s+1}-frame for H and hence $\{K^s f_n\}_{n\in N}$ is a scalable K^{s+1}-frame for H.

Theorem 3.10. *Suppose $\{f_n\}_{n\in N}$ is a scalable frame for H.Let $T \in B(H)$. Then $\{Tf_n\}_{n\in N}$ is a scalable T-frame for H*

Theorem 3.11. *Let $T \in B(H)$. Then $\{f_n\}_{n\in N}$ is a scalable frame for $R(T^*)$ if and only if $\{Tf_n\}_{n\in N}$ is a scalable T-frame for H.*

Proof. suppose $\{f_n\}_{n\in N}$ is a scalable frame for $R(T^*)$. Then,

$$\sum_{n\in N} |\langle g, a_n f_n\rangle|^2 = \|g\|^2$$

for all $g \in R(T^*)$ where $g = T^* f$ for some $f \in H$. This implies

$$\sum_{n\in N} |\langle T^* f, a_n f_n\rangle|^2 = \|T^* f\|^2$$

for all $f \in H$ and

$$\sum_{n\in N} |\langle f, a_n T f_n\rangle|^2 = \|T^* f\|^2$$

for all $f \in H$. Hence $\{Tf_n\}_{n\in N}$ is a scalable T-frame for H. Conversely suppose $\{Tf_n\}_{n\in N}$ is a scalable T-frame for H. Then,

$$\sum_{n\in N} |\langle f, a_n T f_n\rangle|^2 = \|T^* f\|^2$$

for all $f \in H$. This implies

$$\sum_{n\in N} |\langle T^* f, a_n f_n\rangle|^2 = \|T^* f\|^2$$

for all $f \in H$. That is,

$$\sum_{n\in N} |\langle g, a_n f_n\rangle|^2 = \|g\|^2$$

for all $g \in R(T^*)$. Hence $\{f_n\}_{n\in N}$ is a scalable frame for $R(T^*)$. □

Theorem 3.12. *Let $\{f_n\}_{n\in N}$ be a scalable K-frame. Then $\{f_n\}_{n\in N}$ is a scalable $(KK^*)^{\frac{1}{2}}$-frame.*

Proof. Suppose $\{f_n\}_{n\in N}$ is a scalable K-frame for H. This implies,

$$\sum_{n\in N} |\langle f, a_n f_n\rangle|^2 = \|K^* f\|^2$$

for all $f \in H$. Let S be the frame operator of $\{a_n f_n\}_{n\in N}$. Then $S = KK^*$. So we get $\langle S^{\frac{1}{2}} S^{\frac{1}{2}^*} f, f\rangle = \langle KK^* f, f\rangle$ for all $f \in H$ and $\|S^{\frac{1}{2}^*} f\|^2 = \|K^* f\|^2$ for all $f \in H$. Thus we obtain

$$\sum_{n\in N} |\langle f, a_n f_n\rangle|^2 = \|S^{\frac{1}{2}^*} f\|^2$$

for all $f \in H$. Thus $\{a_n f_n\}_{n\in N}$ is a parseval $S^{\frac{1}{2}}$-frame and hence is a parseval $(KK^*)^{\frac{1}{2}}$-frame. That is $\{f_n\}_{n\in N}$ is a scalable $(KK^*)^{\frac{1}{2}}$-frame. □

Theorem 3.13. *Let $\{f_n\}_{n\in N}$ and $\{g_n\}_{n\in N}$ be scalable K-frames for H with scalings $a = \{a_n\}_{n\in N}$ and $b = \{b_n\}_{n\in N}$ respectively. Suppose the analysis operators T_{aF} and T_{bG} of the scaled frames satisfy $T^*_{aF} T_{bG} = 0$. Then $\{a_n f_n + b_n g_n\}_{n\in N}$ is a $2c^2$-scalable K-frame. In particular, $\{a_n f_n + b_n g_n\}_{n\in N}$ is a scalable K-frame.*

Proof. For all $f \in H$ we have,

$$\sum_{n\in N} |\langle f, a_n f_n\rangle|^2 = \|K^* f\|^2$$

and

$$\sum_{n\in N} |\langle f, b_n g_n\rangle|^2 = \|K^* f\|^2.$$

Also since, $T^*_{aF} T_{bG} = 0$, we get, $\sum_{n\in N} a_n b_n \langle f, g_n\rangle f_n = 0$.
Take $c_n = c$ for all n where $c > 0$. Then $\{c_n\}_{n\in N}$ is a non-negative sequence and

$$\sum_{n\in N} |\langle f, c_n(a_n f_n + b_n f_n)\rangle|^2 = \sum_{n\in N} |\langle f, c_n a_n f_n\rangle|^2 + \sum_{n\in N} |\langle f, c_n b_n f_n|^2 +$$

$$\sum 2Re\langle f, c_n a_n f_n\rangle\langle f, c_n b_n g_n\rangle = 2c^2 \|K^* f\|^2.$$

Therefore, $\{a_n f_n + b_n g_n\}_{n\in N}$ is a $2c^2$-scalable K-frame.
If we take $c = \frac{1}{\sqrt{2}}$, then $\{a_n f_n + b_n g_n\}_{n\in N}$ is a scalable K-frame.

□

4 Main results

Theorem 4.1. *Let $\{f_n\}_{n\in N}$ be a K-frame for H with analysis operator T_F and let $a = \{a_n\}_{n\in N}$ be a sequence of non-negative scalars. Then if $G = \{a_n f_n\}_{n\in N}$ is a K-frame for H then $R(T_F) \subset D(D_a)$ and $D_a|_{R(T_F)}$ is bounded.*

Proof. Suppose $\{a_n f_n\}_{n\in N}$ is a K-frame for H and let T_G be the corresponding analysis operator. Then, for any $f \in H$,

$$T_G f = \{\langle f, a_n f_n\rangle\}_{n\in N} = \{a_n \langle f, f_n\rangle\}_{n\in N} = D_a T_F f.$$

Thus $T_G = D_a T_F$ and $R(T_F) \subset D(D_a)$.
Now let $v \in R(T_F)$ so that $v = T_F f$ for some $f \in H$.
Consider,

$$\|D_a v\| = \|D_a T_F f\| = \|T_G f\| \leq A_1 \|f\|^2 \leq A_1 \|T_F^{-1} v\| \leq A_1 \|T_F\|^{-1} \|v\|$$

Thus we get, $D_a|_{R(T_F)}$ is bounded. □

Theorem 4.2. *Let $\{f_n\}_{n\in N}$ be a K-frame for H with analysis operator T_F and let $a = \{a_n\}_{n\in N}$ be a sequence of non-negative scalars. Then the following conditions are equivalent.*
(i)$G = \{a_n f_n\}_{n\in N}$ is a K-frame for H.
(ii)There exist a diagonal operator D_a in $l^2(N)$ such that $R(T_F) \subset D(D_a)$ and $D_a|_{R(T_F)}$ is bounded and $R(K) \subseteq R(D_a T_F^*)$. In particular, in this case frame operator of $G = \{a_n f_n\}_{n\in N}$ is given by $S_G = T_F^* D^2 T_F$.

Proof. Suppose $G = \{a_n f_n\}_{n\in N}$ is a K-frame for H. Then $R(T_F) \subset D(D_a)$ and $D_a|_{R(T_F)}$ is bounded follows from the proof of Theorem 4.1. Since $\{a_n f_n\}$ is a K-frame, we have

$$A\|K^* f\|^2 \leq \sum_{n\in N} |\langle f, a_n f_n\rangle|^2 = \sum_{n\in N} |a_n \langle f, f_n\rangle|^2 = \|a_n \{\langle f, f_n\rangle\}_{n\in N}\|^2 = \|D_a T_F f\|^2.$$

Using Douglas' Majorization theorem, we get, $R(K) \subseteq R((D_a T_F)^*)$.
To prove the converse, let $v \in R(T_F)$. Then $v = T_F f$ for some $f \in H$. Since $D_a|_{R(T_F)}$ is bounded, we have,

$$\|D_a v\| \leq \alpha \|v\|$$

for some $\alpha > 0$ and for all $v \in R(T_F)$. That is,

$$\|D_a T_F f\|^2 \leq \alpha \|T_F f\|^2 \leq \alpha \|T_F\|^2 \|f\|^2.$$

and we get,

$$\sum_{n\in N} |\langle f, a_n f_n\rangle|^2 \leq B\|f\|^2$$

where $B = \alpha \|T_F\|^2$. Also since $R(K) \subseteq R(D_a T_F^*)$, we get

$$A\|K^* f\|^2 \leq \sum_{n\in N} |\langle f, a_n f_n\rangle|^2.$$

Hence $\{a_n f_n\}_{n\in N}$ is a K-frame for H. Also,

$$S_G = T_G^* T_G = (D_a T_F)^* (D_a T_F) = T_F^* D_a^2 T_F.$$

□

Theorem 4.3. *Let $\{f_n\}_{n\in N}$ be a K-frame for H with analysis operator T_F and let $a = \{a_n\}_{n\in N}$ be a sequence of non-negative scalars. Also assume that $\inf_n \|f_n\| > 0$. Then the following conditions are equivalent.*
(i)$G = \{a_n f_n\}_{n\in N}$ is a K-frame for H.
(ii)D_c is bounded and $R(K) \subseteq R(D_a T_F^*)$.

Proof. Suppose $\{a_n f_n\}_{n\in N}$ is a K-frame. Then there exist $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{n\in N} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$$

for all $x \in H$.From the right inequality we get D_a is bounded and from the left inequality we get $R(K) \subseteq R(D_a T_F^*)$.
Conversely suppose, D_a is bounded and $R(K) \subseteq R(D_a T_F^*)$. Then using Theorem 4.2 we get $\{a_n f_n\}_{n\in N}$ is a K-frame for H. □

Theorem 4.4. *Let $\{f_n\}_{n\in N}$ be a K-frame for H. If $\{f_n\}_{n\in N}$ is a scalable K-frame for H, then there exist a non-negative diagonal operator D in $l^2(N)$ such that $KK^* = T_F^* D^2 T_F$.*

Proof. Suppose $\{f_n\}_{n\in N}$ is a scalable K-frame for H. This implies that, there exist $a = \{a_n\}_{n\in N}$ where $a_n \geq 0$ such that $\{a_n f_n\}_{n\in N}$ is a parseval K-frame. Then by Theorem 4.2 frame operator of $\{a_n f_n\}_{n\in N}$ is $S_G = T_F^* D_a^2 T_F$. But frame operator of parseval K-frame is KK^*. Thus we obtain $T_F^* D^2 T_F = KK^*$ where $D = D_a$. □

Remark 4.5. Using Theorem 4.2 it is clear that, if there exist a semi-normalized diagonal operator D_a in $l^2(N)$ such that $KK^* = T_F^* D^2 T_F$, then $\{f_n\}_{n\in N}$ is a scalable K-frame for H.

Theorem 4.6. *Let $\{f_n\}_{n\in N}$ be a K-frame for H such that $\inf_n \|f_n\| > 0$. Then the following conditions are equivalent.*
(i)$\{f_n\}_{n\in N}$ is a scalable K-frame for H.
(ii)There exist a non-negative bounded diagonal operator D in $l^2(N)$ such that $KK^* = T_F^* D^2 T_F$.

Proof. (i)implies(ii) holds from Theorem 4.3 and Theorem 4.4. Conversely suppose that there exist a non-negative bounded diagonal operator D in $l^2(N)$ such that $KK^* = T_F^* D^2 T_F$. Then for all $f \in H$,

$$\langle T_F^* D^2 T_F f, f\rangle = \langle KK^* f, f\rangle.$$

This implies

$$\|DT_F f\|^2 = \|K^* f\|^2$$

and we get

$$\sum_n |a_n\langle f, f_n\rangle|^2 = \|K^* f\|^2.$$

Thus $\{f_n\}_{n\in N}$ is a scalable K-frame for H. □

To provide an illustration of the above theorem we use the same K-frame given in Example 3.2

Example 4.7. We have

$$\{f_n\}_{n\in N} = \{Ke_1, Ke_2, Ke_3\} = \{e_1, e_1, e_2\}$$

is a K-frame for $H = \mathbb{C}^3$. Take $\{a_n\}_{n\in N} = \{1, 1, 1\}$. Then $\{a_n f_n\}_{n\in N}$ is a scalable K-frame for H. Here $\inf_n \|f_n\| > 0$. Let $D : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be the diagonal operator defined by $D(\{v_n\}_{n\in N}) = \{a_n v_n\}_{n\in N}$ where $a_n \geq 0$. We have,

$$\begin{aligned}
T_F{}^* D^2 T_F(f) &= \sum_{j\in J} a_j{}^2 \langle f, Ke_j\rangle Ke_j \\
&= (a_1{}^2 + a_2{}^2)\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2 \\
&= 2\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2,
\end{aligned}$$

and

$$\begin{aligned}
KK^*(f) &= K(K^* f) \\
&= K(\sum_{j\in J} \langle f, Ke_i\rangle e_i) \\
&= \sum_{j\in J} \langle f, Ke_i\rangle Ke_i \\
&= 2\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2.
\end{aligned}$$

That is, there exist a non-negative bounded diagonal operator D in $l^2(N)$ such that $KK^* = T_F^* D^2 T_F$. Now for the converse, suppose there exist a non-negative bounded diagonal operator D in $l^2(N)$ such that $KK^* = T_F^* D^2 T_F$. This implies that

$$({a_1}^2 + {a_2}^2)\langle f, e_1\rangle e_1 + {a_3}^2\langle f, e_2\rangle e_2 = 2\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2$$

and we get ${a_1}^2 + {a_2}^2 = 2$ and ${a_3}^2 = 1$. Then taking $\{a_n\}_{n\in N} = \{a_1, a_2, a_3\} = \{1, 1, 1\}$ we get $\{a_n f_n\}_{n\in N}$ is a scalable K-frame.

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