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Slow convergence of sequences of b-linear functionals in linear n-normed space

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Abstract Different types of convergence of the sequence of bounded b-linear functional in the case of linear n-normed space is being studied. We discuss the rate of convergence of a sequence of bounded b-linear functionals that converges to b-pointwise to a bounded b-linear functional in linear n-normed space. Finally, we give a characterization of almost b-arbitrarily slow convergence in linear n-normed space.

1 Introduction

Let X and Y be two normed linear spaces and $\{T_k\}$ be a sequence of bounded linear operators from X to Y. Then $\{T_k\}$ is said to converges pointwise to a bounded linear operator T provided $T(x) = \lim\limits_{k \to \infty} T_k(x)$. The rate or speed of pointwise convergence of sequence of operators is very useful and interesting concept. The rate of convergence means that weather the convergence is extremely slow, linearly (very fast) or arbitrarily slow. The sequence $\{T_k\}$ is said to be converges to T arbitrarily slowly if it is converges pointwise and for each sequence of real numbers $\{\phi(k)\}$ with $\lim\limits_{k \to \infty} \phi(k) = 0$, there exists $x = x_\phi \in X$ such that $\|T_k(x) - T(x)\| \geq \phi(k)$ for each k. Almost arbitrarily slow convergence is another slowest type pointwise convergence which is more general form of arbitrarily slow convergence. The "Lethargy" theorem characterize between almost arbitrarily slowly convergence and pointwise convergence of sequence of operators. This theorem establish that a sequence of linear linear operators converges almost arbitrarily slowly if and only if it converges pointwise, but not in norm.

Shock [16] gave a definition for a special class of method for obtaining approximate solutions to a particular linear operator equation. Frank Deutsch and Hein Hundal [1] defined arbitrarily slow convergence and they extended Schock's definition to a more general form and established that this Schock's definition was equivalent to almost arbitrarily slow convergence. They proved Lethargy theorem which gives useful conditions that guarantee arbitrarily slow convergence. They also observed that the Lethargy theorem implies that the classical numerical quadrature rules viz., Trapezoidal Rule, Simpson's Rule, and Gaussian quadrature all share the same types of slow convergence: namely, almost arbitrarily slowly convergence.

The notion of linear 2-normed space was introduced by S. Gahler [3]. A survey of the theory of linear 2-normed space can be found in [2]. The concept of 2-Banach space is briefly discussed in [17]. H. Gunawan and Mashadi [7] developed the generalization of a linear 2-normed space for $n \geq 2$.

In this paper, we introduce the different types of convergence of sequence of bounded b-linear functionals and establish some relationship between them in linear n-normed space. We also construct "Lethargy" theorem with the help of bounded b-linear functionals in linear n-normed space.

2 Preliminaries

Definition 2.1. [7] Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field with $\dim X \geq n$, where n is a positive integer. A real valued function $\|\cdot, \cdots, \cdot\| : X^n \to \mathbb{R}$ is called an n-norm on X if

(i) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, \cdots, x_n$ are linearly dependent,

(ii) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutations of $x_1, x_2, \cdots, x_n$,

(iii) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \, \|x_1, x_2, \cdots, x_n\| \quad \forall \alpha \in \mathbb{K}$,

(iv) $\|x + y, x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|y, x_2, \cdots, x_n\|$

hold for all $x, y, x_1, x_2, \cdots, x_n \in X$. The pair $(X, \|\cdot, \cdots, \cdot\|)$ is then called a linear n-normed space. For particular value $n = 2$, the space X is said to be a linear 2-normed space [3].

Throughout this paper, X will denote linear n-normed space over the field $\mathbb{K}$ associated with the n-norm $\|\cdot, \cdots, \cdot\|$.

Definition 2.2. [7] A sequence $\{x_k\} \subseteq X$ is said to converge to $x \in X$ if

$$\lim_{k \to \infty} \|x_k - x, e_2, \cdots, e_n\| = 0$$

for every $e_2, \cdots, e_n \in X$ and it is called a Cauchy sequence if

$$\lim_{l, k \to \infty} \|x_l - x_k, e_2, \cdots, e_n\| = 0$$

for every $e_2, \cdots, e_n \in X$. The space X is said to be complete or n-Banach space if every Cauchy sequence in this space is convergent in X. 2-Banach space [17] is a particular case of n-Banach space for $n = 2$.

Definition 2.3. [15] We define the following open and closed ball in X:

$$B_{\{e_2, \cdots, e_n\}}(a, \delta) = \{x \in X : \|x - a, e_2, \cdots, e_n\| < \delta\} \text{ and}$$

$$B_{\{e_2, \cdots, e_n\}}[a, \delta] = \{x \in X : \|x - a, e_2, \cdots, e_n\| \leq \delta\},$$

where $a, e_2, \cdots, e_n \in X$ and δ be a positive number.

Definition 2.4. [15] A subset G of X is said to be open in X if for all $a \in G$, there exist $e_2, \cdots, e_n \in X$ and $\delta > 0$ such that $B_{\{e_2, \cdots, e_n\}}(a, \delta) \subseteq G$.

Definition 2.5. [15] Let $A \subseteq X$. Then the closure of A is defined as

$$\overline{A} = \left\{ x \in X \mid \exists \{x_k\} \in A \text{ with } \lim_{k \to \infty} x_k = x \right\}.$$

The set A is said to be closed if $A = \overline{A}$.

Definition 2.6. [5] Let W be a subspace of X and $b_2, b_3, \cdots, b_n$ be fixed elements in X and $\langle b_i \rangle$ denote the subspaces of X generated by b_i, for $i = 2, 3, \cdots, n$. Then a map $T : W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{K}$ is called a b-linear functional on $W \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$, if for every $x, y \in W$ and $k \in \mathbb{K}$, the following conditions hold:

(i) $T(x + y, b_2, \cdots, b_n) = T(x, b_2, \cdots, b_n) + T(y, b_2, \cdots, b_n)$

(ii) $T(kx, b_2, \cdots, b_n) = k\, T(x, b_2, \cdots, b_n)$.

A b-linear functional is said to be bounded if there exists a real number $M > 0$ such that

$$|T(x, b_2, \cdots, b_n)| \leq M \, \|x, b_2, \cdots, b_n\| \quad \forall x \in W.$$

The norm of the bounded b-linear functional T is defined by

$$\|T\| = \inf\{M > 0 : |T(x, b_2, \cdots, b_n)| \leq M \, \|x, b_2, \cdots, b_n\| \; \forall x \in W\}.$$

The norm of T can be expressed by any one of the following equivalent formula:

(i) $\| T \| = \sup \{ | T(x, b_2, \cdots, b_n) | : \| x, b_2, \cdots, b_n \| \leq 1 \}$.

(ii) $\| T \| = \sup \{ | T(x, b_2, \cdots, b_n) | : \| x, b_2, \cdots, b_n \| = 1 \}$.

(iii) $\| T \| = \sup \left\{ \dfrac{| T(x, b_2, \cdots, b_n) |}{\| x, b_2, \cdots, b_n \|} : \| x, b_2, \cdots, b_n \| \neq 0 \right\}$.

Also, we have

$$| T(x, b_2, \cdots, b_n) | \leq \| T \| \, \| x, b_2, \cdots, b_n \| \;\; \forall \, x \in W.$$

Let X_F^* denote the Banach space of all bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ with respect to the above norm.

Some properties of bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ have been discussed in [5, 6].

Definition 2.7. [1] Let C_0 denote the collection of all real-valued functions on the positive integers $\mathbb{N}$ that converge to 0. That is,

$$C_0 = \left\{ \phi \mid \phi : \mathbb{N} \to \mathbb{R}, \lim_{k \to \infty} \phi(k) = 0 \right\}.$$

We replace this fundamental set C_0 by the more restrictive set

$$C_0' = \left\{ \phi \mid \phi : \mathbb{N} \to (0, \infty), \phi(k + 1) \leq \phi(k) \text{ for each } k, \lim_{k \to \infty} \phi(k) = 0 \right\}.$$

That is, unlike C_0, the functions in C_0' are also strictly positive and decreasing.

3 Different types of convergence

In this section, we study the different types of convergence of bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$. We establish that a sequence of bounded b-linear functionals converges almost b-arbitrarily slow convergence if and only if it converges b-pointwise, but not in norm.

Definition 3.1. A sequence $\{T_k\} \subset X_F^*$ is said to be converge to $T \in X_F^*$ in norm if $\lim_{k \to \infty} \| T_k - T \| = 0$ and it is said to be b-pointwise convergent if

$$\lim_{k \to \infty} | T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n) | = 0,$$

for each $x \in X$.

Definition 3.2. A sequence $\{T_k\}$ in X_F^* is said to be b-pointwise convergent to $T \in X_F^*$ with order $\phi \in C_0$ if for each $x \in X$, there exists a constant $C_x > 0$ such that

$$| T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n) | \leq C_x \, \phi(k),$$

for each $k \in \mathbb{N}$. Using "big O" i. e., order notation, we can saying that $\{T_k\}$ converges to T b-pointwise with order ϕ if

$$| T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n) | = O(\phi(k)),$$

for each $x \in X$.

Definition 3.3. A sequence $\{T_k\}$ in X_F^* is said to be b-linearly convergent to some $T \in X_F^*$ if there exist constants $\alpha \in [0, 1)$ and $c \in \mathbb{R}$ such that $\| T_k - T \| \leq c \, \alpha^k$ for each k.

Theorem 3.4. *Let $\{T_k\} \subseteq X_F^*$. Consider the following statements:*

(i) *$\{T_k\}$ is b-linearly converges to $T \in X_F^*$.*

(ii) $\{T_k\}$ is converges to $T \in X_F^*$ in norm.

(iii) $\{T_k\}$ is b-pointwise converges to $T \in X_F^*$ with order ϕ, for some $\phi \in C_0$.

(iv) $\{T_k\}$ is b-pointwise converges to $T \in X_F^*$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

Proof. $(i) \Rightarrow (ii)$ Suppose that $\{T_k\}$ is b-linearly converges to T. Then there exist constants $\alpha \in [0, 1)$ and $c \in \mathbb{R}$ such that $\|T_k - T\| \leq c\,\alpha^k$ for each k and this implies that

$$\lim_{k \to \infty} \|T_k - T\| \leq \lim_{k \to \infty} c\,\alpha^k = 0.$$

So, $\{T_k\}$ converges to T in norm.

$(ii) \Rightarrow (iii)$ Now, for each $x \in X$, we have

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \leq \|T_k - T\|\,\|x, b_2, \cdots, b_n\|.$$

Take $\phi(k) = \|T_k - T\|$, for each k and $c_x = \|x, b_2, \cdots, b_n\|$. Then for each $x \in X$, we have

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \leq c_x\,\phi(k) \text{ for each } k.$$

Thus, $\{T_k\}$ is b-pointwise converges to T with order ϕ, for some $\phi \in C_0$.

$(iii) \Rightarrow (iv)$ Suppose (iii) hold. Then for each $x \in X$ there exists $C_x > 0$ such that

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \leq C_x\,\phi(k) \text{ for each } k \in \mathbb{N}$$

$$\Rightarrow \lim_{k \to \infty} |T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| = 0 \text{ for each } x \in X.$$

□

In the next example, we will see that convergence in norm does not imply b-linear convergence.

Example

Let $X = \mathbb{R}^n$ be a linear n-normed space with n-norm defined by

$$\|x_1, x_2, \cdots, x_n\| = \text{abs}\left(\begin{vmatrix} x_{11} & x_{12} & \cdots x_{1n} \\ x_{21} & x_{22} & \cdots x_{2n} \\ \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1}, x_{i2}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Consider the subspaces $\langle b_2 = (0, 1, \cdots, 1)\rangle$, $\langle b_3 = (1, 0, \cdots, 1)\rangle$, $\cdots$, $\langle b_n = (1, 1, \cdots, 0, 1)\rangle$ of X generated by the fixed elements $b_2, b_3, \cdots, b_n$ in X.
Now, for each k, we define $T_k : X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle \to \mathbb{R}$ by

$$T_k\{(x_1, x_2, \cdots, x_n), b_2, \cdots, b_n\} = \frac{1}{k}\begin{vmatrix} x_1 & x_2 & \cdots x_{n-1} & x_n \\ 0 & 1 & \cdots 1 & 1 \\ 1 & 0 & \cdots 1 & 1 \\ \vdots & \ddots & \vdots & \\ 1 & 1 & \cdots 0 & 1 \end{vmatrix}$$

for all $(x_1, x_2, \cdots, x_n) \in X$.
For every $(x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n) \in W$ and $\alpha \in \mathbb{R}$,

$$T_k\{(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n), b_2, \cdots, b_n\}$$

$$= T_1 \{ (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n), b_2, \cdots, b_n \}$$

$$= \frac{1}{k} \begin{vmatrix} x_1 + y_1 & x_2 + y_2 & \cdots x_n + y_n \\ 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \ddots & \vdots \\ 1 & 1 & \cdots 0 \end{vmatrix}$$

$$= \frac{1}{k} \begin{vmatrix} x_1 & x_2 & \cdots x_n \\ 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \ddots & \vdots \\ 1 & 1 & \cdots 0 \end{vmatrix} + \frac{1}{k} \begin{vmatrix} y_1 & y_2 & \cdots y_n \\ 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \ddots & \vdots \\ 1 & 1 & \cdots 0 \end{vmatrix}$$

$$= T_k \{ (x_1, x_2, \cdots, x_n), b_2, \cdots, b_n \} + T_k \{ (y_1, y_2, \cdots, y_n), b_2, \cdots, b_n \}$$

and

$$T_k \{ \alpha (x_1, x_2, \cdots, x_n,), b_2, \cdots, b_n \}$$
$$= T_k \{ (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), b_2, \cdots, b_n \}$$

$$= \frac{1}{k} \begin{vmatrix} \alpha x_1 & \alpha x_2 & \cdots \alpha x_n \\ 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \ddots & \vdots \\ 1 & 1 & \cdots 0 \end{vmatrix} = \frac{\alpha}{k} \begin{vmatrix} x_1 & x_2 & \cdots x_n \\ 0 & 1 & \cdots 1 \\ 1 & 0 & \cdots 1 \\ \vdots & \ddots & \vdots \\ 1 & 1 & \cdots 0 \end{vmatrix}$$

$$= \alpha T_k \{ (x_1, x_2, \cdots, x_n), b_2, \cdots, b_n \}.$$

Also, for each k, we have

$$\| T_k \| = \sup_{\| (x_1, x_2, \cdots, x_n), b_2, \cdots, b_n \| = 1} | T_k \{ (x_1, x_2, \cdots, x_n), b_2, \cdots, b_n \} | = \frac{1}{k}.$$

This shows that for each k, T_k is a bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ with $\| T_k \| = \frac{1}{k}$. Thus $\{ T_k \}$ converges to 0 in norm. If $\{ T_k \}$ b-linearly converged to 0, there exist constants $\alpha \in [0, 1)$ and $c \in \mathbb{R}$ such that $\| T_k \| \leq c \alpha^k$ for each k. Then for each k, $1 \leq c k \alpha^k$. Since $\alpha \in [0, 1)$, the right side of this inequality converges to 0, which is a contradiction.

Theorem 3.5. *Let X be a n-Banach space and $\{ T_k \} \subseteq X_F^*$. Then $\{ T_k \}$ converges to $T \in X_F^*$ in norm if and only if $\{ T_k \}$ converges to T b-pointwise with order ϕ for some $\phi \in C_0$.*

Proof. Suppose that $\{ T_k \}$ converges to T b-pointwise with order ϕ for some $\phi \in C_0$. For each $m \in \mathbb{N}$, we define

$$A_m = \{ x \in X : | T_k (x, b_2, \cdots, b_n) - T (x, b_2, \cdots, b_n) | \leq m \phi(k) \; \forall \, k \in \mathbb{N} \}.$$

By definition of b-pointwise convergence with order ϕ, it follows that for each $x \in X$ is in A_m for some m. So, we can write $X = \bigcup_{m=1}^{\infty} A_m$. Now, it can be easily seen that A_m is closed for each m. By the Baire Category theorem for n-Banach space, there exists $m_0 \in \mathbb{N}$ such that F_{m_0} is not nowhere dense in X i.e., A_{m_0} has nonempty interior. So, there exists a non-empty open ball $B_{\{ e_2, \cdots, e_n \}} (x_0, \delta)$ such that $B_{\{ e_2, \cdots, e_n \}} (x_0, \delta) \subset A_{m_0}$. That is, for all $x \in B_{\{ e_2, \cdots, e_n \}} (x_0, \delta)$

$$| T_k (x, b_2, \cdots, b_n) - T (x, b_2, \cdots, b_n) | \leq m_0 \phi(k)$$

The above can be written in the form

$$\left|T_k - T\left(B_{\{e_2,\cdots,e_n\}}(x_0,\delta), b_2, \cdots, b_n\right)\right| \leq m_0\,\phi(k).$$

Now,

$$\begin{aligned}
&x_0 + \delta B_{\{e_2,\cdots,e_n\}}(0,1)\\
&= \left\{x \in X : x = x_0 + \delta a,\ a \in B_{\{e_2,\cdots,e_n\}}(0,1)\right\}\\
&= \{x \in X : x = x_0 + \delta\, a,\ \|a, e_2, \cdots, e_n\| < 1\}\\
&= \left\{x \in X : \left\|\frac{x - x_0}{\delta}, e_2, \cdots, e_n\right\| < 1\right\}\\
&= \{x \in X : \|x - x_0, e_2, \cdots, e_n\| < \delta\} = B_{\{e_2,\cdots,e_n\}}(x_0,\delta)\\
&\Rightarrow B_{\{e_2,\cdots,e_n\}}(0,1) = \frac{B_{\{e_2,\cdots,e_n\}}(x_0,\delta) - x_0}{\delta}.
\end{aligned}$$

Clearly,

$$|T_k(x_0, b_2, \cdots, b_n) - T(x_0, b_2, \cdots, b_n)| \leq m_0\,\phi(k).$$

This implies that

$$|T_k - T\,(x_0, b_2, \cdots, b_n)| \leq m_0\,\phi(k).$$

Now,

$$\begin{aligned}
&\left|T_k - T\left(B_{\{e_2,\cdots,e_n\}}(0,1), b_2, \cdots, b_n\right)\right|\\
&= \left|T_k - T\left(\frac{B_{\{e_2,\cdots,e_n\}}(x_0,\delta) - x_0}{\delta}, b_2, \cdots, b_n\right)\right|\\
&= \left|\frac{1}{\delta}\,(T_k - T)\left(B_{\{e_2,\cdots,e_n\}}(x_0,\delta) - x_0, b_2, \cdots, b_n\right)\right|\\
&\leq \frac{1}{\delta}\left\{\left|T_k - T(B_{\{e_2,\cdots,e_n\}}(x_0,\delta), b_2, \cdots, b_n)\right| + |T_k - T(x_0, b_2, \cdots, b_n)|\right\}\\
&\leq \frac{2\,m_0}{\delta}\,\phi(k),\ \ \forall\, T \in \mathcal{A}.
\end{aligned}$$

Therefore, for all $T \in \mathcal{A}$, we have

$$|T_k - T(x, b_2, \cdots, b_n)| \leq \frac{2\,m_0}{\delta}\,\phi(k),\ \ \forall\, x \in B_{\{e_2,\cdots,e_n\}}(0,1).$$

It follows that, for each k, we have

$$\begin{aligned}
\|T_k - T\| &= \sup\left\{|T_k - T(x, b_2, \cdots, b_n)| : x \in B_{\{e_2,\cdots,e_n\}}(0,1),\ \forall\, T \in \mathcal{A}\right\}\\
&\leq \frac{2\,m_0}{\delta}\,\phi(k),\ \ \forall\, T \in \mathcal{A}.
\end{aligned}$$

Converse part follows from the implication $(ii) \Rightarrow (iii)$ of the Theorem (3.4).
This completes the proof. □

Definition 3.6. A sequence $\{T_k\}$ in X_F^* is said to be b-arbitrarily slowly convergent to $T \in X_F^*$ if the following two conditions hold:

(i) $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$ for each $x \in X$.

(ii) For each $\phi \in C_0$, there exists $x = x_\phi$ such that

$$|T_k\,(x, b_2, \cdots, b_n) - T\,(x, b_2, \cdots, b_n)| \geq \phi(k) \text{ for each } k \in \mathbb{N}.$$

If the second condition holds for infinitely many $k \in \mathbb{N}$, then the sequence $\{T_k\}$ is said to be almost b-arbitrarily slowly convergent to T.

Theorem 3.7. *A sequence $\{T_k\}$ in X_F^* converges to $T \in X_F^*$ b-arbitrarily slowly if and only if*

(i) $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$ *for each* $x \in X$ *and*

(ii) *For each* $\psi \in C_0'$*, there exists* $x = x_\psi$ *such that*

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \geq \psi(k) \text{ for each } k \in \mathbb{N}.$$

Proof. Form definition (2.7) of C_0 and C_0', it can be easily verify that $C_0' \subset C_0$. Let $\phi \in C_0$. Now, we define $\psi : \mathbb{N} \to (0, \infty)$ by

$$\psi(k) = \max\left\{\frac{1}{k}, \sup_{i \geq n} \phi(i)\right\}.$$

Then for each k, $0 < \psi(k+1) \leq \psi(k)$ and $\lim_{k \to \infty} \psi(k) = 0$. This show that $\psi \in C_0'$ and $\psi(k) \geq \phi(k)$ for all k. Using this above fact, the proof of this theorem is easily verified. □

Remark 3.8. A sequence $\{T_k\}$ in X_F^* converges to $T \in X_F^*$ almost b-arbitrarily slowly if and only if

(i) $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$ for each $x \in X$ and

(ii) For each $\psi \in C_0'$, there exists $x = x_\psi$ such that

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \geq \psi(k) \text{ for infintely many } k \in \mathbb{N}.$$

Definition 3.9. A sequence $\{T_k\}$ in X_F^* is said to be b-Schock slowly convergent to $T \in X_F^*$ if the following two conditions hold:

(i) $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$ for each $x \in X$.

(ii) For each $\phi \in C_0'$, there exists $x = x_\phi$ such that

$$\limsup_k \left(\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)}\right) = \infty.$$

Theorem 3.10. *Let $\{T_k\} \subseteq X_F^*$. Then the following statements are equivalent:*

(i) *$\{T_k\}$ converges to T almost b-arbitrarily slowly.*

(ii) *$\{T_k\}$ converges to T b-pointwise and for each $\phi \in C_0'$, there exists $x = x_\phi$ such that*

$$\limsup_k \left(\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)}\right) > 0.$$

(iii) *$\{T_k\}$ converges to T b-Schock slowly.*

Proof. $(i) \Rightarrow (ii)$ Suppose (i) holds. Then by remark (3.8),

(i) $T_k(x, b_2, \cdots, b_n) \to T(x, b_2, \cdots, b_n)$ for each $x \in X$ and

(ii) For each $\phi \in C_0'$, there exists $x = x_\phi$ such that

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \geq \phi(k) \text{ for infinitely many } k \in \mathbb{N}.$$

This implies that

$$\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)} > 1$$

for infinitely many $k \in \mathbb{N}$. Therefore,

$$\limsup_k \left(\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)}\right) \geq 1 > 0.$$

Thus (ii) holds.

$(ii) \Rightarrow (iii)$ Suppose (ii) holds. Then for each $\phi \in C_0'$, $\psi = \sqrt{\phi}$ is also in C_0', so there exists $x = x_\psi \in X$ such that

$$\alpha_x = \limsup_k \left(\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\sqrt{\phi(k)}} \right) > 0.$$

This shows that

$$\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\sqrt{\phi(k)}} \geq \frac{\alpha_x}{2}$$

for infinitely many $k \in \mathbb{N}$, which implies that

$$\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)} \geq \frac{\alpha_x}{2\sqrt{\phi(k)}}$$

for infinitely many $k \in \mathbb{N}$. Since $\phi(k) \to 0$ as $k \to \infty$, we get that

$$\limsup_k \left(\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)} \right) = \infty$$

and therefore (iii) holds.

$(iii) \Rightarrow (i)$ Assume that (iii) holds. Then for each $\phi \in C_0'$, there exists $x = x_\phi$ such that

$$\limsup_k \left(\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)} \right) = \infty.$$

In particular, we get that

$$\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)} \geq 1$$

for infinitely many $k \in \mathbb{N}$ and this implies that

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \geq \phi(k) \text{ for infinitely } k \in \mathbb{N}.$$

Thus (i) holds. This completes the proof. □

In next theorem, we discuss a characterization of almost b-arbitrarily slow convergence in terms of b-pointwise convergence with order ϕ.

Theorem 3.11. *The following statements are equivalent:*

(i) $\{T_k\}$ *converges to* T *b-pointwise but not almost b-arbitrarily slowly.*

(ii) $\{T_k\}$ *converges to* T *b-pointwise with order* ϕ *for some* $\phi \in C_0$.

Proof. $(i) \Rightarrow (ii)$ Suppose (i) holds. Then there exists $\phi \in C_0$ such that for each $x \in X$,

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| < \phi(k)$$

for sufficiently large k (depending on x). This shows that for each $x \in X$, there exists $k_x \in \mathbb{N}$ such that

$$\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)} < 1 \quad \forall\, k \geq k_x.$$

Take

$$\alpha_x = \max_{1 \leq k \leq k_x} \left(\frac{|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)|}{\phi(k)} \right)$$

and $C_x = \max\{1, C_x\}$. Then

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \leq C_x \phi(k) \text{ for each } k \in \mathbb{N}.$$

This shows that $\{T_k\}$ converges to T b-pointwise with order ϕ.

$(ii) \Rightarrow (i)$ Suppose that $\{T_k\}$ converges to T b-pointwise with order ϕ for some $\phi \in C_0$. Then for each $x \in X$, there exists a constant $C_x > 0$ such that

$$|T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| \leq C_x \phi(k) \text{ for each } k \in \mathbb{N} \tag{3.1}$$

$$\Rightarrow \lim_{k \to \infty} |T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n)| = 0 \text{ for each } x \in X.$$

This shows that $\{T_k\}$ converges to T b-pointwise. If possible suppose that $\{T_k\}$ converges to T almost b-arbitrarily slowly. Now, we define $\psi : \mathbb{N} \to (0, \infty)$ by $\psi(k) = \sqrt{\phi(k) + \frac{1}{k}}$. It is easy to verify that $\psi \in C_0$. Now from the definition of almost b-arbitrarily slowly convergence, there exists $x_\psi \in X$ such that

$$|T_k(x_\psi, b_2, \cdots, b_n) - T(x_\psi, b_2, \cdots, b_n)| \geq \phi(k) \tag{3.2}$$

for infinitely many k. Combining (3.1) and (3.2), we get

$$\sqrt{\phi(k) + \frac{1}{k}} \leq C_x \phi(k) \text{ for infinitely many } k.$$

This implies that

$$1 \leq C_x \sqrt{\frac{1}{\frac{1}{\phi(k)} + \frac{1}{k(\phi(k))^2}}} \text{ for infinitely many } k.$$

Since $\lim_{k \to \infty} \phi(k) = 0$ from above we see that $1 \leq 0$, which is absurd. This contradiction shows that $\{T_k\}$ does not converges to T almost b-arbitrarily slowly. □

Theorem 3.12. *The following statements are equivalent:*

(i) $\{T_k\}$ *converges to* T *almost b-arbitrarily slowly.*

(ii) $\{T_k\}$ *converges to* T *b-pointwise but not b-pointwise with order* ϕ *for any* $\phi \in C_0$.

Proof. The statements of this theorem is the contrapositives statements of Theorem (3.11) and the proof follows from Theorem (3.11). □

In the next theorem, we conclude that when the domain space X is finite-dimensional, then b-pointwise convergence and norm convergence are equivalent.

Theorem 3.13. *Let* X *be a finite dimensional linear n-normed space and* $\{T_k\} \subseteq X_F^*$. *Then* $\{T_k\}$ *converges to* T *b-pointwise if and only if it converges in norm.*

Furthermore, b-arbitrarily slow convergence or almost b-arbitrarily slow convergence is never possible when X *is finite dimensional.*

Proof. Let X be a finite dimensional linear n-normed space with dimension $d \geq n$ and $\{e_1, e_2, \cdots, e_d\}$ be a basis for X. Then $x = \sum_{i=1}^{\infty} a_i e_i$, where $a_i \in \mathbb{R}$ for $i = 1, 2, \cdots, d$. Consider the bounded b-linear functionals $f_1, f_2, \cdots, f_d$ defined on $X \times \langle b_2 \rangle \times \cdots \times \langle b_n \rangle$ such that

$$f_i(e_j, b_2, \cdots, b_n) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \ 1 \leq i, j \leq d \end{cases}$$

Then

$$f_i(x, b_2, \cdots, b_n) = f_i\left(\sum_{j=1}^{d} a_j e_j, b_2, \cdots, b_n\right)$$

$$= \sum_{j=1}^{d} a_j f_i(e_j, b_2, \cdots, b_n) = a_i, (1 \leq i \leq d).$$

Thus, for each $x \in X$, we have

$$x = \sum_{i=1}^{\infty} f_i(x, b_2, \cdots, b_n) e_i.$$

Let us now suppose that $\{T_k\}$ converges to T b-pointwise. We define for each $x \in X$,

$$L_k(x, b_2, \cdots, b_n) = T_k(x, b_2, \cdots, b_n) - T(x, b_2, \cdots, b_n).$$

Then it is easy to verify that L_k is a bounded b-linear functional with

$$\lim_{k \to \infty} L_k(x, b_2, \cdots, b_n) = 0 \text{ for each } x \in X.$$

In particular,

$$\lim_{k \to \infty} L_k(e_i, b_2, \cdots, b_n) = 0 \text{ for each } i = 1, 2, \cdots, d.$$

Let

$$\alpha(k) = \max_{1 \leq i \leq d} |L_k(e_i, b_2, \cdots, b_n)|.$$

Then $\lim_{k \to \infty} \alpha(k) = 0$. Now, for each $x \in X$, we have

$$|L_k(x, b_2, \cdots, b_n)| = \left|L_k\left(\sum_{i=1}^{d} f_i(x, b_2, \cdots, b_n) e_i, b_2, \cdots, b_n\right)\right|$$

$$= \left|\sum_{i=1}^{d} f_i(x, b_2, \cdots, b_n) L_k(e_i, b_2, \cdots, b_n)\right|$$

$$\leq \sum_{i=1}^{d} |f_i(x, b_2, \cdots, b_n)| |L_k(e_i, b_2, \cdots, b_n)|$$

$$\leq \alpha(k) \sum_{i=1}^{d} |f_i(x, b_2, \cdots, b_n)| \leq \alpha(k)\left(\sum_{i=1}^{d} \|f_i\|\right) \|x, b_2, \cdots, b_n\|.$$

Therefore,

$$\|T_k - T\| = \|L_k\| \leq \alpha(k)\left(\sum_{i=1}^{d} \|f_i\|\right) \to 0 \text{ as } k \to \infty.$$

Thus, $\{T_k\}$ converges to T in norm.

Conversely, it is easy to verify that norm convergence of $\{T_k\}$ to T, implies its b-pointwise convergence.

To prove the last part, suppose that $\{T_k\}$ converges to T almost b-arbitrarily slowly. Then $\{T_k\}$ must converges to T b-pointwise and by first part of the proof $\{T_k\}$ must converges to T in norm. According to Theorem (3.4), $\{T_k\}$ converges to T b-pointwise with order ϕ for some $\phi \in C_0$ which is a contradiction of Theorem (3.12). Hence, for finite dimensional linear n-normed space, b-arbitrarily slow convergence or almost b-arbitrarily slow convergence is never possible. □

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