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THE TIMELESSNESS OF MATHEMATICAL STUDY: CASSIUS J. KEYSER 100 YEARS ON

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Abstract This short essay marks the centenary of a paper authored by C.J. Keyser who, in pondering the various strands of a career in mathematics in 1923, raised themes that are still very much relevant to our tertiary sector—in particular, his intuitions underscore something of the ageless nature of mathematical study.

"To humanize the teaching of mathematics means so to present the subject, so to interpret its ideas and doctrines, that they shall appeal, not merely to the computatory faculty or to the logical faculty but to all the great powers and interests of the human mind."

C.J.K. (1912)

1 Short Biography

Cassius Jackson Keyser (b. 15th May, 1862; d. 8th May, 1947) was an American with an enduring interest in mathematical philosophy. Keyser's initial higher education took place at North West Ohio Normal School (now Ohio Northern University), after which he became a school teacher and principal. In 1885 he married a fellow student at the Normal School, completing a second undergraduate degree (a B.Sc.) at the University of Missouri in 1892. He taught there, and also at the New York State Normal School (now S.U.N.Y. New Paltz) and at Washington University, subsequently enrolling as a graduate student at Columbia University where he earned an M.A. in 1896 and Ph.D. in 1901. He spent the rest of his working life at Columbia, becoming Chair of the Department of Mathematics (1910–1916) during his tenure (until retirement) as the Adrain Professor of Mathematics (1904–1927).

Keyser was one of the first Americans to appreciate new and emerging directions in the foundations of mathematics that were being heralded through the ideas of contemporary Europeans Dedekind, Cantor, Peano, Poincaré, Hilbert, Zermelo, Russell, Whitehead, and others. He joined the editorial board of the *Hibbert Journal*—making regular contributions to it and other philosophical journals—in addition to which he acted as Associate Editor of *Scripta Mathematica* and served in 1906 as the Mathematics Editor for the *Encyclopedia Americana*. Together with his Columbia colleague John Dewey, Keyser helped establish the American Association of University Professors (A.A.U.P.) which still thrives today. He was a fellow of the American Academy for the Advancement of Science, and a member of the American Mathematical Society. With just two doctoral students to his name Keyser had no significant research profile of which to boast, and—while he did write on some technical matters and was more than competent enough to teach—he embraced mathematics holistically which produced a drive to write on a range of topics grounded in the philosophy, humanity, spirit and ethos of the discipline. He was a forerunner of the concept of general semantics—which examines substantive structures of mathematics and the sciences, and attempts to apply them to human interactions—influencing names such as John von Neumann and Alfred H.S. Korzybski. With over fifty works to his name, a group of his colleagues and friends formed The Society of Friends of Cassius Jackson Keyser whose purpose was to disseminate the remainder of Keyser's unpublished essays; the first volume was to have been presented at his 85th birthday (which occurred just a week after his death).

Let us see the words of a fine champion of mathematics, writing on affairs close to his heart. If they cause anyone to consult his works further, then this paper may be deemed to be in some way a success for that reason alone.[1]

2 Context and Quotes

Based on an unshakeable view that mathematics is an instrument of acute importance to individual growth and societal enrichment, he wrote extensively throughout his life on its function in personal human development and wider civilised culture aside from its obvious utility. His style was one of unabashed flamboyance in delineating his passion for the virtues and authenticity of mathematics—explaining its critical contributions to personhood and community at large—and he often exhibited a verbose freedom indicative of the period but with his own imprint regaling all. Occasionally, Keyser's narrative would take on a more pragmatic mood, and the article from which the below is drawn is a good example of that. Titled 'Mathematics as a Career', it is 100 years since its appearance in *The Scientific Monthly* (Vol. 17, pp. 489–497) where he gave the topic over to sections chronologically headed 'Interest and Ability', 'Mathematical Research Ability and its Test', 'The Nature, Scope, Vitality and Dignity of Mathematics', 'The Period of Preparation', and 'The Mathematician's Rewards'. As a mathematician myself, I find the language of his expressive manner emphasises how the world has changed since then, but the 1923 piece also confirms that mathematics has a timeless quality in certain primary aspects; this short offering is a place where I can share a sense of its constancy while advertising Keyser as a prolific author worth reading. Words are taken initially from the first of the aforementioned sections in which he weighs the tensions between, and misreading of, ability and interest in the subject that can arise in young minds, offering wise guidance and advice that stand the test of time. We then capture his penultimate passage as he opines on the idea of the doctorate, considering what it entails and with what it equips the graduate student.

2.1 On Undergraduate Study

> "In speaking of mathematical ability I shall mean *native* mathematical ability, in accord with the proper sense of the familiar saying that the mathematician is *born*, not made. Every one knows that mathematical ability and interest in mathematical study often go together. And it is commonly believed that the two things are always associated in such a way that intensity of interest implies a high degree of ability, and that superior ability implies corresponding interest (actual or potential). But that belief is erroneous. Every experienced teacher of mathematics knows very well that a student whose mathematical ability is meagre may yet feel and manifest a lively interest in mathematical studies. On the other hand, it occasionally happens that a student having indubitably great mathematical gifts has relatively slight interest in the subject. What I have said of mathematics is equally true of every other form or field of activity.
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> It is thus evident that what may for a time seem to a student to be an inner summons to mathematics or another subject may or may not be a genuine call thereto. Such a summons requires to be scrutinized carefully, for the choice of a vocation is a momentous act. I do not mean that choice should be deferred in the hope of resolving all doubt, for in such matters complete elimination of uncertainties is impossible. Yet one must decide or drift. But, although the decision must be adventurous, the adventure need not be rash: it may await, not certainty, but the weighing of probabilities.
>
> I believe it safe to say that, with very rare exceptions, a student may not wisely choose mathematics for a vocation unless he has asked, with respect to himself, and has been able to answer affirmatively, the following questions: Is my seeming interest in mathematics genuine? Is it an abiding interest? Does it exceed my interest in every other subject? Is it deep and strong enough to stimulate my powers to their highest possible

[1]The reader may be interested to see the recent essay by this author, 'Reflections On What Mathematics Is and Isn't: Halmos, Keyser, and Others' (*Palest. J. Math.*, Vol. 11, No. 3, pp. 664–699 (2022)), in which Keyser—set principally alongside another major mathematical expositor, Paul R. Halmos—is revealed at length in all of his glory, as it were.

activity and cause mathematical work to be for me, not irksome toil, but a labor of love? Have I sufficient mathematical ability to read a solid mathematical work with fair facility and good understanding? Does my mathematical ability exceed my natural ability in every other subject? Do I possess in fair measure the natural gifts essential to the qualifications of a successful teacher of high-school pupils or college students or university undergraduates?

Of the foregoing questions the first one is intended to guard the student against the danger of mistaking for interest in mathematics such transitory delights as a bright student is likely to have in class-room competitions under the quickening influence of a live teacher. If a student have genuine interest in the subject, he will know it; but if his "interest" be spurious, he may be deceived.

The last one of the questions appears to be justified in view of the fact that in our country young men aspiring to a career in mathematics have seldom been able to escape the necessity, even when they have desired to do so, of giving a good deal of their time and energy to the work of undergraduate instruction, and that relief from such necessity does not seem probable.

Ordinarily, a student may not hope to arrive at trustworthy affirmative answers before his studies have advanced far enough at least to include [certain] substantial courses ... In any case he should seek the counsel of his instructors, especially if they be candid men of mathematical reputation and good judgment." (pp. 489–490).

A century on, nobody could doubt that these observations and insights apply today still. He continued with

"A more difficult problem presents itself in the case of those rare students who have no decisive predilection for a single subject but have talents and interests qualifying them well and equally for adventure in any one of two or more great fields. It is a pity that such a student can not wisely make it his vocation to cultivate all the fields at once or in succession. There is always something tragic in having to specialize, for in a profound sense the subject of all science is one whole. But the whole is too vast and complicated for the limited powers of one man; whence the necessity for division and for concentration upon a fragment—a necessity whose tragic quality is felt with special keenness by a student of diversified gifts and interests. The fields in question may be or seem to be widely sundered, as geology and linguistics, for example; or they may be obviously adjacent, closely related, interpenetrating, as physics and chemistry, for example, or zoology and botany, or philosophy and mathematics. If the fields be intimately related a student of the mentioned type may aim at a career in two or more of them combined *provided* he be endowed with a measure of genius like that of Helmholtz, for example, or Henri Poincaré, who won eminent distinction in astronomy, in physics and in mathematics. But men of such capabilities are exceedingly rare. Ordinarily, a student whose tastes and talents qualify him well and equally for two or more important subjects ought to choose one of them definitely and resolutely as a *vocation*, reserving the others not less definitely and resolutely as *avocations*; for ordinarily such a decision will be most favorable to health and happiness, to depth and breadth of culture, to good citizenship in the commonwealth of science, and to the service of mankind." (pp. 490–491),

where he was remarking on a familiar conundrum for students that has resulted in combined joint and major-minor honours programmes of study, and their like, around the world.

2.2 On Doctoral Study

"What is to be said under this caption with reference to mathematics applies quite as well to every other cardinal subject. The period of preparation usually includes two or three years of university residence devoted to what is called graduate study subsequent to graduation from college. During these years the student will be in fact, if not officially, a candidate for the degree of doctor of philosophy, and his period

> of preparation for a scientific career will usually terminate when he has gained the degree. To gain the degree he must produce a dissertation embodying the result of fairly independent and somewhat original work and must pass an examination, which may be oral or written or both, in the general field (or fields) of his studies.
>
> The gaining of the doctorate is not regarded as conclusive evidence that the student has research ability in the major meaning of the term ... It is regarded, and rightly regarded, as signifying that the student has attained a fairly high degree of scholarly competence in his field of study and that he possesses research ability in at least the minor meaning of the term.[2] On this account the fact of having won the doctorate is distinctly helpful, and in some instances is even essential, in obtaining a position, especially a college or university appointment, and thereafter in obtaining promotion. It is true that not all doctors of philosophy are scientifically productive; on the other hand, some of the most productive scholars within and without the universities have not held the doctorate; it is also true that pursuit of scholarship and pursuit of a degree, though they are compatible, are not the same. Nevertheless, in view of the prevailing practice, a student aspiring to a university career in mathematics or another subject will find it advantageous to make whatever sacrifice may be necessary for gaining the doctorate." (pp. 495–496).

I wholeheartedly agree with Keyser that the completion of a doctorate merely announces that the candidate has completed basic training, so to say, and is ready to embark on a research path if conditions are favourable and the ability, will and application is there—it is akin to moving from a provisional to a full driving licence which grants access to the busy roads ahead and a place among scores of other vehicles, but no more than that.

A further point if I may, motivated by the above. An ex colleague of mine (Dave French, now no longer with us) was an excellent lecturer at his best, cared for his students, and was very knowledgeable mathematically. He had never written a Ph.D. thesis (largely due to the environment in which he had worked during the 1960s through to the 1990s), nor published anything until we began a fruitful collaboration which ran for a decade or so before he succumbed to illness. He would currently be regarded by many as slightly second rate, but nothing could be further from the truth as I found out when I began to work with him and discovered just how theoretically able he was and how informed he kept himself across many parts of mathematics. To contextualise this a little further, there is a contrast between him—and I've known one or two others of similar type—and today's hyperfocused specialist who is narrow in research, neglects (or feels pressurised to relegate) both teaching duties and pastoral oversight of students in favour of maintaining that all important research visibility, and who never allows themself to think about (or shows no interest in reflecting upon) mathematics as just one part of a complex world wherein it has an ever evolving role and engenders recognised emotions and behaviours in its practitioners; one feels this last kind of person is missing out on something really quite pivotal to individual growth beyond the confines of the subject itself.

3 Concluding Remarks

Keyser was a dedicated advocate of what he called 'humanistic' education—whose aim was the "attainment of excellence in the great matters that constitute our common humanity"—over and above 'industrial' or 'vocational' education geared towards practicalities of purpose ("... [paradigms that] differ widely in the conceptions at the heart of them and in the motives that actuate them."). While acknowledging the latter a necessity *per se*, it was humanistic education—for which there could be no substitute in Keyser's eyes—that was central to those paragons of behaviour and mindset towards which everyone should aspire in the name of mankind as it afforded the opportunity to undergo an educational experience that was also aesthetic and spiritual—'whole', in other words, nurturing soul and sensibilities through rewards diffuse and varied. Core to this philosophy was the need to think rationally and clearly, and Keyser believed

[2]This refers to work which, though of merit and some (albeit limited) inventiveness, "requires neither creative genius nor a [particularly] high order of talent". Major ability is, according to Keyser, much rarer and is the business of those mathematicians who achieve greatness in the power and level of their results; his divide between the ordinary and the exceptional is quite unambiguous and intentional.

that mathematics was where this bedrock of logic and intellectual rigour is found—a resource of immeasurable and incontestable value in shaping those "numerous instincts, impulses, propensities and powers" that we all share as a species (the author feels it is helpful to provide, in an appendix, a first hand account of how mathematics fitted into Keyser's analytic model of humanistic education, drawing (as we have just) on a 1922 essay of his.

The modern academic always has to fall in line with those credit systems we have created for ourselves, and the madness we currently live with—which unceasingly demands publications and so called 'impact' as key (almost sole) metrics of our professional worth—is not one that suffocated those of Keyser's era. As a man who enjoyed life and devoted much time to his work (he is said to have been "full of fun and nonsense", and a fiddler who gave up the instrument apparently because it would rob him of time from his beloved mathematics), I salute C.J. Keyser who penned his deepest thoughts for us—writing sometimes with animated exuberance, ebullience and buoyancy, and at other times with a serious solemnity, moral probity and earnest sobriety, but always with honesty, vivacity, passion and integrity; his legacy in print is a rich one.

In attempting to map out some of the unchartered borderland between mathematics and philosophy, Cassius Keyser did have his critics—being accused of writing in an antiquarian manner laden with overly repetitive phrases and with descriptive adjectives piled high at the expense of clarity. Yes, he may have structured his discourse around convoluted and lengthy sentences (a sign, perhaps, that he felt assured in his position among a coterie of highbrow elites), but I, for one, for the most part find his approach to both language and argument engaging and entertaining in equal measure. The messages he sent out, over many years, were the result of (to paraphrase one reviewer of his 1916 collection of essays (see Footnote 3)) the relentless air thrashing of windmills with which he continually fought duels. For Keyser the fundamental nature, the seriousness, the merit, the harmony, and the desirability, of humanistic education formed his predominating thesis. These were things that ran through his veins—he could never leave them alone, nor ignore the prominence of mathematics as a force for human good.

A lovely article by Angus E. Taylor ('Convention and Revolt in Mathematics', *The Math. Teach.*, Vol. 55, pp. 2–9)—readable, informative and erudite—reminds us of both the doubt and confidence in the minds of those who are charged with the superintendence of mathematics, and in possession of a wish to perfect the flow of communication and ideas across a large and diverse assemblage of learners. The following words of his, from this 1962 piece, might well have been written today:

> "There are, it seems to me, several things which it is essential to recognize. First, change is inevitable in mathematics itself and in the teaching of mathematics at any particular level. Second, although the urge to change is usually resisted somewhat, and perhaps a great deal, the resistance to change is not all of one complexion, and it is certainly wrong to label such resistance uniformly as bad. Some of the resistance certainly stems from pure adherence to custom and convention—from human inertia. Some of it stems from fear and ignorance of the new ways which are in prospect. But some of it also stems from the wisdom of experience and from a desire to moderate the forces of revolt. Third, nothing is perfect—neither the outmoded present nor the shining promise of tomorrow.
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> And one final thing must be said: *We must never lose our confidence in the future. We must believe in the certainty of endless vistas for mathematics. And, as we recognize the energies and abilities of the torchbearers of the new, we must welcome the light they bring.*" (p. 9);

Keyser's own considerations and meditations seem to have had some of the attributes of a different kind of torchbearer, even if he didn't describe himself as such—one with a voice whose faint and insistent echoes are still audible if we listen carefully enough.

"I venture to say, regarding time and extent of courses, that, for pupils of fair talent, a collegiate freshman year or even a high school senior year of geometry and algebra, if the subjects be administered in the true mathematical spirit, with due regard to precision of ideas and to the exquisite beauty of perfect demonstration, is sufficient to give a fair vision of the ideal and standard of sound thinking."

C.J.K. (1922)

Appendix: Keyser 1922

In *The Mathematics Teacher* Keyser published an article, with title ‘The Human Worth of Rigorous Thinking’ (Vol. 15, pp. 1–5), where he explained the basis of what he called humanistic education. It is worth adding extra context from this 1922 paper—supplementing the main body of the essay and quoted briefly at the start of Section 3—in order to see more clearly how mathematics is situated within Keyser’s mental framework of education, and to appreciate the prominence he gives to it as something that directly plays into life itself.

Early on, he sets down some basic facts underpinning the ethos of humanistic education:

> “. . . the nature of our common humanity is fairly well characterized by saying that human beings as such possess in some recognizable measure such marks as the following: a sense for language, for expression in speech—the literary faculty; a sense for the past, for the value of experience—the historical faculty; a sense for the future, for natural law, for prediction—the scientific faculty; a sense for fellowship, cooperation, and justice—the political faculty; a sense for the beautiful—the artistic faculty; a sense for logic, for rigorous thinking—the mathematical faculty; a sense for wisdom, for world harmony, for cosmic understanding—the philosophical faculty; and a sense for the mystery of divinity—the religious faculty. Such are the evident tokens and the cardinal constituents of that which in human beings is human. It is essential to note that to each of these senses or faculties there corresponds a certain type of distinctively human activity—a kind of activity in which all human beings, whatever their stations or occupations, are obliged to participate. Like the faculties to which they correspond, these types of activity, though they are interrelated, are yet distinct. Each of them has a character of its own. Above each of the types of activity there hovers an *ideal* of excellence—a guardian angel wooing our loyalty, with a benignant influence superior to every compulsive force and every authority that may command. Nothing more precious can enter a human life than a vision of those angels, and it is the revealing of them that humanistic education has for its function and its aim. Stated in abstract terms the principle is this: Each of the great types of distinctively human activity owns an appropriate standard of excellence; it is the aim and function of humanistic education to lead the pupil into a clear knowledge of these standards and to give him a vivid and abiding sense of their authority in the conduct of life.” (pp. 1–2),

asserting that industrial/vocational education is, rightly conceived, compatible with, and complementary to,

> “. . . the humanistic [counterpart]; it may breathe the humanistic spirit; the two varieties of education are essential to constitute an ideal whole, for human beings possess both individuality and the common humanity of man. Industrial education, when thus regarded as supplementary to humanistic education, is highly commendable; but when it is viewed as an equivalent for the latter or as a good-enough substitute for it, it is ridiculous, vicious and contemptible. For the fact must not be concealed that a species of education which, in producing the craftsman, neglects the man, is, in point of kind and principle, precisely on a level with that sort of training which teaches the monkey and the bear to ride a bicycle or the seal to balance a staff upon its nose or to twirl a disc.” (p. 3),

having written that (pp. 2–3), being

> “. . . directly and primarily concerned with our individualities, . . . it might more properly be called individualistic education. It regards the world as an immense camp of industries where endlessly diversified occupations call for special propensities, gifts and training. And so its aim, its ideal, is to detect in each youth as early as may be the presence of such gifts and propensities as tend to indicate and to qualify him for some specific form of calling or bread-winning craft; then to counsel and guide him in the direction thereof; and finally, by way of education, to teach him those things which, in the honorable sense of the phrase, constitute “the tricks of the trade.” ”.

He then introduces the notion of thought:

> "It is plain that *one* of the great types of distinctively human activity—perhaps the greatest of all the types—is what is known as Thinking. It consists in the handling of ideas as ideas—the forming of concepts, the combining of concepts into higher and higher ones, discerning the relations among concepts, embodying these relations in the forms of judgments or propositions, ordering these propositions in the construction of doctrine regarding life and the world. ... Thinking is not indeed essential to *life*, but it is essential to *human* life. All men and women as human beings are inhabitants of the *Gedankenwelt*—they are native citizens, so to speak, of the world of ideas, the world of thought. They must think in order to be human." (p. 3),

adding

> "And now what shall we say is the ideal of excellence that hovers above thought-activity? What is the angel that woos our loyalty to what is best in that? What is the muse of life in the great art of thinking? An austere goddess, high, pure, serene, cold towards human frailty, demanding perfect precision of ideas, perfect clarity of expression, and perfect allegiance to the eternal laws of thought. In mathematics the name of the muse is familiar: it is Rigor—Logical Rigor—which signifies a kind of silent music, the still harmony of ideas, the intellect's dream of logical perfection." (p. 4),

and stating his belief "... that, as the ideal of excellence in thinking, logical rigor is supremely important, not only in mathematical thinking, but in all thinking and *especially* in just those subjects where precision is least attainable." "Why?", he asks, answering

> "Because without that ideal, thinking is without a just standard for self-criticism; it is without light upon its course; it is a wanderer like a vessel at sea without compass or star. Were it necessary, how easy it would unfortunately be to cite endless examples of such thinking from the multitudinous writings of our time. Indeed, if the pretentious books produced in these troubled years by men without logical insight or a sense of logical obligation were gathered into a heap and burned, they would thus produce, in the form of a bright bonfire, the only light they are qualified to give."

Continuing, he says "Now it so happens that the term mathematics is the name of that discipline which, because it attains more nearly than any other to the level of logical rigor, is better qualified than any other to reveal the prototype of what is best in the quality of thinking as thinking." (p. 4), concluding—in accord with the tenets of humanistic education—that the threshold of instruction crucial to the development of men and women may be described as "... the amount necessary to give them a fair understanding of Rigor as the standard of logical rectitude and therewith, if it may be, the spirit of loyalty to the ideal of excellence in the quality of thought as thought." (pp. 4–5).[3]

In making some concluding remarks, I also digress a little. Some forty years forward a transitional phase in American high school education (from a traditional curriculum to a newly termed 'modern' one) sparked much debate inside a mathematical community which held an obvious vested interest in pertaining affairs, and a gathering of views was effected accordingly (described as "... a modest number from men with mathematical competence, background, and experience and from various geographical locations."). Accompanied by a list of over sixty endorsing

[3]For the record, the January 1922 essay was based on a short address made by Keyser at the previous December 2nd meeting of the New York Section of the Association of Teachers of Mathematics of the Middle Atlantic States and Maryland. It was reproduced posthumously, in full, in 1966, in *The Mathematics Teacher* to publicise the fact that (according to the editor, F. Joe Crosswhite) "For many, rigorous thinking has become the hallmark of modern mathematics." (Editor's Note), and to reach a new generational readership of teachers.

Interestingly, Keyser had already published an article, with the same title, in a 1913 issue of *Science*—an essay of somewhat different content in many parts, though similar in tone (Vol. 38, pp. 789–800; this was the write up of his talk at a Columbia University Mathematical Colloquium on October 13th of that year, and there is an indication that Keyser planned further lectures on this general theme (I have found no accompanying material documented, however)). He also produced a book (again with this same running title—*The Human Worth of Rigorous Thinking: Essays and Addresses*), published by Columbia University Press in 1916 (314 pp.), which was reviewed by Norbert Wiener in June 1917 (see Vol. 14 of *The Journal of Philosophy, Psychology and Scientific Methods*), and in July and August by, resp., Warner Fite (Vol. 26 of *The Philosophical Review*) and G.A. Miller (Vol. 46 of *Science*). Two short (unattributed) reviews appeared in August and December of 1916; I have accessed later reviews in 1920 and 1941 (which latter examined a 1940 3rd edition of the text). Clearly there was demonstrable interest in the work of Keyser for some years, before it inevitably fell away.

signatures—drawn from an impressive pool of (mostly university) instructors/researchers—they were duly summarised in a March 1962 journal article 'On the Mathematics Curriculum of the High School' (*Amer. Math. Month.*, Vol. 69, pp. 189–193; it also appeared, that same month, in *The Mathematics Teacher* (Vol. 55, pp. 191–195)) which mapped out concisely, under a number of headings, those conclusions and suggestions made. It is interesting to see the cultural and practical significance of the subject noted here—along with its interplay with other sciences and fields—and supported fully. On the matter of mathematical thinking, the following short passage is worth documenting for its accuracy both then and now, and also because the question of what and how to teach developing students (juggling the content versus pedagogy balance to provide a solid foundation for all—that is, for non-specialists and would be professional mathematicians alike) was evidently an imperative feature of Keyser's broader mantra:

> "Mathematical thinking is not just deductive reasoning; it does not consist merely in formal proofs. The mental processes which suggest what to prove and how to prove it are as much a part of mathematical thinking as the proof that eventually results from them. Extracting the appropriate concept from a concrete situation, generalizing from observed cases, inductive arguments, arguments by analogy, and intuitive grounds for an emerging conjecture are mathematical modes of thinking. Indeed, without some experience with such "informal" thought processes the student cannot understand the true role of formal, rigorous proof which was so well described by Hadamard:[4] "The object of mathematical rigor is to sanction and legitimatize the conquests of intuition, and there never was any other object for it."
>
> There are several levels of rigor. The student should learn to appreciate, to find and to criticize proofs on the level corresponding to his experience and background. If pushed prematurely to a too formal level he may get discouraged and disgusted. Moreover the feeling for rigor can be much better learned from examples wherein the proof settles genuine difficulties than from hair-splitting or endless harping on trivialities." (p. 190);

the ways in which this (commonly agreed) mental training could translate itself into adult life—mathematical or otherwise—is the kind of thing Keyser was claiming as integral to personal success for any individual as a member of society, and these independent deliberations serve to cement and strengthen his principles in the long history of education. As a small add on, Columbia University's Michael Harris has quite recently reminded us that the very *act* of mathematical thinking brings us into immediate contact with the outside world—a point on which he felt compelled to write ('Visions of That Which is Sought', *Soc. Res.*, Vol. 83, pp. 1033–1059 (2016))

> "The prevalent view of mathematics among philosophers—as I'm told by philosophers who contest this view—is as a purely logical activity, or maybe something that is not an activity but just happens, or even something that just *is*, but in any case, one that in no way depends on the possibility of sensory interaction with an external world. On this model, the ideal mathematician would be free of all sensory distractions, a brain in a vat running on pure logic—in other words, a computer. Nevertheless, mathematicians' talk about our work is invariably sprinkled with sensory metaphors, ..." (p. 1033),

pointing out that in fact these are nearly always visual ones (as history attests); Keyser would have approved, I'm sure. The late Reuben Hersh (1927–2020), American mathematician and academic, is best known for his writings on the nature, practice, and social impact of mathematics. His narrative work, like that of Harris, can be viewed as both challenging and complementing the mainstream philosophy of mathematics and should be recognised as such. It is of note, too, that strands of Keyser's humanistic model for mathematical education still filter through to us in the 21st century. An interesting piece by Alan J. Bishop ('What Would the Mathematics Curriculum Look Like if Instead of Concepts and Techniques, Values Were the Focus?', *in* B. Larvor (Ed.), *Mathematical Cultures: The London Meetings 2012–2014*, Birkhäuser, Cham, pp. 181–188 (2016)) asserts that the mathematics curriculum and its associated pedagogy is drastically in need of fresh impetus and new ideas, citing the faith that we place in old concepts, tech-

[4]French mathematician Jacques S. Hadamard (1865–1963).

niques and syllabus structures as still causing mathematics to be one of the most feared and disliked (even hated) of school subjects despite several limited attempts at reform. His book chapter—outlining a central role for *values* (that is, a grouping of three pairs around the values cluster described as Ideology (Rationalism and Objectism), Sentiment (Control and Progress) and Sociology (Openness and Mystery), whose theoretic basis was formed in the 1950s), ahead of concepts and techniques—sets down the results of a speculative 'thought experiment' to justify a curriculum redesign which, although not researched or empirically tested, offers much for the instructor of mathematics to think about in re-orientating students within the discipline in terms of their attitudes and participation. Evidently, Keyser willingly took on the task of some important precursory and early thinking about humanistic education—and the way mathematics relates to it—to which others have been drawn independently or as a consequence; through their work, he lives on in our contemporary educational setting.

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