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A NEW ITERATIVE SCHEME FOR APPROXIMATION OF FIXED POINTS OF SUZUKI'S GENERALIZED NON-EXPANSIVE MAPPINGS

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Abstract In this paper, we introduce a new iteration scheme, named as the S**-iteration scheme, for approximation of fixed points of Suzuki's generalized non-expansive mappings. We also put forward some weak and strong convergence theorems for Suzuki's generalized non-expansive mappings in the setting of uniformly convex Banach spaces. We prove the stability of our instigated scheme and give numerical examples to show that this scheme is faster than Agarwal, Abbas, Thakur and Ullah iteration schemes. Our results comprehend, improve and consolidate many results in the existing literature.

1 Introduction

Fixed point theory provides very useful tools to solve most of the nonlinear problems, that have application in different fields, as they can be easily transformed into a fixed point problem. After establishing the existence of a fixed point we find its value using iterative processes. Till now many iterative processes have been developed, all of which can not be covered. Banach contraction principle [3], which is the most celebrated result in fixed point theory uses Picard iteration process for approximating the fixed point. The Picard iteration process is useful for the approximation of the fixed point of the contraction mappings but when one is dealing with nonexpansive mappings then it may fail to converge to the fixed point even if the fixed point is unique.

In 1953, Mann [14] introduced a new iterative scheme to approximate the fixed points of nonexpansive mappings. For a nonempty subset $\mathfrak{C}$ of a Banach space $\mathfrak{X}$, let $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a mapping. In this iterative scheme the sequence (t_n) is generated by $t_0 \in \mathfrak{C}$ as:

$$t_{n+1} = (1-\alpha_n)t_n + \alpha_n \mathfrak{T} t_n \text{ for all } n \geq 0, \tag{1.1}$$

where $\alpha_n \in (0,1)$. But the Mann iterative scheme fails to converge to the fixed points of pseudo-contractive mappings.

In 1974, Ishikawa [11] introduced a two step Mann iterative scheme to approximate fixed points of pseudo-contractive mappings, where the sequence (t_n) is generated by $t_0 \in \mathfrak{C}$ as:

$$\left.\begin{array}{l} t_{n+1} = (1-\alpha_n)t_n + \alpha_n \mathfrak{T} s_n \\ s_n = (1-\beta_n)t_n + \beta_n \mathfrak{T} t_n \end{array}\right\}, \tag{1.2}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0,1)$.
Many authors studied Mann and Ishikawa iterative schemes for approximation of fixed point of nonexpansive mappings (for instance [12],[20] and [27]).

In 2000, Noor [15] established another iterative scheme, where the sequence (t_n) is generated by $t_0 \in \mathfrak{C}$ as:

$$\left.\begin{array}{l} t_{n+1} = (1-\alpha_n)t_n + \alpha_n \mathfrak{T} s_n \\ s_n = (1-\beta_n)t_n + \beta_n \mathfrak{T} r_n \\ r_n = (1-\gamma_n)t_n + \gamma_n \mathfrak{T} t_n \end{array}\right\}, \tag{1.3}$$

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2007, Agarwal et al. [2] introduced a two-step iteration process for nearly asymptotically nonexpansive mappings, where the sequence (t_n) is generated from arbitrary $t_0 \in \mathfrak{C}$ by

$$\left.\begin{array}{l} t_{n+1} = (1-\alpha_n)\mathfrak{T} t_n + \alpha_n \mathfrak{T} s_n \\ s_n = (1-\beta_n)t_n + \beta_n \mathfrak{T} t_n \end{array}\right\}, \tag{1.4}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0,1)$. This process converges faster than Mann iteration process for contraction mappings.

In 2014, Abbas and Nazir [1] developed an iterative scheme which is faster than Agarwal et al.'s [2] scheme, where a sequence (t_n) is formulated from arbitrary $t_0 \in \mathfrak{C}$ by

$$\left.\begin{array}{l} t_{n+1} = (1-\alpha_n)\mathfrak{T} s_n + \alpha_n \mathfrak{T} r_n \\ s_n = (1-\beta_n)\mathfrak{T} t_n + \beta_n \mathfrak{T} r_n \\ r_n = (1-\gamma_n)t_n + \gamma_n \mathfrak{T} t_n \end{array}\right\}, \tag{1.5}$$

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2016, Thakur et al. [23] developed an iterative procedure, where a sequence (t_n) is generated iteratively by arbitrary $t_0 \in \mathfrak{C}$ and

$$\left.\begin{array}{l} t_{n+1} = (1-\alpha_n)\mathfrak{T} r_n + \alpha_n \mathfrak{T} s_n \\ s_n = (1-\beta_n)r_n + \beta_n \mathfrak{T} r_n \\ r_n = (1-\gamma_n)t_n + \gamma_n \mathfrak{T} t_n \end{array}\right\}, \tag{1.6}$$

for all $n \geq 0$, where $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

In 2018, Ullah and Arshad [26] developed a new iteration process which converges faster than all the aforementioned process, where the sequence (t_n) is constructed by taking arbitrary $t_0 \in \mathfrak{C}$ and

$$\left.\begin{array}{l} t_{n+1} = \mathfrak{T} s_n \\ s_n = \mathfrak{T}((1-\alpha_n)r_n + \alpha_n \mathfrak{T} r_n) \\ r_n = (1-\beta_n)t_n + \beta_n \mathfrak{T} t_n \end{array}\right\}, \tag{1.7}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0,1)$.

Recently, many iterative schemes have been given by eminent mathematicians leading to faster convergence to the fixed point (see, for instance [25], [13], [6], [24], [10]).

In this paper, we introduce a new three-step iteration process which is faster than Agarwal, Abbas, Thakur, and Ullah iteration processes and prove the convergence results using our iterative scheme for Suzuki's generalized non-expansive mappings in the context of uniformly convex Banach spaces. We also show that our process is analytically stable. With the help of examples, we compare the rate of convergence of our iteration process with the aforementioned iteration processes.

2 Preliminaries

Throughout this paper, $\mathfrak{C}$ is a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$, $\mathbb{N}$ denotes the set of all positive integers, $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a mapping and $F(\mathfrak{T})$ denotes the set of all fixed points of $\mathfrak{T}$.

Definition 2.1. [7] A Banach space $\mathfrak{X}$ is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists a $\delta > 0$ such that for all $x, y \in \mathfrak{X}$,

$$\left.\begin{array}{l}\|x\| \leq 1,\\ \|y\| \leq 1,\\ \|x-y\| > \epsilon\end{array}\right\} \text{ implies } \left\|\frac{x+y}{2}\right\| \leq \delta. \tag{2.1}$$

Definition 2.2. [16] A Banach space $\mathfrak{X}$ is said to satisfy Opial property if for each sequence (t_n) in $\mathfrak{X}$, converging weakly to $p \in \mathfrak{X}$, we have

$$\limsup_{n\to\infty}\|t_n - p\| < \limsup_{n\to\infty}\|t_n - q\|, \tag{2.2}$$

for all $q \in \mathfrak{X}$ such that $p \neq q$.

Definition 2.3. A mapping $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is called a contraction if there exists $\alpha \in (0,1)$, such that

$$\|\mathfrak{T}p - \mathfrak{T}q\| \leq \alpha\|p - q\|, \text{ for all } p, q \in \mathfrak{C}. \tag{2.3}$$

Definition 2.4. A mapping $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is called quasi non-expansive if for all $p \in \mathfrak{C}$ and $q \in F(\mathfrak{T})$, we have

$$\|\mathfrak{T}p - q\| \leq \|p - q\|. \tag{2.4}$$

In 2008, Suzuki introduced the concept of generalized non-expansive mappings as follows.

Definition 2.5. [21] A mapping $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is called Suzuki's generalized non-expansive mapping if for all $p, q \in \mathfrak{C}$, we have

$$\frac{1}{2}\|p - \mathfrak{T}p\| \leq \alpha\|p - q\| \text{ implies } \|\mathfrak{T}p - \mathfrak{T}q\| \leq \|p - q\|. \tag{2.5}$$

Suzuki [21] proved that the generalized non-expansive mapping is weaker than non-expansive mapping and stronger than quasi non-expansive mapping and obtained some fixed points and convergence theorems for Suzuki's generalized non-expansive mappings. Recently, many authors have studied fixed-point theorems for Suzuki's generalized non-expansive mappings (e.g.,[22]).

Senter and Dotson [20] introduced a class of mappings satisfying condition (I).

Definition 2.6. A mapping $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is said to satisfy condition (I), if there exists a nondecreasing function $\mathfrak{f} : [0,\infty) \to [0,\infty)$ with $\mathfrak{f}(0) = 0$ and $\mathfrak{f}(\delta) > 0$, for all $\delta > 0$ such that $\|q - \mathfrak{T}q\| \geq \mathfrak{f}(d(q, F(\mathfrak{T})))$, for all $q \in \mathfrak{C}$, where $d(q, F(\mathfrak{T})) = \inf_{q^* \in F(\mathfrak{T})}\|q - q^*\|$.

Proposition 2.7. *[21] Let $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be any mapping. Then*
(i) If $\mathfrak{T}$ is non-expansive, then $\mathfrak{T}$ is a Suzuki's generalized non-expansive mapping.
(ii) If $\mathfrak{T}$ is a Suzuki's generalized non-expansive mapping and has a fixed point, then $\mathfrak{T}$ is a quasi non-expansive mapping.
(iii) If $\mathfrak{T}$ is a Suzuki's generalized non-expansive mapping, then

$$\|p - \mathfrak{T}q\| \leq 3\|\mathfrak{T}p - p\| + \|p - q\|, \textit{ for all } p, q \in \mathfrak{C}. \tag{2.6}$$

Lemma 2.8. *[21] Suppose $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is Suzuki's generalized non-expansive mapping satisfying Opial property and (t_n) be any real sequence in $\mathfrak{C}$. If (t_n) converges weakly to p for some $p \in \mathfrak{C}$ and $\lim_{n\to\infty}\|\mathfrak{T}t_n - t_n\| = 0$, then $\mathfrak{T}p = p$.*

Lemma 2.9. *[21] Let $\mathfrak{X}$ be a uniformly convex Banach space and $\mathfrak{C}$ a weakly convex compact subset of $\mathfrak{X}$. Assume that $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is Suzuki's generalized non-expansive mapping. Then $\mathfrak{T}$ has a fixed point.*

Lemma 2.10. *[19] Let $\mathfrak{X}$ be a uniformly convex Banach space and (x_n) be any real sequence such that $0 < a \leq x_n \leq b < 1$ for all $n \geq 1$. Suppose that (u_n) and (v_n) are any two sequences of $\mathfrak{X}$ such that $\limsup_{n\to\infty}\|u_n\| \leq r$, $\limsup_{n\to\infty}\|v_n\| \leq r$ and $\limsup_{n\to\infty}\|x_n u_n + (1 - x_n)v_n\| = r$ hold for some $r \geq 0$. Then, $\limsup_{n\to\infty}\|u_n - v_n\| = 0$.*

Definition 2.11. [26] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{C}$ a non-empty closed convex subset of $\mathfrak{X}$. Assume that (t_n) is a bounded sequence in $\mathfrak{X}$. For $p \in \mathfrak{X}$, we set $r(p,(t_n)) = \limsup_{n\to\infty} \|t_n - p\|$. The asymptotic radius of (t_n) relative to $\mathfrak{C}$ is the set $r(\mathfrak{C},(t_n)) = \inf\{r(p,(t_n)) : p \in \mathfrak{C}\}$ and the asymptotic center of (t_n) relative to $\mathfrak{C}$ is given by the following set:

$$\mathfrak{A}(\mathfrak{C},(t_n)) = \{p \in \mathfrak{C} : r(p,(t_n)) = r(\mathfrak{C},(t_n))\}. \tag{2.7}$$

It is known that, in a uniformly convex Banach space, $\mathfrak{A}(\mathfrak{C},(t_n))$ consists of exactly one point.

Definition 2.12. [8] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$. Suppose that $t_0 \in \mathfrak{X}$ and $t_{n+1} = f(\mathfrak{T}, t_n)$ defines an iteration procedure which gives a sequence of points $t_n \in \mathfrak{X}$. Assume that (t_n) converges to the fixed point p. Suppose (s_n) be a sequence in $\mathfrak{X}$ and (ϵ_n) be a sequence in $\mathbb{R}^+ = [0,\infty)$ given by $\epsilon_n = \|s_{n+1} - f(\mathfrak{T}, s_n)\|$. Then the iteration procedure defined by $t_{n+1} = f(\mathfrak{T}, t_n)$ is said to be $\mathfrak{T}$-stable or stable with respect to $\mathfrak{T}$ if $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} s_n = p$.

Definition 2.13. [17] Let $\mathfrak{X}$ be a Banach space and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$. Then $\mathfrak{T}$ is called a contractive mapping on $\mathfrak{X}$ if there exist $L \geq 0$, $b \in [0,1)$ such that for each $p, q \in \mathfrak{X}$

$$\|\mathfrak{T}p - \mathfrak{T}q\| \leq L\|p - \mathfrak{T}p\| + b\|p - q\|. \tag{2.8}$$

By using (7), Osilike [17] established several stability results most of which are generalizations of the results of Rhoades [18] and Harder and Hicks [9].

Lemma 2.14. *[4] If λ is a real number such that $0 \leq \lambda < 1$, and (ϵ_n) is the sequence of positive numbers such that*

$$\lim_{n\to\infty} \epsilon_n = 0,$$

then for any sequence of positive numbers (t_n) satisfying

$$t_{n+1} \leq \lambda t_n + \epsilon_n, \text{ for } n = 1, 2, ...,$$

we have

$$\lim_{n\to\infty} t_n = 0.$$

3 S**-Iteration Process

We introduce a new iteration scheme by generating the sequence (t_n) iteratively, taking arbitrary $t_0 \in \mathfrak{C}$, as

$$\left.\begin{aligned} t_{n+1} &= \mathfrak{T}((1-\mu_n)\mathfrak{T}r_n + \mu_n\mathfrak{T}s_n) \\ s_n &= \mathfrak{T}((1-\nu_n)r_n + \nu_n\mathfrak{T}r_n) \\ r_n &= \mathfrak{T}((1-\xi_n)t_n + \xi_n\mathfrak{T}t_n) \end{aligned}\right\}. \tag{3.1}$$

for all $n \geq 0$, where $(\mu_n), (\nu_n)$ and (ξ_n) are real sequences in the interval $(0,1)$.

We will establish the convergence results for Suzuki's generalized non-expansive mappings for S**-iteration process foremost. Then we will show that S**-iteration process converges faster than all aforementioned iteration processes for contractive mappings due to Berinde [5] and is stable.

4 Convergence Results for Suzuki's Generalized Non-expansive Mappings

In this section, we prove some weak and strong convergence results for the sequence generated by the S**-iteration process for Suzuki's generalized non-expansive mappings in the setting of uniformly convex Banach spaces.

Theorem 4.1. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalized non-expansive mapping with $F(\mathfrak{T}) \neq \emptyset$. For $t_0 \in \mathfrak{C}$, the sequence (t_n) is generated by the S**-iteration process. Then $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$.*

Proof. Let $q \in F(\mathfrak{T})$. From 2.7(ii) and (2.4), we have

$$\begin{aligned}\|r_n - q\| &= \|\mathfrak{T}((1-\xi_n)t_n + \xi_n \mathfrak{T}t_n) - q\| \\ &\le \|(1-\xi_n)t_n + \xi_n \mathfrak{T}t_n - q\| \\ &\le (1-\xi_n)\|t_n - q\| + \xi_n\|\mathfrak{T}t_n - q\| \\ &\le (1-\xi_n)\|t_n - q\| + \xi_n\|t_n - q\| \\ &= \|t_n - q\|,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\|s_n - q\| &= \|\mathfrak{T}((1-\nu_n)r_n + \nu_n \mathfrak{T}r_n) - q\| \\ &\le \|(1-\nu_n)r_n + \nu_n \mathfrak{T}r_n - q\| \\ &\le (1-\nu_n)\|r_n - q\| + \nu_n\|\mathfrak{T}r_n - q\| \\ &\le (1-\nu_n)\|r_n - q\| + \nu_n\|r_n - q\| \\ &= \|r_n - q\| \\ &\le \|t_n - q\|,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\|t_{n+1} - q\| &= \|\mathfrak{T}((1-\mu_n)\mathfrak{T}r_n + \mu_n \mathfrak{T}s_n) - q\| \\ &\le \|(1-\mu_n)\mathfrak{T}r_n + \mu_n \mathfrak{T}s_n - q\| \\ &\le (1-\mu_n)\|\mathfrak{T}r_n - q\| + \mu_n\|\mathfrak{T}s_n - q\| \\ &\le (1-\mu_n)\|r_n - q\| + \mu_n\|s_n - q\| \\ &\le (1-\mu_n)\|t_n - q\| + \mu_n\|t_n - q\| \\ &= \|t_n - q\|.\end{aligned} \tag{4.3}$$

Hence $\lim_{n\to\infty}\|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$. □

Corollary 4.2. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a Banach space $\mathfrak{X}$ and $\mathfrak{T}$ a non-expansive mapping on $\mathfrak{C}$ with $F(\mathfrak{T}) \neq \emptyset$. For $t_0 \in \mathfrak{C}$, the sequence (t_n) be defined by (3.1). Then $\lim_{n\to\infty}\|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$.*

Proof. From Proposition 2.7(i) and Theorem 4.1, we get our result. □

Now we give an example to show that the class of Suzuki's generalized non-expansive mappings is bigger than the class of non-expansive mappings.

Example 4.3. Define $T : [2,3] \to [2,3]$ by

$$T(x) = \begin{cases} 5 - x, & \text{if } x \in [2, \frac{15}{7}), \\ \frac{x+18}{7}, & \text{if } x \in [\frac{15}{7}, 3]. \end{cases} \tag{4.4}$$

Then T is a Suzuki's generalized non-expansive mapping, but T is not a non-expansive mapping.

Proof. Take $x = \frac{214}{100}$ and $y = \frac{15}{7}$, then

$$\|x - y\| = \|\frac{214}{100} - \frac{15}{7}\| = \frac{2}{700},$$

and

$$\begin{aligned}\|Tx - Ty\| &= \|5 - \frac{214}{100} - \frac{141}{49}\| \\ &= \frac{43}{2450} > \frac{2}{700} = \|x - y\|\end{aligned}$$

Thus, T is not a non-expansive mapping. Now we show that T is a Suzuki's generalized non-expansive mapping. Consider the following cases:

Case I. If either $x, y \in [2, \frac{15}{7})$ or $x, y \in [\frac{15}{7}, 3]$, then in both the cases T is non-expansive mapping and hence T is Suzuki's generalized non-expansive mapping.

Case II. Let $x \in [2, \frac{15}{7})$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|x - (5 - x)\| = \frac{1}{2}\|2x - 5\| \in (\frac{5}{14}, \frac{1}{2}]$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have $\frac{1}{2}(5 - 2x) \leq |x - y|$ which gives two possibilities:

(a) Let $x > y$, then $y \in [2, \frac{15}{7}]$ since we have taken $x \in [2, \frac{15}{7}]$. This situation is already included in Case I.

(b) Let $x < y$, then $\frac{1}{2}(5 - 2x) \leq y - x$ which implies $y \geq \frac{5}{2}$ and hence $y \in [\frac{5}{2}, 3]$. Now,

$$\|Tx - Ty\| = \|\frac{y + 18}{7} - 5 + x\| = \|\frac{y + 7x - 17}{7}\| < \frac{1}{7},$$

and

$$\|x - y\| = |x - y| > |\frac{15}{7} - \frac{5}{2}| = \frac{5}{14} > \frac{1}{7}$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.

Case III. Let $x \in [\frac{15}{7}, 3]$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|\frac{x+18}{7} - x\| = \frac{1}{2}\|\frac{18-6x}{14}\| \in [0, \frac{18}{49}]$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have $\frac{18-6x}{14} \leq |x - y|$ which gives two possibilities:

(a) Let $x < y$, then $y \in [\frac{15}{7}, 3]$ since we have taken $x \in [\frac{15}{7}, 3]$. This situation is already included in Case I.

(b) Ley $x > y$, then $\frac{18-6x}{14} \leq x - y$, i.e. $y \leq \frac{20x-18}{14} \implies y \leq \frac{87}{49}$ and $y \leq 3$, so $y \in [2, 3]$. Since $y \in [2, 3]$ and $y \leq \frac{20x-18}{14} \implies \frac{14y+18}{20} \leq x$. As $x \in [\frac{46}{20}, 3]$ and $y \in [\frac{15}{7}, 3]$ is already included in Case I, so we consider $x \in [\frac{46}{20}, 3]$ and $y \in [2, \frac{15}{7})$. Then

$$\|Tx - Ty\| = \|\frac{x + 18}{7} - 5 + y\| = \|\frac{x + 7y - 17}{7}\| < \frac{1}{7},$$

and

$$\|x - y\| = |x - y| > |\frac{46}{20} - \frac{15}{7}| = \frac{11}{70} > \frac{1}{7}$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$.
Thus T is Suzuki's generalised non-expansive mapping.

Using Matlab, we obtain Table 1 and Figure 1 for comparison of the rate of convergence of our iterative scheme with other iterative processes. We take the initial guess $t_0 = 2.1$ and control sequences $\mu_n = \nu_n = \xi_n = 0.8$.

Table 1. Comparison of the rate of convergence with different iteration schemes

Step	Agarwal	Abbas	Thakur	Ullah	S**-iteration
1	2.10000000000000	2.10000000000000	2.10000000000000	2.10000000000000	2.10000000000000
2	2.95028571428571	2.96886530612244	2.98323265306122	2.99833236151603	2.99982116975069
3	2.99679393586005	2.99882432491223	2.99966015569703	2.99999663832975	2.99999996134254
4	2.99979324157791	2.99995560541184	2.99999311196036	2.99999999322345	2.99999999999164
5	2.99998666619155	2.99999832361893	2.99999986039168	2.99999999998634	2.99999999999999
6	2.99999914010541	2.99999993669828	2.99999999717038	2.999999999999973	3.00000000000000
7	2.99999994454557	2.99999999760966	2.99999999994264	3.00000000000000	3.00000000000000
8	2.99999999642375	2.99999999990973	2.99999999999883	3.00000000000000	3.00000000000000
9	2.99999999976936	2.99999999999659	2.99999999999997	3.00000000000000	3.00000000000000
10	2.99999999998512	2.99999999999987	3.00000000000000	3.00000000000000	3.00000000000000
11	2.99999999999904	2.99999999999999	3.00000000000000	3.00000000000000	3.00000000000000
12	2.99999999999993	3.00000000000000	3.00000000000000	3.00000000000000	3.00000000000000
13	2.99999999999999	3.00000000000000	3.00000000000000	3.00000000000000	3.00000000000000
14	3.00000000000000	3.00000000000000	3.00000000000000	3.00000000000000	3.00000000000000
15	3.00000000000000	3.00000000000000	3.00000000000000	3.00000000000000	3.00000000000000

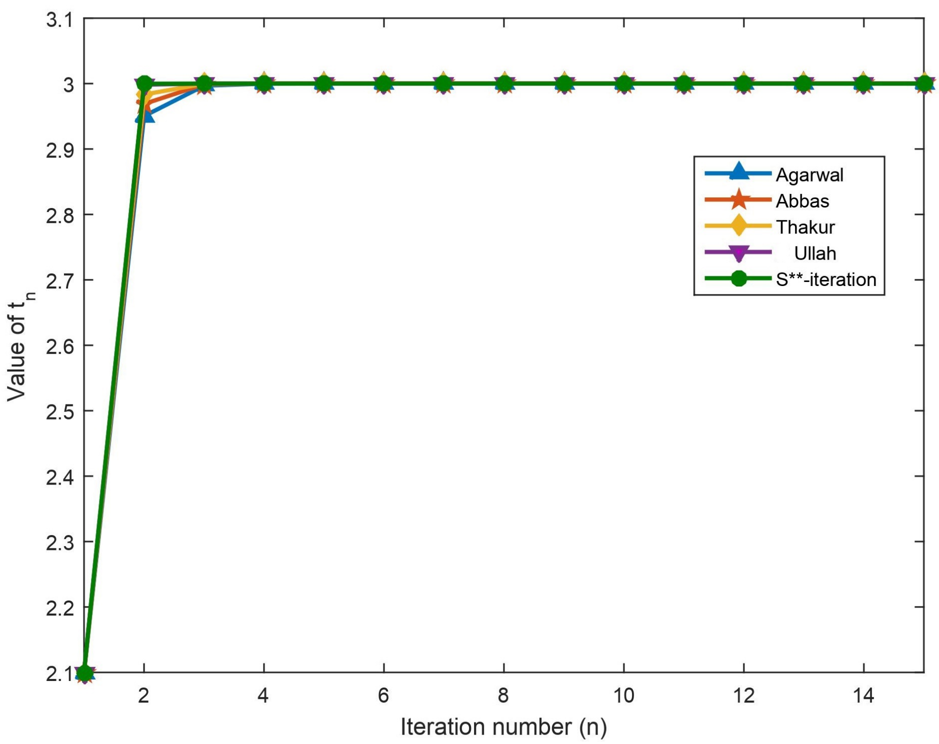

Figure 1. Graphical representation of convergence of iterative schemes.

□

Theorem 4.4. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalized non-expansive mapping. For arbitrary $t_0 \in \mathfrak{C}$ the sequence (t_n) is generated by the S**-iteration process. Then $F(\mathfrak{T}) \neq \emptyset$ if and only if $\lim_{n\to\infty} \|\mathfrak{T}t_n - t_n\| = 0$.*

Proof. Suppose that $F(\mathfrak{T}) \neq \emptyset$ and let $q \in F(\mathfrak{T})$. Then by previous theorem, $\lim_{n\to\infty} \|t_n - q\|$ exists and (t_n) is bounded. Let

$$\lim_{n\to\infty}\|t_n - q\| = c. \tag{4.5}$$

From (4.1) and (4.5), we have

$$\limsup_{n\to\infty}\|r_n - q\| \le \limsup_{n\to\infty}\|t_n - q\| = c. \tag{4.6}$$

By Proposition 2.7(ii), we have

$$\limsup_{n\to\infty}\|\mathfrak{T}t_n - q\| \le \limsup_{n\to\infty}\|t_n - q\| = c. \tag{4.7}$$

On the other hand

$$\begin{aligned}\|t_{n+1} - q\| &= \|\mathfrak{T}((1-\mu_n)\mathfrak{T}r_n + \mu_n\mathfrak{T}s_n) - q\| \\ &\le \|(1-\mu_n)\mathfrak{T}r_n + \mu_n\mathfrak{T}s_n - q\| \\ &\le (1-\mu_n)\|r_n - q\| + \mu_n\|s_n - q\| \\ &\le (1-\mu_n)\|r_n - q\| + \mu_n\|r_n - q\| \\ &\le (1-\mu_n)\|t_n - q\| + \mu_n\|r_n - q\|.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{\|t_{n+1} - q\| - \|t_n - q\|}{\mu_n} &\le [\|r_n - q\| - \|t_n - q\|] \\ \|t_{n+1} - q\| - \|t_n - q\| &\le \frac{\|t_{n+1} - q\| - \|t_n - q\|}{\mu_n} \\ &\le [\|r_n - q\| - \|t_n - q\|] \\ \|t_{n+1} - q\| &\le \|r_n - q\|.\end{aligned}$$

Therefore,

$$c \le \liminf_{n\to\infty}\|r_n - q\|. \tag{4.8}$$

From (4.6) and (4.8), we get

$$\begin{aligned}c &= \lim_{n\to\infty}\|r_n - q\| \\ &= \lim_{n\to\infty}\|(1-\xi_n)t_n + \xi_n\mathfrak{T}t_n - q\| \\ &= \lim_{n\to\infty}\|(1-\xi_n)(t_n - q) + \xi_n(\mathfrak{T}t_n - q)\|.\end{aligned} \tag{4.9}$$

From (4.5), (4.7), (4.9) and Lemma 2.10, we have $\lim_{n\to\infty}\|\mathfrak{T}t_n - t_n\| = 0$.

Conversely, suppose that (t_n) is bounded and $\lim_{n\to\infty}\|\mathfrak{T}t_n - t_n\| = 0$. Let $q \in \mathfrak{A}(\mathfrak{C},(t_n))$. From Proposition 2.7(iii), we get

$$\begin{aligned}r(\mathfrak{T}q,(t_n)) &= \limsup_{n\to\infty}\|t_n - \mathfrak{T}q\| \\ &\le \limsup_{n\to\infty}[3\|\mathfrak{T}t_n - t_n\| + \|t_n - q\|] \\ &\le \limsup_{n\to\infty}\|t_n - q\| \\ &= r(q,(t_n)).\end{aligned}$$

This shows that $\mathfrak{T}q \in \mathfrak{A}(\mathfrak{C}(t_n))$. Since $\mathfrak{X}$ is uniformly convex, $\mathfrak{A}(\mathfrak{C}(t_n))$ is singleton. Thus, $\mathfrak{T}q = q$, i.e. $F(\mathfrak{T}) \neq \emptyset$. □

Theorem 4.5. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ with the Opial property and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalized non-expansive mapping. For arbitrary $t_0 \in \mathfrak{C}$, let the sequence (t_n) be generated by the S**-iteration process with $F(\mathfrak{T}) \neq \emptyset$. Then (t_n) converges weakly to a fixed point of $\mathfrak{T}$.*

Proof. Since $F(\mathfrak{T}) \neq \emptyset$, so by Theorem 4.1 and 4.4, we have that (t_n) is bounded and $\lim_{n\to\infty} \|\mathfrak{T}t_n - t_n\| = 0$. As $\mathfrak{X}$ is uniformly convex so it is reflexive, thus by Eberlin's theorem, there exists a subsequence of (t_n), say (t_{n_i}) which converges weakly to some $q_1 \in \mathfrak{X}$. Now, $\mathfrak{C}$ is a closed and convex subset of $\mathfrak{X}$ so by Mazur's theorem $q_1 \in \mathfrak{C}$. By Lemma 2.8, $q_1 \in F(\mathfrak{T})$. Next we show that (t_n) converges weakly to q_1. Let us assume that it is not true. So there exists a subsequence of (t_n), say (t_{n_j}), such that (t_{n_j}) converges weakly to $q_2 \in \mathfrak{C}$, with $q_1 \neq q_2$. Using Lemma 2.8, we have $q_2 \in F(\mathfrak{T})$. Now, since $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$. Using Theorem 4.4 and Opial property, we have

$$\begin{aligned}\lim_{n\to\infty} \|t_n - q_1\| &= \lim_{i\to\infty} \|t_{n_i} - q_1\| \\ &< \lim_{i\to\infty} \|t_{n_i} - q_2\| \\ &= \lim_{n\to\infty} \|t_n - q_2\| \\ &= \lim_{j\to\infty} \|t_{n_j} - q_2\| \\ &< \lim_{j\to\infty} \|t_{n_j} - q_1\| \\ &= \lim_{n\to\infty} \|t_n - q_1\|,\end{aligned}$$

which is a contradiction. Hence $q_1 = q_2$. This shows that (t_n) converges weakly to a fixed point of $\mathfrak{T}$. □

Theorem 4.6. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a Suzuki's generalized non-expansive mapping. Let (t_n) be defined by the iteration process (3.1) and $F(\mathfrak{T}) \neq \emptyset$. Then the sequence (t_n) converges to a point of $F(\mathfrak{T})$ if and only if* $\liminf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$, *where* $d(t_n, F(\mathfrak{T})) = \inf\{\|t_n - q\| : q \in F(\mathfrak{T})\}$.

Proof. It is obvious that if the sequence (t_n) converges to a point of $F(\mathfrak{T})$ then

$$\lim \inf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0.$$

Now, suppose that $\liminf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$. From Theorem 4.1, we have $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$, so $\lim_{n\to\infty} d(t_n, F(\mathfrak{T}))$ exists and $\liminf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$ by assumption. Now, we will prove that (t_n) is a cauchy sequence in $\mathfrak{C}$. For a given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$d(t_n, F(\mathfrak{T})) < \epsilon/2.$$

In particular, $\inf\{\|t_n - q\| : q \in F(\mathfrak{T})\} < \epsilon/2$. Hence, there exists $q^* \in F(\mathfrak{T})$ such that $\|t_n - q^*\| < \epsilon/2$. Now, for all $m, n \in \mathbb{N}$

$$\|t_{m+n} - t_n\| \le \|t_{m+n} - q^*\| + \|t_n - q^*\| \le 2\|t_n - q^*\| < \epsilon,$$

which shows that (t_n) is a cauchy sequence in $\mathfrak{C}$. Also $\mathfrak{C}$ is given to be a closed subset of $\mathfrak{X}$, therefore there exists $q \in \mathfrak{C}$ such that $\lim_{n\to\infty} t_n = q$. Now, $\lim_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$ gives $d(q, F(\mathfrak{T})) = 0$ which implies that $q \in F(\mathfrak{T})$. □

Corollary 4.7. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a non-expansive mapping. Let (t_n) be defined by the iteration process (3.1) and $F(\mathfrak{T}) \neq \emptyset$. Then the sequence (t_n) converges to a point of $F(\mathfrak{T})$ if and only if* $\liminf_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$, *where* $d(t_n, F(\mathfrak{T})) = inf\{\|t_n - q\| : q \in F(\mathfrak{T})\}$.

Theorem 4.8. *Let $\mathfrak{C}$ be a non-empty closed convex compact subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalized non-expansive mapping. For arbitrary $t_0 \in \mathfrak{C}$, let the sequence (t_n) be generated by the S**-iteration process with $F(\mathfrak{T}) \neq \emptyset$. Then (t_n) converges strongly to a fixed point of $\mathfrak{T}$.*

Proof. From Lemma 2.9, we get $F(\mathfrak{T}) \neq \emptyset$ and so by Theorem 4.4, we get $\lim_{n\to\infty} \|\mathfrak{T}t_n - t_n\| = 0$. By the compactness of $\mathfrak{C}$, there exists a subsequence of (t_n), say (t_{n_i}), converging strongly to q for some $q \in \mathfrak{C}$. Now, by using Proposition 2.7(iii), we get

$$\|t_{n_i} - \mathfrak{T}q\| \leq 3\|\mathfrak{T}t_{n_i} - t_{n_i}\| + \|t_{n_i} - q\|. \tag{4.10}$$

Taking limit $i \to \infty$, we get $\mathfrak{T}q = q$, i.e. $q \in F(\mathfrak{T})$. By using Theorem 4.1, $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$. Thus, (t_n) converges strongly to q. □

Theorem 4.9. *Let $\mathfrak{C}$ be a non-empty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ a Suzuki's generalized non-expansive mapping. For arbitrary $t_0 \in \mathfrak{C}$, let the sequence (t_n) be generated by the S**-iteration process with $F(\mathfrak{T}) \neq \emptyset$. If $\mathfrak{T}$ satisfies condition (I), then (t_n) converges strongly to a fixed point of $\mathfrak{T}$.*

Proof. By Theorem 4.4, $\lim_{n\to\infty} \|t_n - q\|$ exists for all $q \in F(\mathfrak{T})$ and so $\lim_{n\to\infty} d(t_n, F(\mathfrak{T}))$ exists. Let $\lim_{n\to\infty} \|t_n - q\| = \alpha$, for some $\alpha \geq 0$. If $\alpha = 0$, then we are done. Suppose $\alpha > 0$, from condition (I) and the hypothesis, we have

$$f(d(t_n, F(\mathfrak{T}))) \leq \|\mathfrak{T}t_n - t_n\|. \tag{4.11}$$

As $F(\mathfrak{T}) \neq \emptyset$, by Theorem 4.1 we have $\lim_{n\to\infty} \|\mathfrak{T}t_n - t_n\| = 0$. Hence (4.11) implies that

$$\lim_{n\to\infty} f(d(t_n, F(\mathfrak{T}))) = 0. \tag{4.12}$$

Since f is a nondecreasing function, by equation (4.12) we get $\lim_{n\to\infty} d(t_n, F(\mathfrak{T})) = 0$. Thus, we have a subsequence (t_{n_k}) of (t_n) and a sequence (z_k) of $F(\mathfrak{T})$ such that

$$\|t_{n_k} - z_k\| < \frac{1}{2^k}, \text{ for all } k \in \mathbb{N}. \tag{4.13}$$

From equation (4.13),

$$\begin{aligned}
\|t_{n_{k+1}} - z_k\| &\leq \|t_{n_k} - z_k\| < \frac{1}{2^k},\\
\|z_{k+1} - z_k\| &\leq \|z_{k+1} - t_{k+1}\| + \|t_{k+1} - z_k\|\\
&\leq \frac{1}{2^{k+1}} + \frac{1}{2^k}\\
&< \frac{1}{2^{k-1}}.
\end{aligned}$$

Letting $i \to \infty$, we get $\frac{1}{2^{k-1}} \to 0$. Hence (z_k) is a cauchy sequence in $F(\mathfrak{T})$, so it converges to q. As $F(\mathfrak{T})$ is closed, $q \in F(\mathfrak{T})$ and therefore (t_{n_k}) converges strongly to q. Since $\lim_{n\to\infty} \|t_n - q\|$ exists, we have $t_n \to q \in F(\mathfrak{T})$. Hence proved. □

In the next theorem we prove that our iteration process is $\mathfrak{T}$-stable.

Theorem 4.10. *Let $\mathfrak{X}$ be a Banach Space and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ a mapping. Suppose $\mathfrak{T}$ has a fixed point q and (t_n) be a sequence in $\mathfrak{X}$ satisfying (3.1). Then (3.1) is $\mathfrak{T}$-stable.*

Proof. Let (w_n) be an arbitrary sequence in $\mathfrak{X}$ and the sequence which is generated by (3.1) is $t_{n+1} = f(\mathfrak{T}, t_n)$ converging to a unique fixed point q and $\epsilon_n = \|w_{n+1} - f(\mathfrak{T}, w_n)\|$. We show that $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} w_n = q$. First, assume that $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\begin{aligned}
\|w_{n+1} - q\| &= \|w_{n+1} - f(\mathfrak{T}, w_n) + f(\mathfrak{T}, w_n) - q\| \\
&\leq \|w_{n+1} - f(\mathfrak{T}, w_n)\| + \|f(\mathfrak{T}, w_n) - q\| \\
&\leq \|w_{n+1} - \mathfrak{T}((1-\mu_n)\mathfrak{T}u_n + \mu_n \mathfrak{T}v_n)\| + \|\mathfrak{T}((1-\mu_n)\mathfrak{T}u_n + \mu_n \mathfrak{T}v_n) - q\| \\
&\leq \epsilon_n + b[(1-\mu_n)\|\mathfrak{T}u_n - q\| + \mu_n\|\mathfrak{T}v_n - q\|] \\
&\leq \epsilon_n + b^2[(1-\mu_n)\|u_n - q\| + \mu_n\|v_n - q\|] \\
&\leq \epsilon_n + b^2[(1-\mu_n)\|\mathfrak{T}((1-\xi_n)w_n + \xi_n\mathfrak{T}w_n) - q\| \\
&\qquad\qquad + \mu_n\|\mathfrak{T}((1-\nu_n)u_n + \nu_n\mathfrak{T}u_n) - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)\|(1-\xi_n)w_n + \xi_n\mathfrak{T}w_n - q\| + \mu_n\|(1-\nu_n)u_n + \nu_n\mathfrak{T}u_n - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + \mu_n(1-\nu_n(1-b))\|u_n - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| \\
&\qquad\qquad + \mu_n(1-\nu_n(1-b))\|\mathfrak{T}((1-\xi_n)w_n + \xi_n\mathfrak{T}w_n) - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| \\
&\qquad\qquad + \mu_n(1-\nu_n(1-b))b\|(1-\xi_n)w_n + \xi_n\mathfrak{T}w_n - q\|] \\
&\leq \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| \\
&\qquad\qquad + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))\|w_n - q\|] \\
&= \epsilon_n + b^3[(1-\mu_n)(1-\xi_n(1-b)) + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))]\|w_n - q\|.
\end{aligned} \tag{4.14}$$

Since $b \in [0,1)$ and $(\mu_n), (\nu_n)$ and (ξ_n) are in $[0,1]$, we have

$$b^3[(1-\mu_n)(1-\xi_n(1-b)) + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))] < 1. \tag{4.15}$$

Hence by Lemma 2.14, we have $\lim_{n\to\infty}\|w_n - q\| = 0$, which gives $\lim_{n\to\infty} w_n = q$. Conversely, suppose that $\lim_{n\to\infty} w_n = q$. Then

$$\begin{aligned}
\epsilon_n &= \|w_{n+1} - f(\mathfrak{T}, w_n)\| \\
&= \|w_{n+1} - q + q - f(\mathfrak{T}, w_n)\| \\
&\leq \|w_{n+1} - q\| + \|\mathfrak{T}((1-\mu_n)\mathfrak{T}u_n + \mu_n\mathfrak{T}v_n) - q\| \\
&\leq \|w_{n+1} - q\| + b[(1-\mu_n)\|\mathfrak{T}u_n - q\| + \mu_n\|\mathfrak{T}v_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^2[(1-\mu_n)\|u_n - q\| + \mu_n\|v_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^2[(1-\mu_n)\|\mathfrak{T}((1-\xi_n)w_n + \xi_n\mathfrak{T}w_n) - q\| \\
&\qquad\qquad + \mu_n\|\mathfrak{T}((1-\nu_n)u_n + \nu_n\mathfrak{T}u_n) - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)\|(1-\xi_n)w_n + \xi_n\mathfrak{T}w_n - q\| + \mu_n\|(1-\nu_n)u_n + \nu_n\mathfrak{T}u_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| + \mu_n(1-\nu_n(1-b))\|u_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| \\
&\qquad\qquad + \mu_n(1-\nu_n(1-b))\|\mathfrak{T}((1-\xi_n)w_n + \xi_n\mathfrak{T}w_n) - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| \\
&\qquad\qquad + \mu_n(1-\nu_n(1-b))b\|(1-\xi_n)w_n + \xi_n\mathfrak{T}w_n - q\|] \\
&\leq \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b))\|w_n - q\| \\
&\qquad\qquad + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))\|w_n - q\|] \\
&= \|w_{n+1} - q\| + b^3[(1-\mu_n)(1-\xi_n(1-b)) + b\mu_n(1-\nu_n(1-b))(1-\xi_n(1-b))]\|w_n - q\|.
\end{aligned} \tag{4.16}$$

Taking limit as $n \to \infty$ in (4.16) gives $\lim_{n\to\infty} \epsilon_n = 0$. □

Now, we give another example to reconfirm that the convergence of our iteration scheme is faster than other iteration schemes.

Example 4.11. Let $\mathfrak{X} = \mathbb{R}$, $\mathfrak{C} = [1, 30]$ and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a mapping defined by $\mathfrak{T}t = \sqrt{t^2 - 7t + 42}$ for all $t \in \mathfrak{C}$. For $t_0 = 10$ and $\mu_n = \nu_n = \xi_n = 3/4, n = 1, 2, 3, \ldots$. From Table 2 we can see that all the iteration procedures are converging to $q^* = 6$.

Table 2. Comparison of the rate of convergence with different iteration schemes

Step	Agarwal	Abbas	Thakur	Ullah	S**-iteration
1	10.0000000000000	10.0000000000000	10.0000000000000	10.0000000000000	10.0000000000000
2	7.85879965424009	7.48299381981887	7.19365372863816	6.50327292180584	6.17058820431927
3	6.68921521738991	6.39390416650416	6.23393093417649	6.03083955536974	6.00307677148481
4	6.21650474463483	6.08687765191737	6.03851294451616	6.00170547093161	6.00005334404782
5	6.06293951132202	6.01810044948545	6.00610953889799	6.00009371908266	6.00000092420172
6	6.01781666894804	6.00372234157488	6.00096320048653	6.00000514824794	6.00000001601188
7	6.00500351572644	6.00076340654292	6.00015170386309	6.00000028280202	6.00000000027741
8	6.00140196803230	6.00015647717348	6.00002388960866	6.00000001553478	6.00000000000481
9	6.00039257571411	6.00003206977691	6.00000376193069	6.00000000085335	6.00000000000008
10	6.00010990841270	6.00000657250009	6.00000059239429	6.00000000004688	6.00000000000000
11	6.00003076923152	6.00000134698623	6.00000009328476	6.00000000000258	6.00000000000000
12	6.00000861382931	6.00000027605478	6.00000001468962	6.00000000000014	6.00000000000000
13	6.00000241142726	6.00000005657536	6.00000000231318	6.00000000000001	6.00000000000000
14	6.00000067507432	6.00000001159469	6.00000000036426	6.00000000000000	6.00000000000000
15	6.00000018898567	6.00000000237625	6.00000000005736	6.00000000000000	6.00000000000000
16	6.00000005290615	6.00000000048699	6.00000000000903	6.00000000000000	6.00000000000000
17	6.00000001481097	6.00000000009981	6.00000000000142	6.00000000000000	6.00000000000000
18	6.00000000414629	6.00000000002046	6.00000000000022	6.00000000000000	6.00000000000000
19	6.00000000116075	6.00000000000419	6.00000000000004	6.00000000000000	6.00000000000000
20	6.00000000032495	6.00000000000086	6.00000000000001	6.00000000000000	6.00000000000000
21	6.00000000009097	6.00000000000018	6.00000000000000	6.00000000000000	6.00000000000000
22	6.00000000002547	6.00000000000004	6.00000000000000	6.00000000000000	6.00000000000000
23	6.00000000000713	6.00000000000001	6.00000000000000	6.00000000000000	6.00000000000000
24	6.00000000000199	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
25	6.00000000000056	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
26	6.00000000000016	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
27	6.00000000000004	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
28	6.00000000000001	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
29	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
30	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
31	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
32	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
33	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
34	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000
35	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000	6.00000000000000

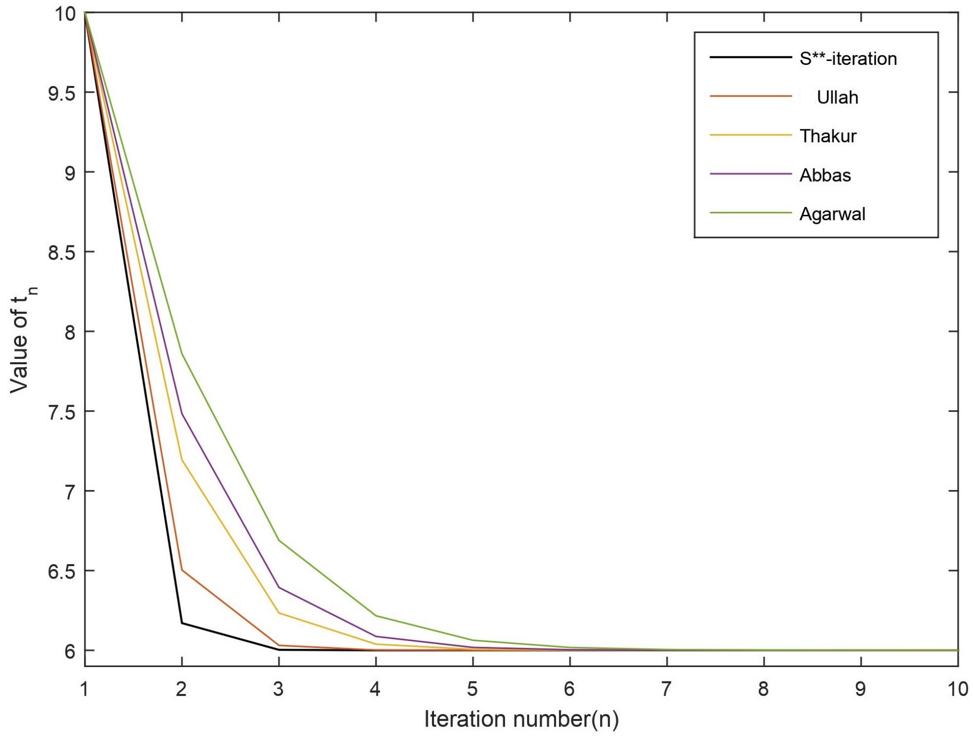

Figure 2. Graphical representation of convergence of iterative schemes.

Figure 2 and Table 2 show that our iterative scheme (3.1) converges faster than Agarwal, Abbas, Thakur and Ullah iterative schemes for Suzuki's generalized non-expansive mappings. The number of iterations in which these iterative schemes attain the fixed point is given in Table 3.

Table 3. Number of iterations in which the fixed point is attained.

Iterative method	Number of iterations
Agarwal	29
Abbas	24
Thakur	21
Ullah	14
S**-iteration	10

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