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APPROXIMATION BY BÉZIER-GENERALIZED BERNSTEIN-DURRMEYER POLYNOMIALS OPERATORS

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Abstract In this paper, we define Bézier variant of generalized Bernstein-Durrmeyer type operators of second order, introduced by Ana et al. Then, we find error estimate in terms of terms of Ditzian Totik modulus of smoothness. Next, we study rate of approximation for larger class function of bounded variation.

1 Introduction

For $C[0,1]$, the space of continuous functions defined on $[0,1]$, and for positive integers n, the sequence of Bernstein operators $B_n(f) : C[0,1] \longrightarrow C[0,1]$ is defined by

$$B_n(f;x) := \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1],$$

where $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$ for $0 \le k \le n$. The polynomials $p_{n,k}(x)$ satisfy following recurrence relation:

$$p_{n,k}(x) = (1-x)p_{n-1,k}(x) + xp_{n-1,k-1}(x), 0 < k < n. \tag{1.1}$$

Although $B_n(f;x)$ are not suitable to approximate integrable function as such, the Bernstein polynomial operators have some interesting properties (see [4]) e.g. preservation of convexity, Lipschitz constant and monotonicity etc. In order to approximate bounded and integrable functions on $[0,1]$, Durrmeyer [9] and Lupaş [17] introduced its integral modifications as follows:

$$D_n(f;x) := (n+1) \int_0^1 \sum_{k=0}^{n} p_{n,k}(x) p_{n,k}(u) f(u)\, \mathrm{d}u.$$

These operators are obtained by a suitable modification of Bernstein operators, wherein the values $f(k/n)$ are replaced by the average values

$$(n+1) \int_0^1 p_{n,k}(u) f(u)\, \mathrm{d}u$$

of f in the interval $[0,1]$. The operators $D_n(f;x)$ have been extensively studied by Derriennic [6] and other authors ([2],[3],[5],[7],[9],[10],[11],[13],[16],[18],[20],[21]). It is observed that the convergence of operators $D_n(f;x)$ to $f(x)$ is uniform. Also, it turns out that the order of approximation by the operators $D_n(f;x)$ is at best, $O(n^{-1})$ however smooth the function may be.

It is known that [12] linear positive operators e.g. Bernstein operators, Baskakov operators, integral variants of Bernstein operators and many other operators are saturated by order $O(n^{-1})$. Consequently, it is not possible to improve the rate of convergence by these operators except for linear functions. Recently a new approach to improve the rate of approximation by decomposition of the its weight function $p_{n,k}(x)$ was recently introduced by Khosravian-Arab et al.[15].

The authors of [15] applied this technique to the Bernstein polynomials operators and were able to improve the order of approximation from linear order $O(n^{-1})$ to the quadratic order $O(n^{-2})$ and cubic order $O(n^{-3})$ also. Subsequently, this technique was also applied to the Bernstein-Durrmeyer operators $D_n(f;x)$ by Ana Maria et al. in [1].

For a Lebesgue integrable function f on $[0,1]$ Acu et al. [1] introduced generalized Bernstein Durrmeyer polynomial operators of order II as follows:

$$D_n^{M,2}(f;x) := (n+1)\sum_{k=0}^{n} p_{n,k}^{M,2}(x)\int_0^1 p_{n,k}(u)f(u)\,\mathrm{d}u, n \geq 3$$

where

$$p_{n,k}^{M,2}(x) = b(x,n)p_{n-2,k}(x) + d(x,n)p_{n-2,k-1}(x) + b(1-x,n)p_{n-2,k-2}(x), \tag{1.2}$$

and $p_{n-2,k}(x) = 0$ if $k < 0$ and $k > n-2$ and the function $b(x,n) = b_2(n)x^2 + b_1(n)x + b_0(n)$ and $d(x,n) = d_0(n)x(1-x)$, where $b_i(n), i = \overline{0,2}$ $d_0(n)$ are the unknown sequences to be determined suitably by imposing the identities $D_n^{M,2}(e_i;x) = e_i(x)$ for $i = 0,1$ and for $i = 2$. So we get the condition $2b_2(n) - b_0(n) = 0$, $b_2(n) + 2b_0(n) + b_1(n) = 1$. Particularly, if we take $b_2(n) = b_0(n) = 1$, $b_1(n) = -2$ and $d_0(n) = 2$ then the fundamental polynomials (1.2) for the operators $D_n^{M,2}(f;x)$ are reduced to fundamental polynomials for Bernstein polynomials in (1.1). Moreover, if we assume sequences such that $b_0(n) = \frac{3}{2}, b_1(n) = -n$ and $b_2(n) = n-2$ and $d_0(n) = 2(n-2)$ then we have

Lemma 1.1. *[1] We have*

$$D_n^{M,2}((u-x);x) = 0,$$

$$D_n^{M,2}((u-x)^2;x) = \frac{20x(1-x)}{(n+2)(n+3)} - \frac{3}{(n+2)(n+3)}.$$

Remark 1.2. By an application of Lemma 1.1, we have

$$D_n^{M,2}((u-x)^2;x) \leq \frac{C}{n+2}\varphi^2(x), x \in [0,1], \tag{1.3}$$

where $\varphi^2(x) = x(1-x)$ and C is positive constant.

Very recently, Kajla and Acar [14] has applied this method to $\alpha-$ Bernstein operators. Motivated to study of various variants of these operators, we define the Bézier variant of the operator $D_n^{M,2}(f;x)$ for a larger class, namely the class of functions of bounded variation. For $\mu \geq 1$ and $f \in L_B[0,1]$, the class of bounded and integrable functions on $[0,1]$, we define Bézier variant of the operator $D_n^{M,2}(f;x)$ as

$$D_{n,\mu}^{M,2}(f;x) := (n+1)\sum_{k=0}^{n} Q_{n,k}^{(\mu)}(x)\int_0^1 p_{n,k}(u)f(u)\,\mathrm{d}u,$$

where

$$Q_{n,k}^{(\mu)}(x) \quad = (J_{n,k}^{M,2}(x))^{\mu} - (J_{n,k+1}^{M,2}(x))^{\mu}, J_{n,k}^{M,2}(x) \quad = \sum_{j=k}^{n} p_{n,j}^{M,2}(x).$$

For $\mu = 1$, this family of Bézier operators yield the operators $D_{n,\mu}^{M,2}(f;x)$ studied by Acu et al. [1]. Let

$$W_{n,\mu}^{M,2}(x,u) = (n+1)\sum_{k=0}^{n} Q_{n,k}^{(\mu)}(x)p_{n,k}(u).$$

Then $D_{n,\mu}^{M,2}(f;x) := \int_0^1 W_{n,\mu}^{M,2}(x,u)f(u)\mathrm{d}u.$

Lemma 1.3. *If $f \in C[0,1]$ then, $\|D_n^{M,2}(f)\| \leq \|f\|$, where $\|.\|$ denotes the sup-norm on $[0,1]$.*

Proof. The proof follows by easy calculations. □

Lemma 1.4. *If* $f \in C[0,1]$ *then,* $\|D^{M,2}_{n,\mu}(f)\| \leq \mu\|f\|$.

Proof. Using the inequality $|a^\mu - b^\mu| \leq \mu|a-b|$ with $0 \leq a, b \leq 1, \mu \geq 1$ and definition of $D^{M,2}_{n,\mu}(f)$, we get for $\mu \geq 1$

$$0 < [(J^{M,2}_{n,k}(x))^\mu - (J^{M,2}_{n,k+1}(x))^\mu] \leq \mu[J^{M,2}_{n,k}(x) - J^{M,2}_{n,k+1}(x)] = \mu p^{M,2}_{n,k}(x). \tag{1.4}$$

Applying $D^{M,2}_{n,\mu}(f;x)$ and Lemma 1.4, we get

$$\|D^{M,2}_{n,\mu}(f)\| \leq \mu\|D^{M,2}_{n}(f)\| \leq \mu\|f\|.$$

□

We will need the Ditzian-Totik modulus of smoothness for the weight $\varphi(x) = \sqrt{x(1-x)}$ defined by

$$\omega_\varphi(f,t) = \sup_{0<h\leq t} \sup_{x\pm h\varphi(x)/2\geq 0} \{|f(x+h\varphi(x)/2) - f(x-h\varphi(x)/2)|\}$$

and appropriate Peetre's $K-$functional is defined as

$$K_\varphi(f,t) = \inf_{g\in W^2_\varphi}\{\|f-g\| + t\|\varphi g'\| + t^2\|g'\|\},$$

where $t > 0$, the norm $\|.\|$ is the sup-norm on $[0,1]$ and $W^2_\varphi = \{g : g \in AC_{loc}, \|\varphi g'\| < \infty, \|g'\| < \infty\}$. By ([8], Theorem 3.1.2), it is known that $K_\varphi(f,t) \sim \omega_\varphi(f,t)$ and there exists an absolute constant $C > 0$ such that

$$C^{-1}\omega_\varphi(f,t) \leq K_\varphi(f,t) \leq C\omega_\varphi(f,t).$$

2 Direct Theorem

Now, we derive a direct result for Bézier operators in terms of Ditzian Totik modulus of smoothness $\omega_\varphi(f,t)$.

Theorem 2.1. *Let* $\varphi(x) = \sqrt{x(1-x)}$, $\mu \geq 1$ *and* $x \in [0,1]$. *If* $f \in C[0,1]$ *then*

$$|D^{M,2}_{n,\mu}(f;x) - f(x)| \leq C\omega_\varphi\left(f, \sqrt{\frac{1}{n+2}}\right),$$

where C *is any absolute constant.*

Proof. We know that $g(t) = g(x) + \int_x^t g'(u)\mathrm{d}u$. So

$$|D^{M,2}_{n,\mu}(g;x) - g(x)| = \left|D^{M,2}_{n,\mu}\left(\int_x^t g'(u)\mathrm{d}u, x\right)\right|. \tag{2.1}$$

For any $x, t \in (0,1)$, we have

$$\left|\int_x^t g'(u)\mathrm{d}u\right| \leq \|\varphi g'\| \left|\int_x^t \frac{1}{\varphi(u)}\mathrm{d}u\right|.$$

Therefore,

$$\begin{aligned}
\left|\int_x^t \frac{1}{\varphi(u)}\mathrm{d}u\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}\mathrm{d}u\right| \\
&\leq \left|\int_x^t \left(\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}\right)\mathrm{d}u\right| \\
&\leq 2\left(|\sqrt{t}-\sqrt{x}|+|\sqrt{1-t}-\sqrt{1-x}|\right) \\
&= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}}+\frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right) \\
&< 2|t-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right) \\
&< \frac{2\sqrt{2}|t-x|}{\varphi(x)}. \qquad (2.2)
\end{aligned}$$

Combining (2.1)-(2.2) and using Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
|D_{n,\mu}^{M,2}(g;x)-g(x)| &\leq 2\sqrt{2}\|\varphi g'\|\varphi^{-1}(x)D_{n,\mu}^{M,2}(|t-x|;x) \\
&\leq 2\sqrt{2}\|\varphi g'\|\varphi^{-1}(x)\left(D_{n,\mu}^{M,2}((t-x)^2;x)\right)^{1/2}.
\end{aligned}$$

Now using inequality (1.3), we obtain

$$|D_{n,\mu}^{M,2}(g;x)-g(x)| < C\sqrt{\frac{1}{(n+2)}}\|\varphi g'\|. \qquad (2.3)$$

Using Lemma 1.1 and inequality (2.3), we obtain

$$\begin{aligned}
|D_{n,\mu}^{M,2}(f;x)-f| &\leq |D_{n,\mu}^{M,2}(f-g;x)|+|f-g|+|D_{n,\mu}^{M,2}(g;x)-g(x)| \\
&\leq C\left(\|f-g\|+\sqrt{\frac{1}{(n+2)}}\|\varphi g'\|\right). \qquad (2.4)
\end{aligned}$$

Taking infimum on both sides of (2.4) over all $g \in W_\varphi^2$, we reach to

$$|D_{n,\mu}^{M,2}(f;x)-f| \leq CK_\varphi\left(f,\sqrt{\frac{1}{(n+2)}}\right)$$

Using relation $K_\varphi(f,t) \sim \omega_\varphi(f,t)$, we get the required result. □

Ozarslan and Aktuglu [19] is considered the Lipschitz-type space with two parameters $\alpha_1 \geq 0, \alpha_2 > 0$, which is defined as

$$Lip_M^{(\alpha_1,\alpha_2)}(\zeta) = \left\{f \in C[0,1] : |f(t)-f(x)| \leq C_0\frac{|t-x|^\zeta}{(t+\alpha_1x^2+\alpha_2x)^{\zeta/2}} : t \in [0,1], x \in (0,1)\right\},$$

where $0 < \zeta \leq 1$, and C_0 is any absolute constant.

Theorem 2.2. *Let $f \in Lip_M^{(\alpha_1,\alpha_2)}(\zeta)$. Then for all $x \in (0,1]$, there holds:*

$$|D_{n,\mu}^{M,2}(f;x)-f| \leq C\left(\frac{\mu\varphi^2(x)}{(n+2)(\alpha_1x^2+\alpha_2x)}\right)^{\frac{\zeta}{2}},$$

where C is any absolute constant.

Proof. Using Lemma 1.1 and (1.4) and Hölder's inequality with $p = \frac{2}{\zeta}$ and $p = \frac{2}{2-\zeta}$, we get

$|D^{M,2}_{n,\mu}(f;x) - f(x)|$

$$\begin{aligned}
&\leq (n+1)\sum_{k=0}^{n} Q^{(\mu)}_{n,k}(x)\int_0^1 p_{n,k}(u)|f(u)-f(x)|\,\mathrm{d}u \\
&\leq (n+1)\sum_{k=0}^{n} Q^{(\mu)}_{n,k}(x)\left(\int_0^1 p_{n,k}(u)|f(u)-f(x)|^{\frac{2}{\zeta}}\,\mathrm{d}u\right)^{\frac{\zeta}{2}} \\
&\leq \left[(n+1)\sum_{k=0}^{n} Q^{(\mu)}_{n,k}(x)\int_0^1 p_{n,k}(u)|f(u)-f(x)|^{\frac{2}{\zeta}}\,\mathrm{d}u\right]^{\frac{\zeta}{2}} \times \\
&\quad \left[(n+1)\sum_{k=0}^{n} Q^{(\mu)}_{n,k}(x)\int_0^1 p_{n,k}(u)\,\mathrm{d}u\right]^{\frac{2-\zeta}{2}} \\
&= \left[(n+1)\sum_{k=0}^{n} Q^{(\mu)}_{n,k}(x)\int_0^1 p_{n,k}(u)|f(u)-f(x)|^{\frac{2}{\zeta}}\,\mathrm{d}u\right]^{\frac{\zeta}{2}} \\
&\leq C\left((n+1)\sum_{k=0}^{n} Q^{(\mu)}_{n,k}(x)\int_0^1 p_{n,k}(u)\frac{(u-x)^2}{(u+\alpha_1x^2+\alpha_2x)}\mathrm{d}u\right)^{\frac{\zeta}{2}} \\
&\leq \frac{C}{(\alpha_1x^2+\alpha_2x)^{\frac{\zeta}{2}}}\left((n+1)\sum_{k=0}^{n} Q^{(\mu)}_{n,k}(x)\int_0^1 p_{n,k}(u)(u-x)^2\mathrm{d}u\right)^{\frac{\zeta}{2}} \\
&\leq \frac{C}{(\alpha_1x^2+\alpha_2x)^{\frac{\zeta}{2}}}[D^{M,2}_{n,\mu}((u-x)^2;x)]^{\frac{\zeta}{2}} \\
&\leq C\left(\frac{\mu\varphi^2(x)}{(n+2)(\alpha_1x^2+\alpha_2x)}\right)^{\frac{\zeta}{2}}.
\end{aligned}$$

Hence, the desired result follows. □

Remark 2.3. The class of $Lip^{(\alpha_1,\alpha_2)}_M(\zeta)$ of functions is larger than the class $C[0,1]$ of continuous functions, therefore theorem 2.2 is more general and provides error estimate also for the class $C[0,1]$. It follows by theorems 2.1 and 2.2 that the rate of approximation is $O(n^{-1/2})$ and $O(n^{-\zeta/2})$ for the classes $C[0,1]$ and $\mathrm{Lip}^{(\alpha_1,\alpha_2)}_M(\zeta)$ respectively.

3 Rate of convergence

We will make use of the notion of variation of a bounded function f defined on a certain interval $[a,b]$. The total variation $\bigvee_a^b(f)$ of the function f is defined by

$$\bigvee_a^b(f) := \sup_{p\in\mathcal{P}}\sum_{j=0}^{n}|f(t_j)-f(t_{j-1})|,$$

where p is the finite set $\{t_0,t_1,...,t_n\}$ with the conditions $a = t_0, b = t_n$ and $t_{j-1} < t_j$ for $j = \overline{1,n}$. The supremum is taken over the class $\mathcal{P}$ of the partitions p. A function f is said to be of bounded variation if $\bigvee_a^b(f) < \infty$. It is plain that the class $B[a,b]$ of functions of bounded variation is larger than the class $C[a,b]$.

If we define

$$\kappa_{n,\mu}(x,y) = \int_0^y W^{M,2}_{n,\mu}(x,t)\mathrm{d}t,$$

then it is obvious that $\kappa_{n,\mu}(x,1) = 1$.

Lemma 3.1. *Let $x \in (0,1)$ and $C > 2$ then for sufficiently large n we have*

$$\kappa_{n,\mu}(x,y) = \int_0^y W_{n,\mu}^{M,2}(x,t)\,dt \leq \frac{C\mu x(1-x)}{n(x-y)^2}, 0 < y < x$$

$$1 - \kappa_{n,\mu}(x,z) = \int_z^1 W_{n,\mu}^{M,2}(x,t)\,dt \leq \frac{C\mu x(1-x)}{n(z-x)^2}, x < z < 1.$$

Remark 3.2. We have

$$\begin{aligned}\kappa_{n,\mu}(x,y) &= \int_0^y W_{n,\mu}^{M,2}(x,t)\,\mathrm{d}t \leq \int_0^y W_{n,\mu}^{M,2}(x,t)\frac{(t-x)^2}{(y-x)^2}\,\mathrm{d}t \\ &= \frac{D_{n,\mu}^{M,2}((t-x)^2;x)}{(y-x)^2} \leq \frac{\mu D_n^{M,2}((t-x)^2;x)}{(y-x)^2} \leq \frac{\mu}{n+2}.\frac{\varphi^2(x)}{(y-x)^2}.\end{aligned}$$

Now, we discuss the approximation properties of functions having derivative of bounded variation on $[0,1]$. Let $B[0,1]$ denote the class of differentiable functions g defined on $[0,1]$, whose derivative g' is of bounded variation on $[0,1]$. The functions $g \in B[0,1]$ is expressed as $g(x) = \int_0^x h(t)\,\mathrm{d}t + g(0)$, where $h \in B[0,1]$, i.e., h is a function of bounded variation on $[0,1]$.

Theorem 3.3. *Let $f \in B[0,1]$ then for $\mu \geq 1, 0 < x < 1$ and sufficiently large n we have*

$$\begin{aligned}\left|D_{n,\mu}^{M,2}(f;x) - f(x)\right| &\leq \left(\frac{1}{\mu+1}|f'(x+) + \mu f'(x-)| + |f'(x+) - f'(x-)|\right)\frac{\mu}{n+2}\varphi^2(x) \\ &+ \frac{\mu}{n+2}.\frac{\varphi^2(x)}{x^2}\sum_{k=1}^{\sqrt{n}}\left(\bigvee_{x-\frac{x}{k}}^{x}(f')_x\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(f')_x\right)\left(\bigvee_{x-\frac{x}{k}}^{x}(f')_x\right) \\ &+ \frac{\mu}{n+2}.\frac{\varphi^2(x)}{1-x}\sum_{k=1}^{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{1-x}{k}}(f')_x\right) + \frac{1-x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{n}}}(f')_x\right).\end{aligned}$$

$$(f')_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x \\ f'(t) - f'(x+), & x < t \leq 1. \end{cases}$$

Proof. As we know that $D_{n,\mu}^{M,2}(1;x) = 1$. Therefore, we have

$$\begin{aligned}D_{n,\mu}^{M,2}(f;x) - f(x) &= \int_0^1 W_{n,\mu}^{M,2}(x,u)(f(u) - f(x))\,\mathrm{d}u \\ &= \int_0^1 W_{n,\mu}^{M,2}(x,u)\int_x^u f'(t)\,\mathrm{d}t\,\mathrm{d}u \qquad (3.1)\end{aligned}$$

Since $f \in B[0,1]$, we may write

$$\begin{aligned}f'(t) &= \frac{f'(x+) + \mu f'(x-)}{\mu+1} + (f')_x(t) + \frac{f'(x+) - f'(x-)}{2}\left(\mathrm{sign}(t-x) + \frac{\mu-1}{\mu+1}\right) \\ &+ \delta_x(t)\left(f'(t) - \frac{f'(x+) + \mu f'(x-)}{2}\right) \qquad (3.2)\end{aligned}$$

where

$$\text{sign}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0. \end{cases}$$

and

$$\delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \neq x. \end{cases}$$

Putting the value of $f'(t)$ from (3.2) in (3.1), we get estimates corresponding to four terms of (3.2), say K_1, K_2, K_3 and K_4 respectively. Obviously,

$$K_4 = \int_0^1 \left(\int_x^u \left(f'(t) - \frac{f'(x+) + \mu f'(x-)}{2} \right) \delta_x(t)\, \mathrm{d}t \right) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u = 0.$$

Now, using Cauchy's Schwarz inequality and Lemma 3.1, we obtain

$$\begin{aligned} K_1 &= \int_0^1 \left(\int_x^u \frac{f'(x+) + \mu f'(x-)}{\mu+1}\, \mathrm{d}t \right) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \\ &= \frac{f'(x+) + \mu f'(x-)}{\mu+1} \int_0^1 (u-x) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \\ &= \frac{f'(x+) + \mu f'(x-)}{\mu+1} D_{n,\mu}^{M,1}((u-x);x) \\ &\leq \frac{1}{\sqrt{n+2}} \frac{f'(x+) + \mu f'(x-)}{\mu+1} \varphi(x). \end{aligned}$$

Next, again using Cauchy's Schwarz inequality and Lemma 3.1, we obtain

$$\begin{aligned} K_3 &= \int_0^1 \left(\int_x^u \left(\frac{f'(x+) - f'(x-)}{2} \right) \left(\text{sign}(t-x) + \frac{\mu-1}{\mu+1} \right) \mathrm{d}t \right) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \\ &= \left(\frac{f'(x+) - f'(x-)}{2} \right) \left[-\int_0^x \left(\int_u^x \left(\text{sign}(t-x) + \frac{\mu-1}{\mu+1} \right) \mathrm{d}t \right) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \right. \\ &+ \left. \int_x^1 \left(\int_x^u \left(\text{sign}_\mu(t-x) + \frac{\mu-1}{\mu+1} \right) \mathrm{d}t \right) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \right] \\ &\leq |f'(x+) - f'(x-)| \int_0^1 |u-x| W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \\ &\leq |f'(x+) - f'(x-)|\, D_{n,\mu}^{M,1}(|u-x|;x) \\ &\leq \frac{1}{\sqrt{n+2}} |f'(x+) - f'(x-)|\, \varphi(x). \end{aligned}$$

Now, we estimate K_2 as follows:

$$\begin{aligned} K_2 &= \int_0^1 \left(\int_x^u (f')_x(t)\,\mathrm{d}t \right) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \\ &= \int_0^x \left(\int_x^u (f')_x(t)\,\mathrm{d}t \right) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \\ &+ \int_x^1 \left(\int_x^u (f')_x(t)\,\mathrm{d}t \right) W_{n,\mu}^{M,2}(x,u)\mathrm{d}u \\ &= K_5 + K_6, \text{say.} \end{aligned}$$

Using Lemma 3.1 and definition of $\kappa_{n,\mu}(x,u)$, we may write

$$K_5 = \int_0^x \left(\int_x^u (f')_x(t)\,\mathrm{d}t \right) \frac{d}{\mathrm{d}u}\kappa_{n,\mu}(x,u)\mathrm{d}u$$

Integrating by parts, we obtain

$$\begin{aligned} |K_5| &\leq \int_0^x |(f')_x(u)|\kappa_{n,\mu}(x,u)\mathrm{d}u \\ &\leq \int_0^{x-x/\sqrt{n}} |(f')_x(u)|\kappa_{n,\mu}(x,u)\mathrm{d}u + \int_{x-x/\sqrt{n}}^x |(f')_x(u)|\kappa_{n,\mu}(x,u)\,\mathrm{d}u \\ &= K_7 + K_8, \text{say.} \end{aligned}$$

In view of facts $(f')_x(x) = 0$ and $\kappa_{n,\mu}(x,u) \leq 1$, we get

$$\begin{aligned} K_8 &= \int_0^x |(f')_x(u) - (f')_x(x)|\kappa_{n,\mu}(x,u)\,\mathrm{d}u \\ &\leq \int_{x-x/\sqrt{n}}^x \left(\bigvee_u^x (f')_x \right) \mathrm{d}u \\ &\leq \left(\bigvee_u^x (f')_x \right) \int_{x-x/\sqrt{n}}^x \mathrm{d}u = \frac{x}{\sqrt{n}} \left(\bigvee_u^x (f')_x \right). \end{aligned}$$

Using Lemma 3.1, definition of $\kappa_{n,\mu}(x,u)$, and transformation $u = x - \frac{x}{t}$ we may write

$$\begin{aligned} K_7 &\leq \frac{\mu}{n+2}\varphi^2(x) \int_0^{x-x/\sqrt{n}} |(f')_x(u) - (f')_x(x)| \frac{\mathrm{d}u}{(u-x)^2} \\ &\leq \frac{\mu}{n+2}\varphi^2(x) \int_0^{x-x/\sqrt{n}} \left(\bigvee_u^x (f')_x \right) \frac{\mathrm{d}u}{(u-x)^2} \\ &\leq \frac{\mu}{n+2} \frac{\varphi^2(x)}{x^2} \int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{t}}^x (f')_x \right) \mathrm{d}t \\ &\leq \frac{\mu}{n+2} \frac{\varphi^2(x)}{x^2} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \left(\bigvee_{x-\frac{x}{t}}^x (f')_x \right). \end{aligned}$$

Combining the estimates of I_7 and I_8, we have

$$|K_5| \leq \frac{\mu}{n+2}\frac{\varphi^2(x)}{x^2}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{t}}^{x}(f')_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{u}^{x}(f')_x\right)$$

In order to estimate K_6, we use integration by parts, Lemma 3.1 and transformation $z = x+\frac{1-x}{\sqrt{n}}$. Therefore, we proceed as follows:

$$\begin{aligned}
|K_6| &= \left|\int_x^1\left(\int_x^u (f')_x(t)\,\mathrm{d}t\right)W_{n,\mu}^{M,2}(x,u)\,\mathrm{d}u\right| \\
&= \left|\int_x^z\left(\int_x^u (f')_x(t)\,\mathrm{d}t\right)\frac{\partial}{\partial u}(1-\kappa_{n,\mu}(x,u))\,\mathrm{d}u+\int_z^1\left(\int_x^u (f')_x(t)\,\mathrm{d}t\right)\frac{\partial}{\partial u}(1-\kappa_{n,\mu}(x,u))\,\mathrm{d}u\right| \\
&= \left|\left[\int_x^u (f')_x(t)\,\mathrm{d}t(1-\kappa_{n,\mu}(x,u))\right]_x^z-\int_x^z (f')_x(u)(1-\kappa_{n,\mu}(x,u))\,\mathrm{d}u\right. \\
&+ \left.\left[\int_x^u (f')_x(t)\,\mathrm{d}t(1-\kappa_{n,\mu}(x,u))\right]_z^1-\int_z^1 (f')_x(u)(1-\kappa_{n,\mu}(x,u))\,\mathrm{d}u\right| \\
&= \left|\int_x^z (f')_x(u)(1-\kappa_{n,\mu}(x,u))\,\mathrm{d}u+\int_z^1 (f')_x(u)(1-\kappa_{n,\mu}(x,u))\,\mathrm{d}u\right| \\
&\leq \frac{\mu}{n+2}.\varphi^2(x)\int_z^1\left(\bigvee_x^u(f')_x\right)(u-x)^{-2}\,\mathrm{d}u+\int_x^z\left(\bigvee_x^u(f')_x\right)\mathrm{d}u \\
&\leq \frac{\mu}{n+2}.\varphi^2(x)\int_{x+\frac{1-x}{\sqrt{n}}}^1\left(\bigvee_x^u(f')_x\right)(u-x)^{-2}\,\mathrm{d}u+\frac{1-x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(f')_x\right).
\end{aligned}$$

Now, substituting $t = \frac{1-x}{t-x}$, we get

$$\begin{aligned}
|K_6| &\leq \frac{\mu}{n+2}.\varphi^2(x)\int_1^{\sqrt{n}}\left(\bigvee_x^{x+\frac{1-x}{t}}(f')_x\right)(1-x)^{-1}\,\mathrm{d}t+\frac{1-x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(f')_x\right) \\
&\leq \frac{\mu}{n+2}\frac{\varphi^2(x)}{1-x}\sum_{k=1}^{\sqrt{n}}\left(\bigvee_x^{x+\frac{1-x}{k}}(f')_x\right)+\frac{1-x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{n}}}(f')_x\right).
\end{aligned}$$

Combining the estimates of $K_1 - K_8$, we get the desired result. Hence the proof follows. □

Remark 3.4. For a function having jump discontinuity of finite magnitude, theorem 2.1 and 2.2 need not be true. However, the notion of the total variation $\bigvee_a^b(f)$ allows one to incorporate such functions belonging to the larger class $B[0,1]$.

4 Conclusion and Future Scope

The approximation of integrable functions by a linear operators require suitable modification of the kernal or replacement of the values $f(k/n)$, $k=\overline{0,n}$. The operator $D_{n,\mu}^{M,2}(f;x)$ approximate such functions with the rate of convergence $O(n^{-1/2})$. Further, these operators are suitable to estimate the functions of a larger class, namely bounded variation. The Bézier variant operator $D_{n,\mu}^{M,2}(f;x)$ can be utilized to plot Bézier curves corresponding to the integrable function f in $[0,1]$ and higher order approximation by the $D_{n,\mu}^{M,2}(f;x)$ is another problem of interest.

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