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Integral product: definition and stability

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Abstract We reformulate the integral product of semigroups in a general framework and give a clarification on some hypotheses which are imposed in/on its definition. We make precise the notion of $\pi-$integrability of a given family of semi groups $(T(t,s))_{s\geq 0}, t \in [0,\tau]$ as used recently to give an alternative treatment of some evolution equations mainly maximal regularity and invariance problems. More precise, we prove, via examples, that the uniform $\pi-$integrability on all subintervals of fixed horizon $[0,\tau]$, is necessary, contrary to additive integral.

1 Introduction

The integral product, as introduced first time by Vito Volterra [19] in connection with the differential equation, became an efficient analytic tool to deal otherwise some important results such as solvability of evolution equations and maximal regularity [15] of their solutions. It constitutes a suitable field to reformulate interesting classical product formulas(e.g Chernoff [21], Trotter-Kato [20] or [10]) and some conjectures concerning regularity results ([4], [11] and [15] or more recently [5]).

For a complete historical point of view, A. Slavick summarized([22]) and developed ([23]) principal results concerning the integral product and its various applications mainly in physics and probability. For some applications to positivity, see the masterpiece [9].

The modern transcription of integral product, as reformulated in [16], is $\pi-$*integral* of a given family of semi groups $(T(t,s))_{s\geq 0}, t \in [0,\tau]$, see definition (1.1) below. It was first introduced by H. Laasri in [11] as a key notion to study well-posedness of linear non-autonomous evolutionary Cauchy problem $\dot{u}(t) + A(t)u(t) = f(t),\ \ t \in [0,\tau], \quad u(0) = u_0$, where each operator $A(s),\ s \in [0,\tau]$ generates a $\mathcal{C}_0$ semigroup $(\mathcal{T}_s(t))_{t\geq 0}$.
The main interest of this restitution of the integral product is certainly a new treatment of *maximal regularity* in different spaces and the establishment of alternative proofs of classical results such as Lions' theorem ([15]) and criteria of invariance of closed convex subsets (e.g [12], [13]and [16]) which play an important role in positivity conservation.
Unfortunately, the definition imposes that the convergence of walks:
$\P_\Lambda(c,d) := \mathcal{T}_l(d-\lambda_l)\mathcal{T}_{l-1}(\lambda_l-\lambda_{l-1})...\mathcal{T}_m(\lambda_{m+1}-\lambda_m)\mathcal{T}_{m-1}(\lambda_m-c)$ must occur for all ordered subdivisions $0 = \lambda_0 < \lambda_1 < \lambda_2.... < \lambda_n = \tau$ of $[0,\tau]$ but uniformly on the subintervals $[c,d]$ of $[0,\tau]$. It is an open question whether it is possible to warrant the $\pi-$integrability without assuming this constraining hypothesis. In this paper, a negative answer is given.

Throughout this paper, E denotes a Banach space and n is an integer. Let $[0,\tau]$ be an interval and Λ_θ an adapted subdivision

$$0 = \lambda_0 < \lambda_1 < ... < \lambda_n \ < \lambda_{n+1} = \tau.$$

A subordinate subdivision is a finite net $\theta := \{\theta_0, \theta_1, ..., \theta_n\}$, $k = 0, 1, ..., n$ such that for all k,

one has $\theta_k \in [\lambda_k, \lambda_{k+1}[$. The positive real number $|\Lambda_\theta| = \max|\lambda_{i+1} - \lambda_i|$, $i \in \{0, 1, \dot{n}\}$ is the mesh or a modulus of subdivision Λ_θ .
Let $(c, d) \in \Delta = \{(u, v) \in [0, \tau]^2,\ u \leq v\}$ and k_0 et k_1 two integers which satisfy $\lambda_{k_0} \leq c \leq \lambda_{k_0+1}$ and $\lambda_{k_1} \leq d < \lambda_{k_1+1}$.

Consider a family of semigroups $T(t, .)_{t\in[0,\tau]}$ and the corresponding *walks* $\mathcal{P}_{\Lambda_\theta}(c, d, s)$ defined on E by the following finite product:

$$\mathcal{P}_{\Lambda_\theta}(c, d, s)x = T(\theta_{k_1}, (d - \lambda_{k_1})s)T(\theta_{k_1-1}, (\lambda_{k_1} - \lambda_{k_1-1})s)... \\ ...T(\theta_{k_0+1}, (\lambda_{k_0+2} - \lambda_{k_0+1})s)T(\theta_{k_0}, (\lambda_{k_0+1} - c)s)x, \tag{1.1}$$

$$\mathcal{P}_{\Lambda_\theta}(c, d, s)x = x \text{ if } c = d. \tag{1.2}$$

The mapping $(c, d, s) \in \Delta \times \mathbb{R}_+ \mapsto \mathcal{P}_{\Lambda_\theta}(c, d, s)$ is strongly continuous as a composition of finite number of strongly continuous mappings.

Definition 1.1. [11, Definition 1.1.1] The family $t \longmapsto T(t, .)$ is $\pi-$integrable on $[0, \tau]$ if for all $x \in E$ the limit of $\mathcal{P}_{\Lambda_\theta}(c, d, s)x$ exists uniformly on s in compact sets of $\mathbb{R}_+$ and on $(c, d) \in \Delta$ when $|\Lambda_\theta| \to 0$, in which case we put

$$\mathcal{P}(c, d, s)x := \lim_{|\Lambda_\theta|\to 0} \mathcal{P}_{\Lambda_\theta}(c, d, s)x. \tag{1.3}$$

Naturally $\mathcal{P}(c, c, s)x = x$ for all $c \in [0, \tau]$ and $s \geq 0$. This convention is due to the fact: $\lim_{c\to d} \mathcal{P}_{\Lambda_\theta}(c, d, s)x = x$ (see [11], Proposition 1.1.5). If we reverse the order of composition, we obtain another aspect of $\pi-$integrability which is the dual of the first. Here, we do not distinguish between the two notions and we focus the attention on the following question:

Is the uniformity on $[c, d]$ in Definition 1.1 necessary ? In other words, are there mappings $T(t, .)$ which are $\pi-$integrable on $[0, \tau]$ but not $\pi-$integrable on some subinterval $[a, b]$ of the horizon $[0, \tau]$?

In the context of classical additive integration, this question seems to be not worth to be asked, but the answer for multiplicative case may not be immediate. Indeed, this is not obvious and in contrast to expectations, the answer is negative as it will be shown in the remainder of this paper.

The method of frozen coefficient to solve evolution equation problems of which the model is as follows:

$$\dot{u}(t) + A(t)u(t) = f(t), \quad u(0) = x, \tag{1.4}$$

where the operators $A(t),\ t \in [0, T]$, are time dependent, involves on approximating (1.4) by the piecewise non-autonomous problem:

$$\dot{u}_\wedge(t) + A_\wedge(t)u_\wedge(t) = f(t), \quad u_\wedge(0) = x. \tag{1.5}$$

Here we assume that all of operators $A(t)$ with domains $D(t)$ are defined on dense subspace $D \subset \bigcap_{t\geq 0} D(t)$ and that $A_\wedge$, $u_\wedge$ are given by:

$$A_\wedge(t) = A_k \quad \text{and} \quad u_\wedge(t) = u(t) \quad \text{where} \quad \lambda_k \leq t < \lambda_{k+1}.$$

The choice of $A_\wedge$ in (1.5) and then A_k is crucial and gives difference between various approximation methods of problem (1.4). The most known is the Yosida one (see[2]) but recently, an integral formula was suggested by El-Mennaoui, Laasri and Kuyantuo [3] and El-Mennaoui, Laasri [4] and seems to be more convenient for our purpose.
If the mapping $t \mapsto A(t)$ is so smooth, and especially if it is measurable and *relatively continuous* (see [1]), the problem (1.5) is well-posed in the sense of [24, Definition 1.4]. Its solution is given by walks $\mathcal{P}_{\Lambda_\theta}(0, \tau, s)$, introduced above, and the limit of finite product in (4.1) converges weakly to a limit which solves uniquely the problem (1.4) (see [15]). From there was born the

idea of $\pi-$integrability which treats existence and behavior of $\mathcal{P}(0,\tau,s) = \lim_{|\wedge|\to 0} \mathcal{P}_{\Lambda_\theta}(0,\tau,s)$. So definition (1.1) makes sense and treatment of **Question**.1 is so plausible.
This paper is organized as follows: After the given summary above on $\pi-$integrability, the next section is devoted to construct convenient spaces and define appropriate operators. Finally, we highlight the main result and give some interpretations for this unpredictable situation.

2 Preliminaries

Some constructions are necessary to give examples illustrating different regularity situations. Thus, we begin by putting up some convenient spaces and operators (see [8]) as suggested in the following proposition:

Proposition 2.1. *There exist a Banach E space and a family of semigroups $(T(t,.))_t$ on E such that*

- *The semigroups $(T(t,.))_t$ commute pairwise.*
- *The generators $A(t)$ of $(T(t,.))_t$ are defined on a common subspace F densely embedded in E.*
- *The mapping $t \mapsto A(t)$ is Lipschitz continuous.*

Proof. The proof of Proposition 2.1 will be given on three steps:

First, Let $(a_k)_{k\geq 1}$ be a sequence of numbers in $]0,\frac{1}{3}]$ and b_k, J_k as $b_k = 1 - a_k$, $J_k = [0, b_k]$. The mapping $h_k : \ J_k \mapsto \mathbb{R}$ defined by

$$h_k(x) = \chi_{[0,a_k[\cup[2a_k,b_k[} + 2\chi_{[a_k,2a_k[}$$

is a probability density . Let μ_k be the measure associated with h_k. It is allowed to define the product space $\Omega = \prod_{k\geq 1} J_k$. By Kolmogorov's Theorem ([**?**]), Ω may be endowed with probability measure ν which extends the finite distributions $\mu_{t_1,t_2,\ldots t_n} = \mu_1(C_1)\mu_2(C_2)....\mu_n(C_n)$ for all finite cylinder $C_1 \times C_2 \times C_3..... \times C_n$ to infinite one:

$$C_1 \times C_2 \times C_3..... \times C_n \times J_{n+1} \times J_{n+2}....$$

Henceforth, one can define L^1 space: $E = L^1(\Omega, \nu)$.

Second Consider for all integer j the operator A_j defined as

$$\begin{aligned} A_j : D(A_j) \subset \Sigma &\to \Sigma \\ f &\longmapsto -\frac{\partial f}{\partial x_j} \end{aligned}$$

where

$$\Sigma = \mathcal{C}^0(\mathbb{R}, E) \quad \text{and} \quad D(A) = \{f \in \Sigma\ ;\ f \text{ and } f'_j \in \Sigma\}.$$

The space E is as in step 1.
Let us now define a family $((T_k)_{k\geq 1})$ of nilpotent semigroups on E which coincide with the classical left translation:

$$\begin{cases} T_k(s)f(x_1, x_2, ...) = & f(x_1, x_2, ...x_{k-1}, x_k - s, x_{k+1}, ...)\chi_{J_k}(x_k - s) \\ f \in E, & \end{cases}$$

It is easy to see that A_j is the infinitesimal generator of the semigroup T_j still denoted as usual: $\forall s \geqslant 0,\ T_j(s) = e^{sA_j}$.

Third: A suitable combination A of the operators $(A_k)_{k\geqslant 1}$

For all $k \geq 1$ and each $s \in [0, \frac{1}{3}]$, we define t_k , β_k as $t_k = 1 - \frac{1}{2^{k-1}}$ and $\beta_k(s) = -(s - t_k)(s - t_{k+1})\chi_{[t_k,t_{k+1}[}(s)$. Consider the sequence $(a_k)_k$ given by

$$a_k = \frac{1}{2}\int_{t_k}^{t_{k+1}} \beta(s)ds = \frac{1}{12.2^{-3k}}$$

The numbers $(a_k)_k$ satisfy all conditions required for a numerical sequence by the step 1. So the associated spaces E and F then operators $(A_k, D(A_k))$ given in the step 2 are well defined. It yields that the operators $A(s)$ given by

$$A(s) = \begin{cases} \beta(s)A_k & \text{with domain} \quad D(A(s)) \quad \text{if } s \in]t_k, t_{k+1}[\\ 0 & \text{otherwise} \end{cases}$$

are well defined. Compactly, one can write $A(s) = \sum_k \beta(s)A_k\chi_{]t_k;t_{k+1}[}$.
As operators A_k, the operators $A(s)$ remain generators of commutative family of $\mathcal{C}_0-$semigroups $(e^{uA(s)})_{(u\geq 0, s\in[0,\frac{1}{3}])}$. Moreover, the family $(e^{uA(s)})_u$ is uniformly bounded and each domain $D(A(s))$ contains F as a subspace. (i.e $F \subset \bigcap_s D(A(s))$.) On the other hand, one verifies easily that the mapping $t \mapsto A(t)$ is strong Lipschitz continuous, which means that:

$$\exists K > 0 \quad \text{tel que} \quad \|A(t)f - A(s)f\| \leq K|t - s|$$

□

The semigroups $(e^{\cdot A_k})_{k\geq 1}$ seen in the second step have interesting properties summarized in the following proposition:

Proposition 2.2. *(i) The family $(e^{\cdot A_k})_k$ is commutative.*

(ii)

$$\|e^{sA_k}\| = \begin{cases} 2, & \text{for} \quad 0 \leqslant s \leqslant a_k \\ 1, & \text{for} \quad 2a_k \leqslant s \leqslant b_k \\ 0, & \text{otherwise} \end{cases}$$

(iii) $\|\prod_{k=1}^n e^{sA_k}\| = 2^n$ *for all* $s_1, s_2, ...s_k \in]0, 2a_k[$

(iv) $F = \{f \in \bigcap_{k\geq 1} D(A_k)\ ;\ \sup^k \|A_k f\| < \infty\}$ *is continuously and densely embedded in* E.

(v) The space F*, endowed with the norm* $\|f\|_F = \max\{\|f\|_E; \sup\|_{k\geq 1}A_k f\|\}$*, is a Banach space.*

Remark 2.3. i)The property 4 is true when F is endowed with the norm given by property 5.
ii) The proofs of above properties may be found in [8, Lemma 3.1].

3 Integral product and stability

The notion of stability is important to ensure well-posedness of evolution problems. In literature there are many definitions of stability. The most suitable for integral product of semigroups is:

Definition 3.1. A family of C_0-semigroups $\{T(t,.), t \in [a,b]\}$ is *exponentially stable* if there exist $M > 0$ and $\omega \in \mathbb{R}$ such as

$$\|\mathcal{P}_{\Lambda_\theta}(c,d,s)\| \leq Me^{s\omega} \ (s \geq 0), \tag{3.1}$$

for all $(c,d) \in \Delta$ and all subdivision Λ_θ de $[a,b]$.
The family $\{T(t,.), t \in [a,b]\}$ is *stable* if

$$\|\mathcal{P}_{\Lambda_\theta}(c,d,s)\| \leq M \quad (0 \leq s \leq 1), \tag{3.2}$$

for all $(c,d) \in \Delta$ and all subdivision Λ_θ de $[a,b]$.

Remark 3.2. • The operator $\mathcal{P}_{\Lambda_\theta}$ denotes the finite product in Definition1.1.

- The exponential stability implies stability. The two notions are equivalent if semi-groups $T(t,.)$ commute.
- These definitions fit with those of Kato [6], [7] and Schnaubelt [8].

With additional assumption, stability warrants that the linear Cauchy evolution problem

$$\text{nCP}(0,\tau)\begin{cases} \dot{u}(t) = & A(t)u(t),\ t\in[0,\tau]\subset\mathbb{R}_+, \\ u(0) = & x\in E, \end{cases}$$

is well posed. The following theorem gets more precise this result:

Theorem 3.3. *Assume that*

i) *The semigroups $T(t,.)_{t\in[0,\tau]}$ commute pairwise*

ii) *The mapping $t\mapsto A(t)x$ is continuous for all x in some dense subspace $D\subset\bigcap_{t\in[0,\tau]}D(A(t))$*

ii) *The family of semigroups $T(t,.)_{t\in[0,\tau]}$ is stable in the sense of Definition 3.1*

Then

- *The function $t\mapsto T(t,.)$ is $\pi-$integrable.*
- *The $\pi-$integral $P(0,\tau,s)=\lim\limits_{\wedge_\theta\to 0}P_{\wedge_\theta}$ solves uniquely the Cauchy problem* nCP$(0,\tau)$

In fact, there are weaker hypotheses to ensure well-posedness of nCP$(0,\tau)$, but for our purpose, under assumptions i), ii) and iii) in Theorem 3.3, resolvability of the Cauchy problem suffices to emphasize our result. It is known for example that the first hypothesis may be weakened by assuming only that the mapping $t\mapsto A(t)x$ is relatively continuous for all x in a common dense domain of all operators $A(t)$ (see [1]).

Proof. of Theorem3.3
For the first item, and according to [11, Proposition1.4.5], it suffices to prove that

$$\{x\in\bigcap_{t\in[0,\tau]}D(A(t)),t\mapsto A(t)x\quad\text{is Riemann integrable on }\ [0,\tau]\}$$

is a dense subset of E. This is obvious by assumption $ii)$.
The second one is the reason for which the $\pi-$integrability is introduced. One may see for example [3].

□

A natural question arises: is the family $T(t,.)$ stable when it is $\pi-$integrable? The answer is the object of Proposition 1.5.8 of [16]. Unfortunately, the stability alone or even exponential stability does not suffice to warrant well-posedness of nCP$(0,\tau)$.

4 The main result

The idea is to constrain the first semigroup $T(1,s)_s$ in the product $P_{\wedge_\theta}$ to be so smooth to ensure that all orbits $T(1,s)x$ embed in all domains of other operators. It is a kind of a most regular beginning of the first operator and its semigroup. The following theorem makes the result more precise

Theorem 4.1. *Consider the family $T(t,s)=e^{sA_1}\chi_{[-1,0]}(s)+e^{sA(s)}\chi_{[0,1]}(s)$. Then*

- *The mapping $(t,s)\in[-1,1]^2\mapsto T(t,s)$ is $\pi-$integrable.*
- *The family $(t,s)\in[0,1]^2\mapsto T(t,s)$ is not stable thus not $\pi-$integrable .*

The operators $A(t)$ refer to those seen at section2.

Proof. A direct computation leads to

$$\mathcal{P}_{\Lambda_\theta}(-1,1,s)x = T_1(s)T(\theta_{k_1},\lambda_{k_1}s)T(\theta_{k_2},(\lambda_{k_2}-\lambda_{k_1})s)... \\ ...T(\theta_{k_{p-1}},(\lambda_{k_p}-\lambda_{k_{p-1}})s)T(\theta_{k_p},(1-\lambda_{k_p})s)x, \quad (4.1)$$

for all subdivision Λ_k and all subordinate one $\Theta = \theta_{k_1} < \theta_{k_2} < \theta_{k_3} < ... < \theta_{k_p} < 1$ of $[-1;1]$. Here, $T_1(.)$ denotes the semigroup $e^{\cdot A_1}$ which vanishes for all $s > \frac{3}{4}$ according to the second point of proposition 2.2. The $\pi-$integrability on $[-1;1]$ is so evident because the finite products are nulls.
The second result in the Theorem 4.1 is an immediate consequence of the third point in Proposition 2.2. □

Remark 4.2. The construction is so natural in the following sense: it is enough to compose an irregular product of semigroups (e.g unstable ones) with a most regular operator which leads all walks to its domain. It will be the case if, for example, we consider a holomorphic semigroup T on $[0;a]$ as the first term of product and let any pathologic semigroups $(S(t,.)_t$ be composed when $t \in [a;\tau]$. the $\pi-$integrability on the whole interval $[0;\tau]$ is possible but difficult on $[a;\tau]$.

Now, if one projects to give more examples, it will be enough to consider instead of $(e^{tA_1})_{t\geq 0}$, any rescaled version of unbounded semigroup. For instance, the Riemann-Liouville semigroup of operators $J(t)_{t\geq 0}$ given, for every $f \in L^1[0,1]$ by

$$J(z)f(t) = \int_0^t \frac{(t-s)^{z-1}}{\Gamma(z)} f(s)\mathrm{d}s \ (f \in L^1[0,1]). \quad (4.2)$$

According to [14], this semigroup is not bounded and its norm when it acts on $L^2[0,1]$ is given by $\|J(t)\| = e^t$. So when by rescaling, e.g considering $Tj(t) = J(jt)$ for all $j \in \mathbb{N}$, we may follow the same way to construct other examples.

A most difficult question is to ensure $\pi-$integrability without assuming uniformity on the parameter s, in Definition 1.1. It will be our next investigation.

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