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AN EFFECTIVE TAYLOR WAVELETS BASIS FOR THE EVALUATION OF NUMERICAL DIFFERENTIATION

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Abstract Numerical differentiation has been used to deal with many applications in physics and engineering. Therefore, an appropriate approach is very important for the evaluation of numerical differentiation. This paper suggests a numerical approach to evaluate the numerical differentiation of any function that is known only at discrete points. The suggested approach is based on approximation of the unknown function in terms of series of Taylor wavelets basis. The error estimation and convergence behaviour of the suggested approach is illustrated by performing some numerical examples. This approach gives accurate results with exact solutions.

1 Introduction

Numerical differentiation is the procedure of determining the derivatives of a function at given values of the independent variable in the domain. It is the most frequently needed method in computational physics. Numerical differentiation [1, 2] has been used for solving various differential and integral equations, and it has many applications in engineering and science [3]. It is relatively common to use the estimation of a function's derivative when solving the problems of engineering. When training neural networks, the computation of the derivative is also applied for gradient techniques. Moreover, numerical differentiation is also used in data analysis, dynamics, solid mechanics, heat transfer, kinematics simulation, computational fluid dynamics, machine learning and complex system optimization [3, 4, 5]. Many numerical approaches have been studied to evaluate numerical differentiation like Newton's backward and forward interpolation formula, Sterling's formula, Newton's divided difference (NDD) formula, Lagrange's formula, etc.

In recent years, wavelet approaches have made their way to solve integral and differential equations arising in several engineering and scientific problems. Wavelets theory [6, 7, 8] is a relatively new and emerging field in mathematical research and is being widely employed as an effective tool in various real time applications. Wavelets are mathematical functions which have been extensively used in signal processing for waveform representation and segmentations, human vision, quick algorithms for easy implementations, prediction of earthquakes, image compression, and many other areas of applied and pure mathematics [9, 10, 11].

The applications of different wavelet such as Hermite wavelet, Bernoulli wavelet, Muntz wavelet, Chebyshev wavelet and Legendre wavelet show that wavelets are powerful tool for solving the problems of applied science and engineering. Several wavelet techniques and approximation schemes have been investigated to solve integral equations and differential equations. Some of them are mentioned in details. Lepik [12] introduced the Haar wavelet scheme for determining the approximate solution of differential equations. Babolian and Zadeh [13] solved the second order differential equations through Chebyshev wavelet basis with collocation nodes. A method by means of Galerkin scheme and Meyer wavelets established by Dou et al. for solving ill-posed problem in [14]. Mohammadi [15] provided a scheme based on extended Legendre wavelets for solving differential equations such as singular boundary value problem. In [16], authors proposed a new technique through Hermite wavelets for solving the class of boundary value problems. Youssri et al. [17] solved the Lane Emden type differential equation using an approach based on ultraspherical wavelets. Hermite wavelets scheme is discussed by Khashem [18] for the estimate solution of Bratu's problem. In [19], author used trigonometric basis with Galerkin scheme for the approximation of differentiation. Singh and Kaur [20] presented Hermite wavelet basis to determine the numerical approximation of differentiation. In [21], author introduced the Hermite extension scheme for the evaluation of differentiation. See, [22, 23, 24, 25, 26] and the references therein for more descriptions of wavelet techniques and their applications.

Influenced by the best performance of these schemes, we will employ the similar technique for numerical differentiation with Taylor wavelets. The main aim of the present article is to find the numerical differentiation by using Taylor wavelets basis. In this research, a direct numerical technique based on Taylor wavelets introduces for evaluating numerical differentiation. The proposed scheme contains of expanding the unknown function through Taylor wavelets basis with the wavelet coefficients. After computing the unknown function at given tabular points, a set of algebraic equations is obtained for estimating the wavelet coefficients.

This article is structured as follows: Section 2 includes the definition of Taylor wavelets basis. In Section 3, a direct numerical approach based on Taylor wavelets basis is proposed for evaluating numerical differentiation. In Section 4, function approximation through Taylor wavelets has been explained. In Section 5, we present the error bound and convergence of the proposed method. Section 6 presents some test problems illustrating the accuracy of the suggested approach. Finally, the conclusion remark is provided in the last Section 7.

2 Taylor wavelets

Taylor wavelets $\psi_{n,m}(t) = \psi(k,t,m,n)$ has the four arguments in which $k \in \mathbb{Z}^+$, t is the normalized time, m is the order of Taylor polynomial and $n = 1, 2, 3, ..., 2^{k-1}$.

The definition of Taylor wavelets on the interval $[0, 1)$ is as follows [27]

$$\psi_{n,m}(t) = \begin{cases} \sqrt{(2m+1)}\, 2^{\frac{k-1}{2}} T_m(2^{k-1}t - n + 1), & t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\ 0, & otherwise \end{cases},$$

where $m = 0, 1, 2, 3, ..., M-1$ and $M \in \mathbb{N}$. Here, $T_m(t)$ is the Taylor polynomial of degree m given by

$$T_m(t) = t^m. \tag{2.1}$$

Now, generating the Taylor wavelets basis on the interval $[0, \beta)$ for a positive number β by using $\psi_{n,m}(t)$ as

$$\psi^{\beta}_{n,m}(t) = \begin{cases} \sqrt{(2m+1)}\, 2^{\frac{k-1}{2}} T_m(2^{k-1}\frac{t}{\beta} - n + 1), & t \in \left[\frac{(n-1)\beta}{2^{k-1}}, \frac{n\beta}{2^{k-1}}\right) \\ 0, & otherwise \end{cases}, \tag{2.2}$$

where $T_m(t)$ is given in Eq. (2.1).

For $k = 1, M = 6$ and $\beta = 5$, we obtain the Taylor wavelets basis on $[0, 5)$ as

$$\psi^5_{1,0}(t) = 1,$$

$$\psi^5_{1,1}(t) = \frac{\sqrt{3}}{5}t,$$

$$\psi^5_{1,2}(t) = \frac{\sqrt{5}}{25}t^2,$$

$$\psi^5_{1,3}(t) = \frac{\sqrt{7}}{125}t^3,$$

$$\psi^5_{1,4}(t) = \frac{\sqrt{9}}{625}t^4,$$

$$\psi^5_{1,5}(t) = \frac{\sqrt{11}}{3125}t^5.$$

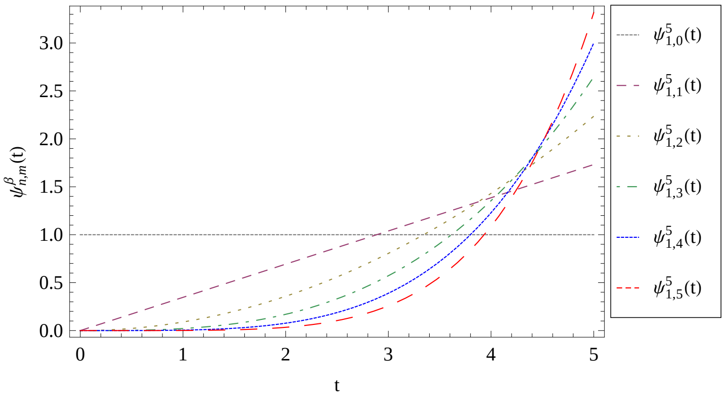


Figure 1. Graph of Taylor wavelets basis on the interval $[0,5)$ for $M=6$ and $k=1$.

Figure 1 shows the graph of Taylor wavelets basis for $M=6$ and $k=1$.

Now, using these Taylor wavelets basis, the function approximation is stated in the following section.

3 Function approximation via Taylor wavelets

A function $f(t) \in L^2[0,\beta]$ can be expanded by Taylor wavelets series as

$$f(t) \approx \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} u_{n,m}\psi^{\beta}_{n,m}(t), \tag{3.1}$$

where $u_{n,m}$ be the unknown wavelet coefficients with respect to Taylor wavelets $\psi^{\beta}_{n,m}(t)$.

If the series in Eq. (3.1) is truncated, then the truncated series can be expressed as

$$\begin{aligned} f(t) &\approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} u_{n,m}\psi^{\beta}_{n,m}(t) \\ &= U^T\Psi(t) = \Psi^T(t)U, \end{aligned} \tag{3.2}$$

where U and $\Psi(t)$ are $2^{k-1}M \times 1$ order matrices given by

$$U^T = \left[u_{1,0}, u_{1,1}, \ldots, u_{1,M-1}, u_{2,0}, u_{2,1}, \ldots, u_{2,M-1}, \ldots, u_{2^{k-1},0}, u_{2^{k-1},1}, \ldots, u_{2^{k-1},M-1}\right], \tag{3.3}$$

and

$$\Psi(t) = \left[\psi^{\beta}_{1,0}, \psi^{\beta}_{1,1}, \ldots, \psi^{\beta}_{1,M-1}, \psi^{\beta}_{2,0}, \psi^{\beta}_{2,1}, \ldots, \psi^{\beta}_{2,M-1}, \ldots, \psi^{\beta}_{2^{k-1},0}, \psi^{\beta}_{2^{k-1},1}, \ldots, \psi^{\beta}_{2^{k-1},M-1}\right]^T \tag{3.4}$$

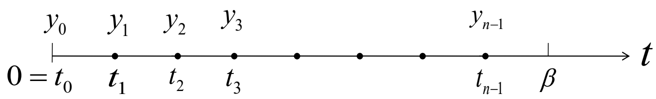
The following section presents the Taylor wavelets scheme to evaluate numerical differentiation by using an approximation method.

4 Wavelets scheme for differentiation

Suppose we are provided the following data set of values of $y(t)$ with respect to discrete values of $t \in [0,\beta)$ as

t	t_0	t_1	t_2	$\cdots$	t_{n-1}
$y=y(t)$	$y_0=y(t_0)$	$y_1=y(t_1)$	$y_2=y(t_2)$	$\cdots$	$y_{n-1}=y(t_{n-1})$

The above tabular data are represented in the number line as



The function $y(t)$ is given only at the tabular points $t_0, t_1, t_2, t_3, ..., t_{n-1}$. These tabular points or arguments may or may not be equidistant.

To determine the derivatives of $y(t)$ from the above data, the procedure of suggested wavelets scheme is provided stepwise as:
(I) First, approximate the unknown function $y(t)$ in the terms of Taylor wavelets series by using Eq. (3.2) as

$$y(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} u_{n,m}\psi^{\beta}_{n,m}(t) = U^T\Psi(t), \tag{4.1}$$

where U and $\Psi(t)$ are given in Eqs. (3.3) and (3.4) respectively.
After expanding the summation, we get $y(t)$ as

$$\begin{aligned} y(t) =& u_{1,0}\psi^{\beta}_{1,0}(t) + u_{1,1}\psi^{\beta}_{1,1}(t) + ... + u_{1,M-1}\psi^{\beta}_{1,M-1}(t) + u_{2,0}\psi^{\beta}_{2,0}(t) + u_{2,1}\psi^{\beta}_{2,1}(t) \\ &+ ... + u_{2,M-1}\psi^{\beta}_{2,M-1}(t) + ... + u_{2^{k-1},0}\psi^{\beta}_{2^{k-1},0}(t) + u_{2^{k-1},1}\psi^{\beta}_{2^{k-1},1}(t) + ... + u_{2^{k-1},M-1}\psi^{\beta}_{2^{k-1},M-1}(t). \end{aligned} \tag{4.2}$$

(II) By using the tabular points $t_0, t_1, t_2, t_3, ..., t_{n-1}$ in Eq. (4.2), we get the following set of equations as

$$\begin{aligned} y_0 =& y(t_0) \\ =& u_{1,0}\psi^{\beta}_{1,0}(t_0) + u_{1,1}\psi^{\beta}_{1,1}(t_0) + ... + u_{1,M-1}\psi^{\beta}_{1,M-1}(t_0) + u_{2,0}\psi^{\beta}_{2,0}(t_0) + u_{2,1}\psi^{\beta}_{2,1}(t_0) \\ &+ ... + u_{2,M-1}\psi^{\beta}_{2,M-1}(t_0) + ... + u_{2^{k-1},0}\psi^{\beta}_{2^{k-1},0}(t_0) + u_{2^{k-1},1}\psi^{\beta}_{2^{k-1},1}(t_0) + ... + u_{2^{k-1},M-1}\psi^{\beta}_{2^{k-1},M-1}(t_0), \end{aligned} \tag{4.3}$$

$$\begin{aligned} y_1 =& y(t_1) \\ =& u_{1,0}\psi^{\beta}_{1,0}(t_1) + u_{1,1}\psi^{\beta}_{1,1}(t_1) + ... + u_{1,M-1}\psi^{\beta}_{1,M-1}(t_1) + u_{2,0}\psi^{\beta}_{2,0}(t_1) + u_{2,1}\psi^{\beta}_{2,1}(t_1) \\ &+ ... + u_{2,M-1}\psi^{\beta}_{2,M-1}(t_1) + ... + u_{2^{k-1},0}\psi^{\beta}_{2^{k-1},0}(t_1) + u_{2^{k-1},1}\psi^{\beta}_{2^{k-1},1}(t_1) + ... + u_{2^{k-1},M-1}\psi^{\beta}_{2^{k-1},M-1}(t_1), \end{aligned} \tag{4.4}$$

$$\begin{aligned} y_2 =& y(t_2) \\ =& u_{1,0}\psi^{\beta}_{1,0}(t_2) + u_{1,1}\psi^{\beta}_{1,1}(t_2) + ... + u_{1,M-1}\psi^{\beta}_{1,M-1}(t_2) + u_{2,0}\psi^{\beta}_{2,0}(t_2) + u_{2,1}\psi^{\beta}_{2,1}(t_2) \\ &+ ... + u_{2,M-1}\psi^{\beta}_{2,M-1}(t_2) + ... + u_{2^{k-1},0}\psi^{\beta}_{2^{k-1},0}(t_2) + u_{2^{k-1},1}\psi^{\beta}_{2^{k-1},1}(t_2) + ... + u_{2^{k-1},M-1}\psi^{\beta}_{2^{k-1},M-1}(t_2), \end{aligned} \tag{4.5}$$

$$\vdots$$

$$\begin{aligned} y_{n-1} =& y(t_{n-1}) \\ =& u_{1,0}\psi^{\beta}_{1,0}(t_{n-1}) + u_{1,1}\psi^{\beta}_{1,1}(t_{n-1}) + ... + u_{1,M-1}\psi^{\beta}_{1,M-1}(t_{n-1}) + u_{2,0}\psi^{\beta}_{2,0}(t_{n-1}) + u_{2,1}\psi^{\beta}_{2,1}(t_{n-1}) \\ &+ ... + u_{2,M-1}\psi^{\beta}_{2,M-1}(t_{n-1}) + ... + u_{2^{k-1},0}\psi^{\beta}_{2^{k-1},0}(t_{n-1}) + u_{2^{k-1},1}\psi^{\beta}_{2^{k-1},1}(t_{n-1}) \\ &+ ... + u_{2^{k-1},M-1}\psi^{\beta}_{2^{k-1},M-1}(t_{n-1}). \end{aligned} \tag{4.6}$$

(III) Solve the set of algebraic equations in Eqs. (4.3)-(4.6), we can obtain unknown wavelet coefficients vector U.
(IV) Substituting the value of U in Eq. (4.1), we determine the wavelets approximation of $y(t)$.
(V) Now, differentiating Eq. (4.1) with respect to t for determining the derivatives of $y(t)$ as

$$\begin{aligned} y'(t) =& U^T \Psi'(t) \\ y''(t) =& U^T \Psi''(t) \\ y'''(t) =& U^T \Psi'''(t) \end{aligned}$$

In this way, we can find the successive higher derivatives of $y(t)$.
Procedure completed.

The algorithm for the proposed scheme is expressed as

Algorithm:

Input: $k, M, n, \beta; t_0, t_1, t_2, t_3, ..., t_{n-1}, y_0, y_1, y_2, y_3, ..., y_{n-1}$.
Step 1. Define the Taylor wavelets $\Psi(t)$ by Eqs. (3.4) and (2.2).
Step 2. Introduce the vector U using Eq. (3.3).
Step 3. Approximate the unknown function $y(t)$ in term of $\Psi(t)$ from Eq. (4.1).
Step 4. Extract the set of linear algebraic equations (4.3)-(4.6) at tabular points $t_0, t_1, t_2, t_3, ..., t_{n-1}$.
Step 5: Solve the set of algebraic equations obtained in Step 4 and evaluate the vectors U.
Output: The approximated Taylor wavelets solutions of $y(t)$ and its derivatives

$$\begin{aligned} y(t) =& U^T \Psi(t) \\ y'(t) =& U^T \Psi'(t) \\ y''(t) =& U^T \Psi''(t) \\ y'''(t) =& U^T \Psi'''(t). \end{aligned}$$

In the following section, error formulae, lemma and theorem are presented which supports the convergence of the suggested scheme.

5 Error calculation and convergency of Taylor wavelets

Some of the error formulae are presented to analyse the errors in the results of computation.

(a) The absolute error between the wavelet approximation function $y_{wave}(t)$ and exact function $y(t)$ is estimated as

$$E_{Abs}\{y, y_{wave}\} = |y(t) - y_{wave}(t)|.$$

Similarly, we can estimate the absolute error between the derivatives of wavelet approximation function and exact function as

$$E_{Abs}\{y^{(j)}, y^{(j)}_{wave}\} = |y^{(j)}(t) - y^{(j)}_{wave}(t)|,$$

where y^j denotes the jth derivative of y.

(b) To estimate the computational error of approximate wavelet solution, the following relative error are used.

$$E_j = \frac{\left\| y^j(t) - y^j{}_{wave}(t) \right\|_{L^2[0,\beta]}}{\left\| y^j(t) \right\|_{L^2[0,\beta]}}; \quad j = 0, 1, 2, ... \tag{5.1}$$

where $y^j(t)$ denotes the jth derivative of $y(t)$.

The following Lemma provide us a necessary part which support the error bound of the approximate solution in Theorem 5.2.

Lemma 5.1. *Let $y(t) \in \boldsymbol{C}^m[\alpha,\beta]$ and $P_m(t)$ be the polynomial of degree m which interpolates $y(t)$ at the Chebyshev grids on $[\alpha,\beta]$, then the interpolation error is given by*

$$|y(t) - P_m(t)| \leq \frac{2}{\Gamma(m+1)} \left(\frac{\beta-\alpha}{4}\right)^m \max_{\xi\in[\alpha,\beta]} |y^m(\xi)| .$$

Proof: *See References [28, 29].*

The error bound of the approximate solution through Taylor wavelets series is given in the following theorem.

Theorem 5.2. *Let $y(t) \in \boldsymbol{C}^m[0,\beta]$ and $y(t) \simeq U^T\Psi(t)$ is the best approximation of $y(t)$, then the error bound of approximate solution by using Taylor wavelets series would be obtained as*

$$||y(t) - U^T\Psi(t)||_{L^2[0,\beta]} \leq \frac{\tau}{\Gamma(m+1)2^{mk+m-1}\beta^{-m-1/2}},$$

where $\tau = \max\limits_{\xi\in[0,\beta]} |y^m(\xi)|$.

Proof: *We divide the interval $[0,\beta]$ into subinterval $\left[\frac{(n-1)\beta}{2^{k-1}}, \frac{n\beta}{2^{k-1}}\right]$ with $n \in \{1,2,3,...,2^{k-1}\}$. Since $U^T\Psi(t)$ is the best approximation of $y(t)$, then by using Lemma 5.1, we have*

$$\begin{aligned}
||y(t) - U^T\Psi(t)||^2_{L^2[0,\beta]} &= \int_0^\beta \left(y(t) - U^T\Psi(t)\right)^2 dt \\
&= \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)\beta}{2^{k-1}}}^{\frac{n\beta}{2^{k-1}}} \left(y(t) - U^T\Psi(t)\right)^2 dt \\
&\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)\beta}{2^{k-1}}}^{\frac{n\beta}{2^{k-1}}} \left(y(t) - P_m(t)\right)^2 dt \\
&\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)\beta}{2^{k-1}}}^{\frac{n\beta}{2^{k-1}}} \left[\frac{2}{\Gamma(m+1)} \left(\frac{\frac{\beta}{2^{k-1}}}{4}\right)^m \max_{\xi\in\left[\frac{(n-1)\beta}{2^{k-1}},\frac{n\beta}{2^{k-1}}\right]} |y^m(\xi)|\right]^2 dt \\
&\leq \sum_{n=1}^{2^{k-1}} \int_{\frac{(n-1)\beta}{2^{k-1}}}^{\frac{n\beta}{2^{k-1}}} \left[\frac{2}{\Gamma(m+1)} \left(\frac{\frac{\beta}{2^{k-1}}}{4}\right)^m \max_{\xi\in[0,\beta]} |y^m(\xi)|\right]^2 dt \\
&\leq \int_0^\beta \left(\frac{2\beta^m}{\Gamma(m+1)4^m 2^{mk-m}}\tau\right)^2 dt \\
&\leq \left(\frac{2\tau\beta^m}{\Gamma(m+1)4^m 2^{mk-m}}\right)^2 \beta.
\end{aligned}$$

By taking square root, we obtain the upper bound of the error as

$$||y(t) - U^T\Psi(t)||_{L^2[0,\beta]} \leq \frac{\tau}{\Gamma(m+1)2^{mk+m-1}\beta^{-m-1/2}}.$$

Remark 5.3. It is clearly seen that the approximation error of function $y(t)$, depend on the term $1/(\Gamma(m+1)2^{mk+m-1}\beta^{-m-1/2})$ rapidly decays.
When $m \to$ fixed and $k \to \infty$, then

$$\left|\frac{n\beta}{2^{k-1}} - \frac{(n-1)\beta}{2^{k-1}}\right| \to 0,$$

this implies

$$\int_{\frac{(n-1)\beta}{2^{k-1}}}^{\frac{n\beta}{2^{k-1}}} (y(t) - U^T\Psi(t))^2 dt \to 0.$$

Therefore,

$$\lim_{k\to\infty} ||y(t) - U^T\Psi(t)||_{L^2[0,\beta]} = 0.$$

Also, when $k \to$ fixed and $m \to \infty$, then

$$\lim_{m\to\infty} ||y(t) - U^T\Psi(t)||_{L^2[0,\beta]} = 0.$$

This shows the convergence of approximation of Taylor wavelets to $y(t)$.

6 Method Implementation

The suggested approach is applied on several examples to examine the applicability and performance of the approach and the results obtained are compared with exact solution. All numerical results are achieved by software Mathematica 7.

Example 1

Consider the data given in Table 1:

Table 1. Data for Example 1

t	0	1	3	4
$y = y(t)$	-12	0	12	24

Using the Lagrange interpolation formula, we obtain the polynomial $y(t)$ as

$$y(t) = t^3 - 6t^2 + 17t - 12.$$

We solve this example for $k = 1$ and $M = 4$ by using the procedure given in section 4 and obtained the function which is exactly same as the function obtained through Lagrange interpolation formula. The first, second and third derivatives of $y(t)$ is obtained as

$$\begin{aligned} y'(t) &= 3t^2 - 12t + 17, \\ y''(t) &= 6t - 12, \\ y'''(t) &= 6. \end{aligned}$$

The estimated errors of derivatives of $y(t)$ by Taylor wavelets basis are listed in the Table 2. The graphs of approximate and exact solutions are given in the Figures 2-4.

Table 2. Estimated absolute errors of derivatives of $y(t)$ in Example 1.

t	$y'(t)$	$y''(t)$	$y'''(t)$
0	7.1×10^{-15}	1.0×10^{-14}	6.2×10^{-15}
1	8.8×10^{-16}	5.3×10^{-15}	6.2×10^{-15}
3	0	5.3×10^{-15}	6.2×10^{-15}
4	7.1×10^{-15}	1.0×10^{-14}	6.2×10^{-15}

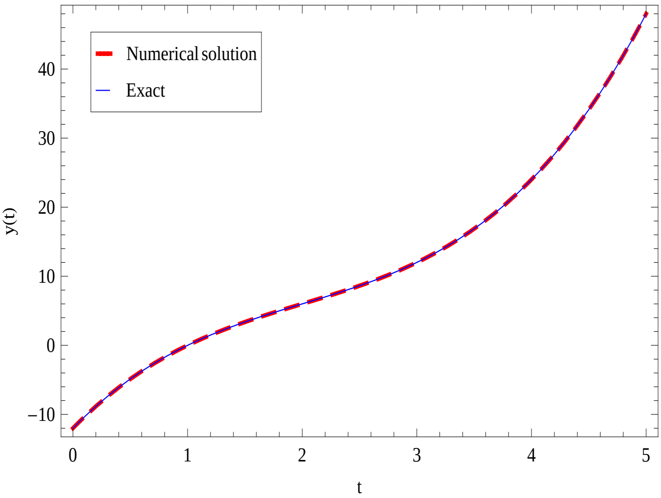


Figure 2. $y(t)$ and its wavelet approximation in Example 1.

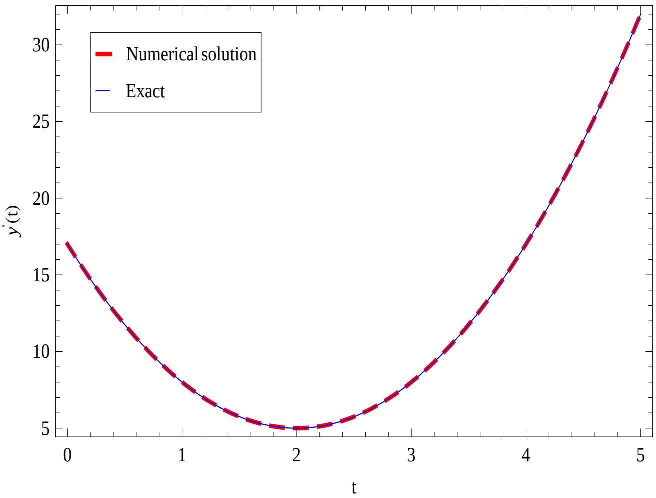


Figure 3. $y'(t)$ and its wavelet approximation in Example 1.

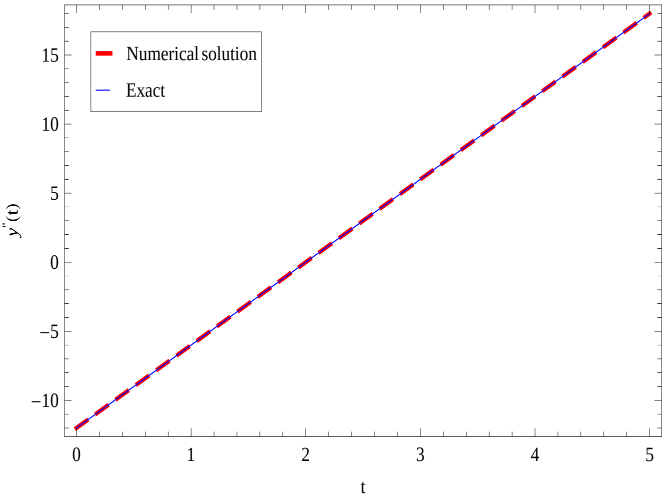


Figure 4. $y''(t)$ and its wavelet approximation in Example 1.

Example 2

Consider the data given in Table 3:

Table 3. Data for Example 2

t	0	0.2	0.4	0.6	0.8
$y = y(t)$	1	1.008	1.064	1.216	1.512

Using the Newton forward or backward interpolation formula, we get the polynomial $y(t)$ as

$$y(t) = t^3 + 1.$$

We solve this example for $k = 1$ and $M = 5$ by using the procedure given in section 4 and obtained the function which is exactly same as the function obtained by Newton forward or backward interpolation formula. The first, second and third derivatives of $y(t)$ is obtained as

$$y'(t) = 2.6 \times 10^{-14}t^3 + 3t^2 + 1.1 \times 10^{-14}t - 6.9 \times 10^{-16},$$
$$y''(t) = 7.8 \times 10^{-14}t^2 + 6t + 1.1 \times 10^{-14},$$
$$y'''(t) = 1.5 \times 10^{-13}t + 6.$$

The estimated errors of derivatives of $y(t)$ by Taylor wavelets basis are listed in the Table 4. The graphs of approximate and exact solutions are given in the Figures 5-7.

Table 4. Estimated absolute errors of derivatives of $y(t)$ in Example 2.

t	$y'(t)$	$y''(t)$	$y'''(t)$
0	6.9×10^{-16}	1.1×10^{-14}	6.6×10^{-14}
0.2	4.0×10^{-16}	9.0×10^{-16}	3.5×10^{-14}
0.4	8.3×10^{-17}	3.0×10^{-15}	4.1×10^{-15}
0.6	4.0×10^{-16}	7.6×10^{-16}	2.7×10^{-14}
0.8	1.9×10^{-16}	7.7×10^{-15}	5.8×10^{-14}

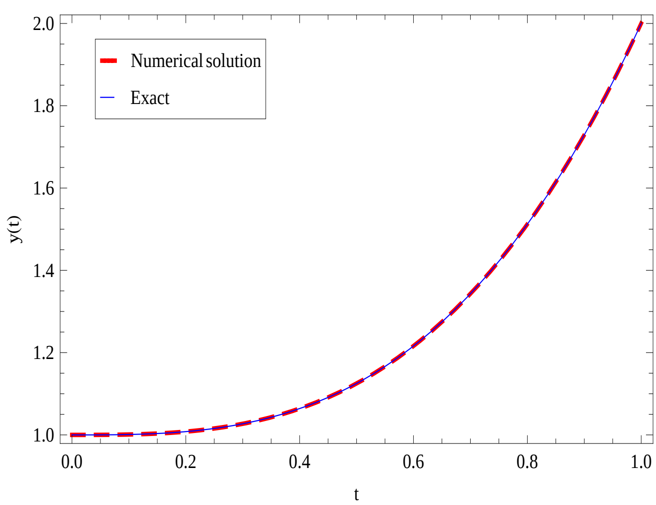


Figure 5. $y(t)$ and its wavelet approximation in Example 2.

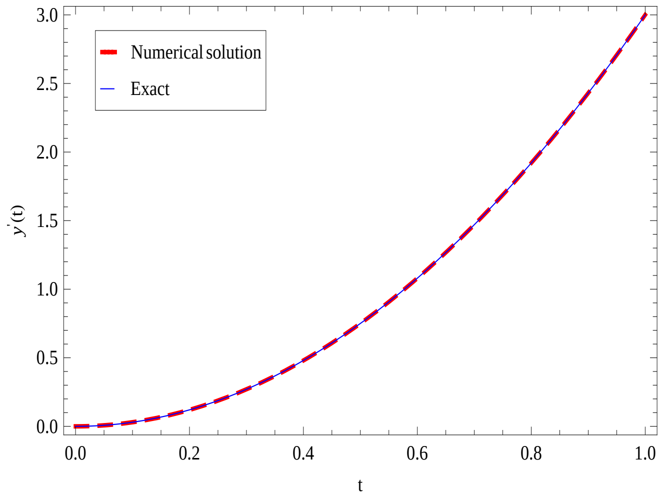


Figure 6. $y'(t)$ and its wavelet approximation in Example 2.

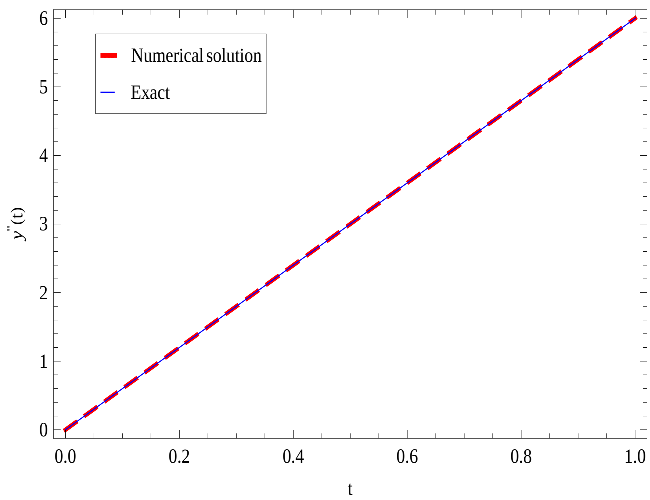


Figure 7. $y''(t)$ and its wavelet approximation in Example 2.

Example 3

Consider the data given in Table 5:

Table 5. Data for Example 3

t	0	1	2	4
$y = y(t)$	1	1	2	5

Using the NDD interpolation formula, we obtain the polynomial $y(t)$ as

$$y(t) = -\frac{1}{12}t^3 + \frac{3}{4}t^2 - \frac{2}{3}t + 1.$$

We solve this example for $k = 1$ and $M = 4$ by using the procedure given in section 4 and obtained the function which is exactly same as the function obtained by NDD interpolation formula. The first, second and third derivatives of $y(t)$ is obtained as

$$y'(t) = -0.25t^2 + 1.5t - 0.6,$$
$$y''(t) = -0.5t + 1.5,$$
$$y'''(t) = -0.5.$$

The estimated errors of derivatives of $y(t)$ by Taylor wavelets basis are listed in the Table 6. The graphs of approximate and exact solutions are given in the Figures 8-10.

Table 6. Estimated absolute errors of derivatives of $y(t)$ in Example 3.

t	$y'(t)$	$y''(t)$	$y'''(t)$
0	1.3×10^{-15}	1.9×10^{-15}	9.9×10^{-16}
1	1.9×10^{-16}	1.0×10^{-15}	9.9×10^{-16}
2	7.7×10^{-16}	1.1×10^{-16}	9.9×10^{-16}
4	8.8×10^{-16}	1.7×10^{-15}	9.9×10^{-16}

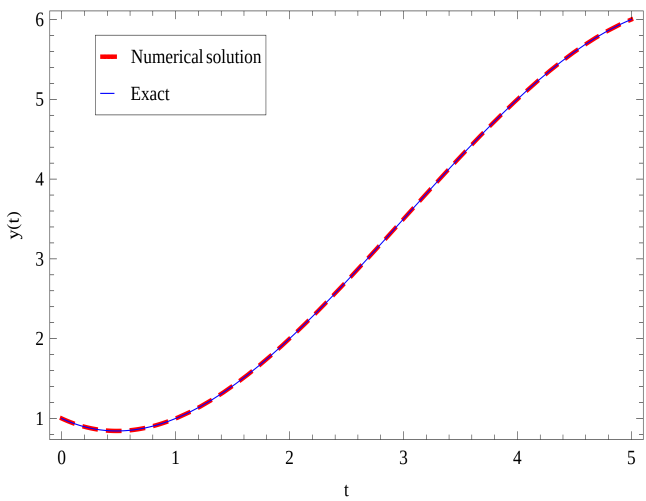


Figure 8. $y(t)$ and its wavelet approximation in Example 3.

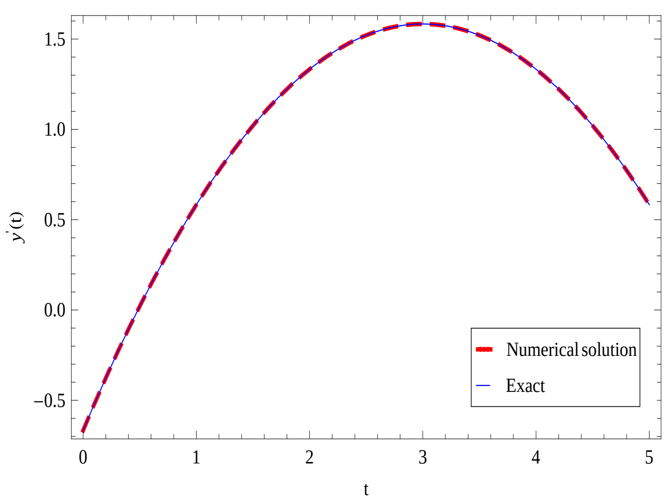


Figure 9. $y'(t)$ and its wavelet approximation in Example 3.

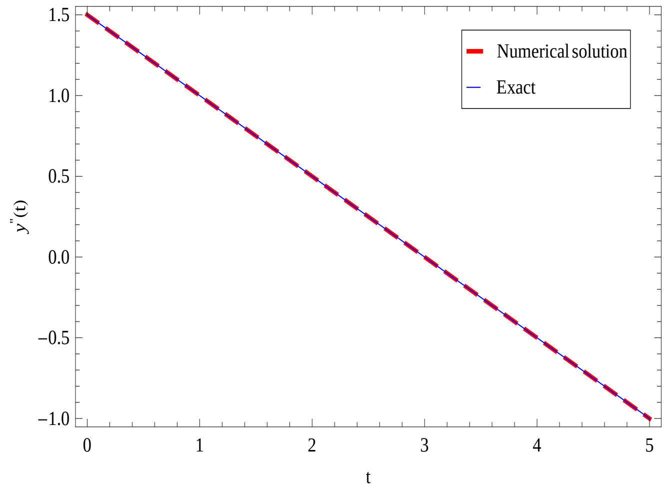


Figure 10. $y''(t)$ and its wavelet approximation in Example 3.

Example 4

Consider the data given in Table 7:

Table 7. Data for Example 4

t	0	1	3	4	5	7	10
$y = y(t)$	1	3	31	69	131	351	1011

Using the NDD interpolation formula, we obtain the polynomial $y(t)$ as

$$y(t) = t^3 + t + 1.$$

We solve this example for $k = 1$ and $M = 7$ by using the procedure given in section 4 and obtained the function which is exactly same as the function obtained by NDD interpolation formula. The first, second and third derivatives of $y(t)$ is obtained as

$$\begin{aligned}
y'(t) &= -2.0\times10^{-16}t^5 + 5.2\times10^{-15}t^4 - 4.8\times10^{-14}t^3 + 3t^2 - 2.6\times10^{-13}t + 1,\\
y''(t) &= -1.0\times10^{-15}t^4 + 2.1\times10^{-14}t^3 - 1.4\times10^{-13}t^2 + 6t - 2.6\times10^{-13},\\
y'''(t) &= -4.0\times10^{-15}t^3 + 6.3\times10^{-14}t^2 - 2.8\times10^{-13}t + 6.
\end{aligned}$$

The estimated errors of derivatives of $y(t)$ by Taylor wavelets basis are listed in the Table 8. The graphs of approximate and exact solutions are given in the Figures 11-13.

Table 8. Estimated absolute errors of derivatives of $y(t)$ in Example 4.

t	$y'(t)$	$y''(t)$	$y'''(t)$
0	9.3×10^{-14}	2.6×10^{-13}	3.6×10^{-13}
1	3.1×10^{-14}	2.1×10^{-14}	1.3×10^{-13}
3	2.9×10^{-14}	2.3×10^{-14}	4.0×10^{-14}
4	3.2×10^{-14}	1.8×10^{-14}	3.6×10^{-14}
5	3.8×10^{-16}	4.0×10^{-14}	3.9×10^{-15}
7	3.7×10^{-14}	2.2×10^{-14}	5.3×10^{-14}
10	8.2×10^{-14}	8.5×10^{-14}	2.4×10^{-13}

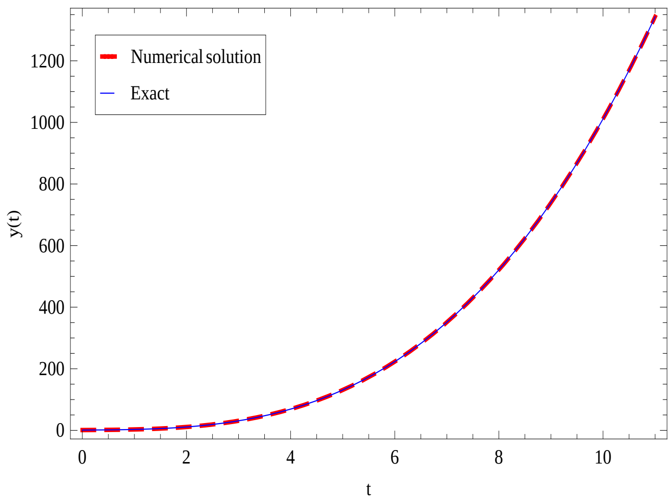


Figure 11. $y(t)$ and its wavelet approximation in Example 4.

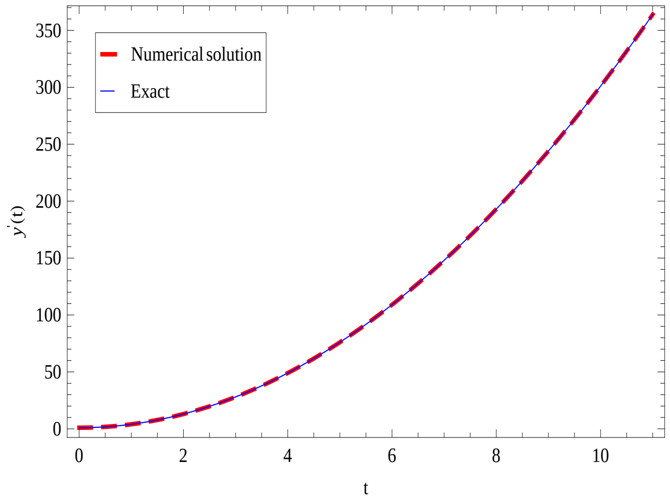


Figure 12. $y'(t)$ and its wavelet approximation in Example 4.

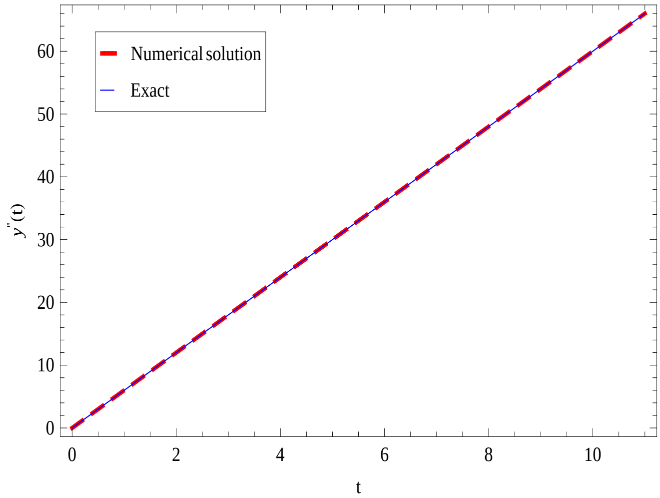


Figure 13. $y''(t)$ and its wavelet approximation in Example 4.

Example 5

Find the 1st, 2nd and 3rd derivatives of the function $y = y(t)$ which satisfy the following data given in Table 9:

Table 9. Data for Example 5

t	0	0.1	0.3	0.5	0.7	0.9
$y = y(t)$	5	5.0012	5.0432	5.25	5.8232	7.0412

Also, compute $y''(0.4)$ and $y'''(0.2)$.

These tabular points are taken from the polynomial $y(t)$ given by $y(t) = 2t^4 + t^3 + 5$. We solve this example for $k = 1$ and $M = 6$ by using the procedure given in section 4 and obtained the function which is exactly same as the original function. The 1st, 2nd and 3rd derivatives of $y(t)$ is obtained as

$$\begin{aligned}
y'(t) &= -1.3 \times 10^{-13}t^4 + 8t^3 + 3t^2 - 1.0 \times 10^{-13}t + 1.2 \times 10^{-14}, \\
y''(t) &= -5.5 \times 10^{-13}t^3 + 24t^2 + 6t - 1.0 \times 10^{-13}, \\
y'''(t) &= -1.6 \times 10^{-12}t^2 + 48t + 6.
\end{aligned}$$

The estimated errors of derivatives of $y(t)$ by Taylor wavelets basis are listed in the Table 10. From the obtained results, the value of $y''(0.4) = 6.24$ and $y'''(0.2) = 15.6$. The graphs of approximate and exact solutions are given in the Figures 14-16.

Table 10. Estimated absolute errors of derivatives of $y(t)$ in Example 5.

t	$y'(t)$	$y''(t)$	$y'''(t)$
0	1.2×10^{-14}	1.0×10^{-13}	3.5×10^{-13}
0.1	3.8×10^{-15}	6.5×10^{-14}	3.4×10^{-13}
0.3	2.8×10^{-15}	5.3×10^{-15}	2.3×10^{-13}
0.5	5.2×10^{-16}	2.0×10^{-14}	4.7×10^{-15}
0.7	1.2×10^{-15}	1.6×10^{-14}	3.7×10^{-13}
0.9	1.2×10^{-14}	1.4×10^{-13}	8.8×10^{-13}

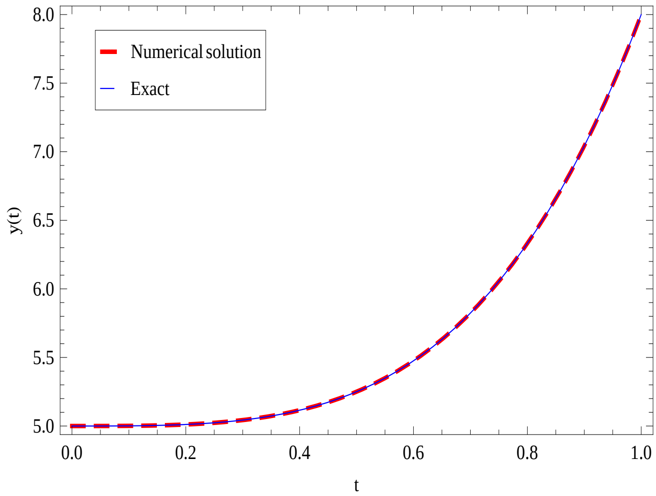


Figure 14. $y(t)$ and its wavelet approximation in Example 5.

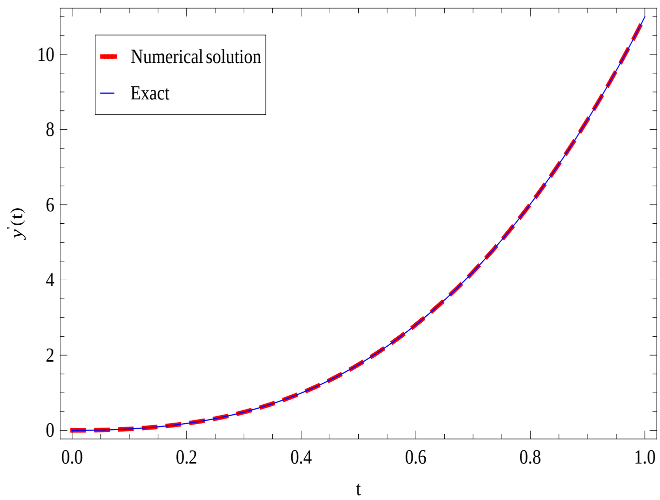


Figure 15. $y'(t)$ and its wavelet approximation in Example 5.

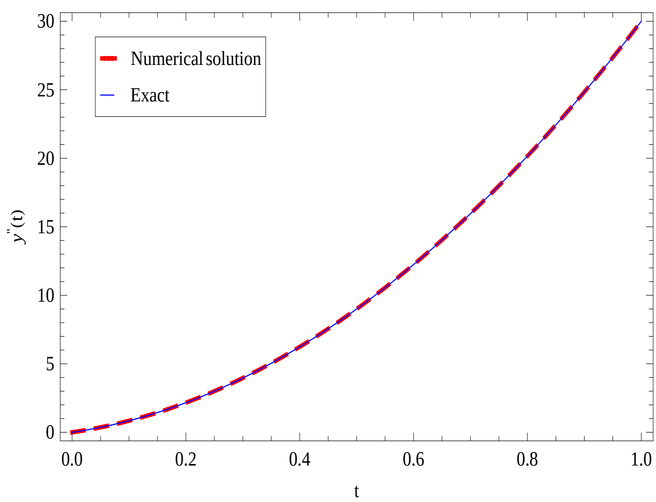


Figure 16. $y''(t)$ and its wavelet approximation in Example 5.

Example 6

Find the 1st, 2nd and 3rd derivatives of the function $y = y(t)$ which satisfy the following data given in Table 11:

Table 11. Data for Example 6

t	0	1	2	5
$y = y(t)$	2	3	12	147

Also, compute $y''(3)$ and $y'''(4)$.
Using the NDD interpolation formula, we obtain the polynomial $y(t)$ as

$$y(t) = t^3 + t^2 - t + 2.$$

We solve this example for $k = 1$ and $M = 4$ by using the procedure given in section 4 and obtained the function which is exactly same as the function obtained by NDD interpolation formula. The first, second and third derivatives of $y(t)$ is obtained as

$$y'(t) = 3t^2 + 2t - 1,$$
$$y''(t) = 6t + 2,$$
$$y'''(t) = 6.$$

The estimated errors of derivatives of $y(t)$ by Taylor wavelets basis are listed in the Table 12. From the obtained results, the value of $y''(3) = 20$ and $y'''(4) = 6$. The graphs of approximate and exact solutions are given in the Figures 17-19.

Table 12. Estimated absolute errors of derivatives of $y(t)$ in Example 6.

t	$y'(t)$	$y''(t)$	$y'''(t)$
0	4.2×10^{-15}	9.3×10^{-15}	5.3×10^{-15}
1	2.4×10^{-15}	3.9×10^{-15}	5.3×10^{-15}
2	3.7×10^{-15}	1.3×10^{-15}	5.3×10^{-15}
5	2.4×10^{-14}	1.7×10^{-14}	5.3×10^{-15}

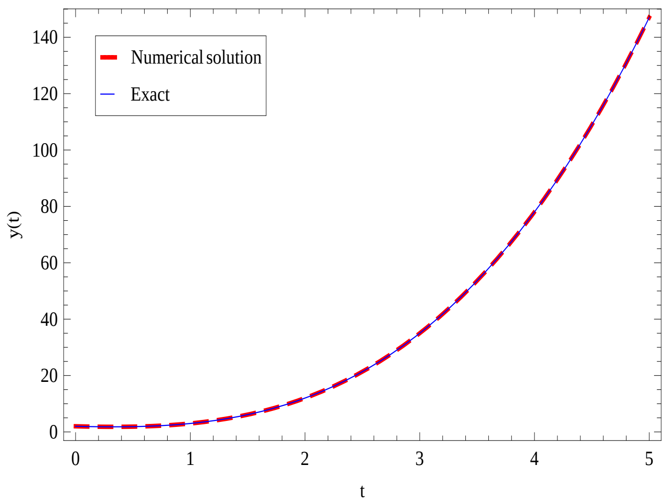


Figure 17. $y(t)$ and its wavelet approximation in Example 6.

We have also presented the relative errors E_j; $j = 0, 1, 2, 3$, by using Eq. (5.1) for all examples in Table 13.

Table 13. The relative errors E_j of Examples 1-6.

E_j	E_0	E_1	E_2	E_3
Example 1	6.9×10^{-16}	4.8×10^{-16}	8.8×10^{-16}	1.0×10^{-15}
Example 2	3.7×10^{-17}	5.8×10^{-16}	2.2×10^{-15}	7.7×10^{-15}
Example 3	1.8×10^{-16}	9.1×10^{-16}	1.8×10^{-15}	1.9×10^{-15}
Example 4	1.5×10^{-16}	2.8×10^{-16}	2.6×10^{-15}	2.5×10^{-14}
Example 5	1.1×10^{-16}	1.8×10^{-15}	5.4×10^{-15}	1.4×10^{-14}
Example 6	1.4×10^{-16}	2.9×10^{-16}	5.0×10^{-16}	8.8×10^{-16}

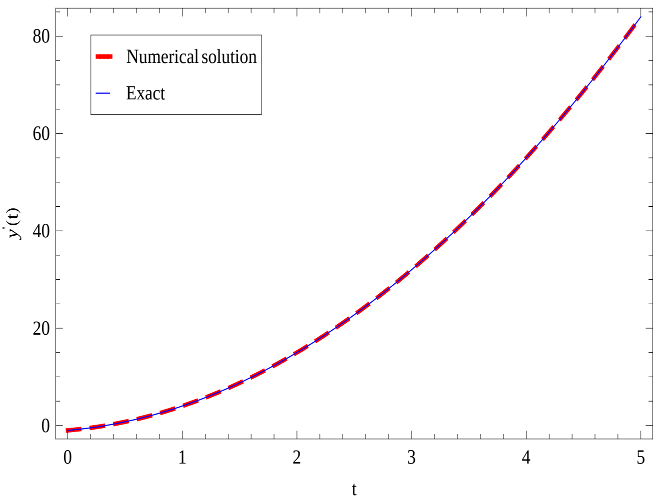


Figure 18. $y'(t)$ and its wavelet approximation in Example 6.

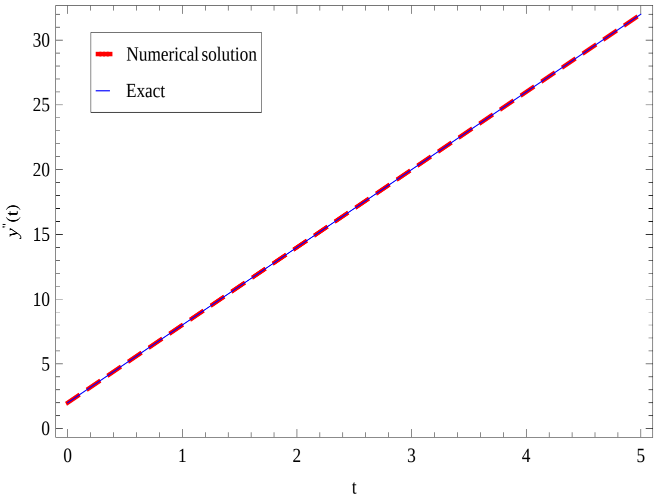


Figure 19. $y''(t)$ and its wavelet approximation in Example 6.

7 Conclusion

In this study, the Taylor wavelets series scheme has been applied successfully to compute numerical differentiation, which is easy to apply directly. Numerical tests show that the Taylor wavelet is a powerful mathematical tool for evaluating numerical differentiation and performs well as indicated through the least values of error obtained. It is also concluded from Figures 1-19 and Tables 1-13 that the Taylor wavelet series expansion converges very fast to the function due to the fast convergency of Taylor wavelets basis. The proposed scheme has less calculation and takes less time to compute numerical differentiation. The key advantage of this approach is that it can deal with the numerical differentiation of any function at arbitrary intervals, where the function is known only at tabular points. Moreover, this scheme can be applied for solving two and three dimensional numerical differentiation.

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