Palestine Journal of Mathematics

Vol. 12(1)(2023) , 569–580

Some growth analysis of composite *p*-adic entire functions on the basis of their generalized relative order (α, β)

Tanmay Biswas and Chinmay Biswas

Communicated by Suheil Khoury

MSC 2020 Classifications: 12J25, 30D35, 30G06, 46S10.

Keywords and phrases: *p*-adic entire function, growth, composition, generalized relative order (α, β), generalized relative lower order (α, β).

The authors are thankful to the Editor/referee for his / her valuable suggestions towards the improvement of the paper.

Abstract In this paper we wish to investigate some interesting results associated with the comparative growth properties of composite *p*-adic entire functions using generalized relative order (α, β) and generalized relative lower order (α, β), where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

1 Introduction, Definitions and Notations

Let us consider an algebraically closed field $\mathbb{K}$ of characteristic zero complete with respect to a *p*-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$) . For any $\Lambda \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x - \Lambda| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x - \Lambda| < R\}$ are denoted by $d(\Lambda, R)$ and $d(\Lambda, R^-)$ respectively. Also $C(\Lambda, r)$ denotes the circle $\{x \in \mathbb{K} : |x - \Lambda| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represents the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$, i.e., the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [18, 19, 20, 22]. During the last several years the ideas of *p*-adic analysis have been studied from different aspects and many important results were gained (see [4] to [17]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r, therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Taking this into account the (p, q)-th order and (p, q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are defined as follows:

Definition 1.1. [8] Let $f \in \mathcal{A}(\mathbb{K})$ and p, q are any two positive integers. Then the (p, q)-th order $\varrho^{(p,q)}(f)$ and (p, q)-th lower order $\lambda^{(p,q)}(f)$ of f are respectively defined as:

$$\varrho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}.$$

Definition 1.1 avoids the restriction $p \geq q$ of the original definition of (p, q)-th order (respectively (p, q)-th lower order) of entire functions introduced by Juneja et al. [21] in complex context.

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\varrho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\varrho^{(2,1)}(f) = \varrho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\varrho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [14].

Now let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow+\infty$ as $x\to+\infty$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x_0\leq x\to+\infty$ for each $c\in(0,+\infty)$, i.e., α is slowly increasing function. Clearly $L^0\subset L$.

The concept of generalized order (α,β) of entire function in complex context was introduced by Sheremeta [23] where $\alpha,\beta\in L$. In complex context, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different direction. For the purpose of further applications of generalized order (α,β) of entire function in complex context, Biswas et al. [5, 6] rewrite the definition of generalized order (α,β) of an entire function considering $\alpha,\beta\in L^0$. For details about generalized order (α,β) and generalized lower order (α,β), one may see [5, 6]. Considering theses ideas, Biswas et al. [2, 3] have defined the generalized order (α,β) and generalized lower order (α,β) of an entire function $f\in\mathcal{A}(\mathbb{K})$ respectively in the following way:

Definition 1.2. [2, 3] Let $f\in\mathcal{A}(\mathbb{K})$ and $\alpha,\beta\in L^0$. The generalized order (α,β) and generalized lower order (α,β) of f denoted by $\varrho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively are defined as:

$$\varrho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(|f|(r))}{\beta(r)}\text{ and }\lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(|f|(r))}{\beta(r)}.$$

If $\alpha(r)=\log^{[p]}r$ and $\beta(r)=\log^{[q]}r$, then Definition 1.1 is a special case of Definition 1.2.

The notion of relative order was first introduced by Bernal [1]. In order to make some progress in the study of *p*-adic analysis, Biswas [7] has introduced the definitions of relative order $\varrho_g(f)$ and relative lower order $\lambda_g(f)$ of entire function $f\in\mathcal{A}(\mathbb{K})$ with respect to another entire function $g\in\mathcal{A}(\mathbb{K})$ in the following way:

$$\varrho_g(f)=\limsup_{r\to+\infty}\frac{\log\widehat{|g|}(|f|(r))}{\log r}\text{ and }\lambda_g(f)=\liminf_{r\to+\infty}\frac{\log\widehat{|g|}(|f|(r))}{\log r}.$$

In the case of relative order, it therefore seems reasonable to define suitably the generalized relative order (α,β) of entire function belonging to $\mathcal{A}(\mathbb{K})$. With this in view, Biswas et al.[3] have introduced the definitions of generalized relative order (α,β) and generalized relative lower order (α,β) of an entire function $f\in\mathcal{A}(\mathbb{K})$ with respect to another entire function $g\in\mathcal{A}(\mathbb{K})$ denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively, in the follows way:

Definition 1.3. [3]. Let $f,g\in\mathcal{A}(\mathbb{K})$ and $\alpha,\beta\in L^0$. The generalized relative order (α,β) and generalized relative lower order (α,β) of f with respect to g denoted by $\varrho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively are defined as:

$$\varrho_{(\alpha,\beta)}[f]_g=\limsup_{r\to+\infty}\frac{\alpha(\widehat{|g|}(|f|(r)))}{\beta(r)}\text{ and }\lambda_{(\alpha,\beta)}[f]_g=\liminf_{r\to+\infty}\frac{\alpha(\widehat{|g|}(|f|(r)))}{\beta(r)}.$$

Here, in this paper, we investigate some interesting results associated with the comparative growth properties of composite *p*-adic entire functions using generalized relative order (α,β) and generalized relative lower order (α,β) where $\alpha,\beta\in L^0$. Further we assume that throughout the present paper α_1, α_2, β_1, and β_2 always denote the functions belonging to L^0.

2 Lemma

In this section we present the following lemma which can be found in [13] or [14] and will be needed in the sequel.

Lemma 2.1. *Let $f,g\in\mathcal{A}(\mathbb{K})$. Then for all sufficiently large positive numbers of r the following equality holds*

$$|f(g)|(r)=|f|(|g|(r)).$$

3 Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let* f, g, $h \in \mathcal{A}(\mathbb{K})$ *be such that* $\varrho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$.
(i) *If either* $\beta_1(r) = B\exp(\alpha_2(r))$ *where* B *is any positive constant or* $\lim_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$, *then*

$$\lim_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = 0.$$

(ii) *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = 0.$$

Proof. Since $\varrho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h$ we can choose $\varepsilon(> 0)$ is such a way that

$$\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon. \tag{3.1}$$

Since $\widehat{|h|}(r)$ is an increasing function of r, it follows from Lemma 2.1 and for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) \leqslant (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(|g|(\beta_2^{-1}(\log r))). \tag{3.2}$$

Now the following three cases may arise .
Case I. Let $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant. Then we have from (3.2) for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) \leqslant B(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\exp(\alpha_2(|g|(\beta_2^{-1}(\log r))))$$

$$i.e.,\ \alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) \leqslant B(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon) r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.3}$$

Case II. Let $\lim_{r\to+\infty} \frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$. Then for all sufficiently large positive numbers of r, we get from (3.2) that

$$\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) < (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon) r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.4}$$

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$. Then for all sufficiently large positive numbers of r we get from (3.2) that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))) \leq (1 + o(1))\alpha_2(|g|(\beta_2^{-1}(\log r)))$$

$$i.e.,\ \alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))) \leqslant r^{(1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{3.5}$$

Also from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]_h$, we get for all sufficiently large positive numbers of r that

$$\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))) \geqslant r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)}. \tag{3.6}$$

Now combining (3.3) of Case I and (3.6) we get for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} \leq \frac{B(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon) r^{(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)}}. \tag{3.7}$$

Therefore in view of (3.1) it follows from (3.7) that

$$\lim_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = 0.$$

Similar conclusion can also be derived from (3.4) of Case II and (3.6).

Hence the first part of the theorem follows.

Further combining (3.5) of Case III and (3.6) we obtain for all sufficiently large positive numbers of r that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} \leq \frac{r^{(1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)}}. \tag{3.8}$$

Therefore in view of (3.1) we get from above that

$$\lim_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = 0.$$

Hence the second part of the theorem follows from above.

Thus the theorem follows. □

Theorem 3.2. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$ be such that $\lambda_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim_{r\to+\infty}\frac{\exp(\alpha_2(r))}{\beta_1(r)} = +\infty$, then

$$\liminf_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = 0.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then

$$\liminf_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = 0.$$

The proof of Theorem 3.2 is omitted as it can be carried out in the line of Theorem 3.1.

Theorem 3.3. *Let $f, g, h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h < \lambda_{(\alpha_2,\beta_2)}[g] < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim_{r\to+\infty}\frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$, then

$$\lim_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = +\infty.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = +\infty.$$

Proof. Let us choose $0 < \varepsilon < \lambda_{(\alpha_1,\beta_1)}[f]_h$. Now for all sufficiently large positive numbers of r we get from Lemma 2.1 that

$$\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) \geqslant (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(|g|(\beta_2^{-1}(\log r))). \tag{3.9}$$

Now the following three cases may arise.

Case I. Let $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant. Then from (3.9) we obtain for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) \geqslant B(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)r^{(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.10}$$

Case II. Let $\lim_{r\to+\infty}\frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$. Then from (3.9) it follows for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))) > (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)r^{(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.11}$$

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$. Then from (3.9) it follows for all sufficiently large positive numbers of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))) \geq (1+o(1))\alpha_2(|g|(\beta_2^{-1}(\log r)))$$

$$i.e., \ \exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))))) \geq r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}. \tag{3.12}$$

Again from the definition of $\varrho_{(\alpha_1,\beta_1)}[f]_h$ we get for all sufficiently large positive numbers of r that

$$\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r))))) \leq r^{(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)}. \tag{3.13}$$

Now combining (3.10) of Case I and (3.13) we get for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} \geq \frac{B(\lambda_{(\alpha_1,\beta_1)}[f]_h-\varepsilon)r^{(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}}{r^{(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)}}.$$

Since $\varrho_{(\alpha_1,\beta_1)}[f]_h < \lambda_{(\alpha_2,\beta_2)}[g]$, it follows from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = +\infty.$$

Similar conclusion can also be derived from (3.11) of Case II and (3.13).

Therefore the first part of the theorem follows.

Again combining (3.12) of Case III and (3.13) we obtain for all sufficiently large positive numbers of r that

$$\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} \geq \frac{r^{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)}}{r^{(\varrho_{(\alpha_1,\beta_1)}[f]_h+\varepsilon)}}$$

$$i.e., \lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = +\infty,$$

Therefore the second part of the theorem follows from above.

Hence the theorem follows. □

Theorem 3.4. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h < \lambda_{(\alpha_2,\beta_2)}[g] < +\infty$.*
(i) If either $\beta_1(r) = B\exp(\alpha_2(r))$ where B is any positive constant or $\lim_{r\to+\infty}\frac{\exp(\alpha_2(r))}{\beta_1(r)} = 0$, then

$$\limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = +\infty.$$

(ii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, then

$$\limsup_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\log r)))))} = +\infty.$$

The proof of Theorem 3.4 is omitted as it can be carried out in the line of Theorem 3.3.

Theorem 3.5. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda_{(\alpha_1,\beta_1)}[f]_h \leq \varrho_{(\alpha_1,\beta_1)}[f]_h < +\infty$ and $0 < \lambda_{(\alpha_2,\beta_2)}[g] \leq \varrho_{(\alpha_2,\beta_2)}[g] < +\infty$.*
(i) If $\beta_1(r) = \alpha_2(r)$, then

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))}\\ &\leqslant \min\left\{\varrho_{(\alpha_2,\beta_2)}[g], \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}\right\}\end{aligned}$$

and

$$\max\left\{\lambda_{(\alpha_2,\beta_2)}[g], \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h \cdot \varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}\right\} \leq \limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h \cdot \varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

(ii) If $\beta_1(\alpha_2^{-1}(r)) \in L^0$, *then*

$$\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\varrho_{(\alpha_1,\beta_1)}[f]_h} \leq \liminf_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \leqslant 1$$

$$\limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \leq \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

(iii) If $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h} \leq \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \min\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}, \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}\right\} \leq \max\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}, \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}\right\} \leq \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}.$$

Proof. From the definitions of generalized relative order (α_1, β_1) and generalized relative lower order (α_1, β_1) of f with respect to h, we have for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f|(r))) \leq (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(r), \tag{3.14}$$

$$\alpha_1(\widehat{|h|}(|f|(r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(r) \tag{3.15}$$

and also for a sequence of positive numbers of r tending to infinity we get that

$$\alpha_1(\widehat{|h|}(|f|(r))) \geq (\varrho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(r). \tag{3.16}$$

Similarly for a sequence of positive numbers of r tending to infinity we obtain that

$$\alpha_1(\widehat{|h|}(|f|(r))) \leq (\lambda_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(r). \tag{3.17}$$

Now in view of Lemma 2.1, we have for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(|g|(r)) \tag{3.18}$$

and also we get for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant (\lambda_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(|g|(r)). \tag{3.19}$$

Similarly, in view of Lemma 2.1, it follows for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(|g|(r)) \tag{3.20}$$

and also we obtain for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq (\varrho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(|g|(r)). \tag{3.21}$$

Now the following two cases may arise:
Case I. Let $\beta_1(r) = \alpha_2(r)$.

Now we have from (3.18) for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r), \tag{3.22}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r). \tag{3.23}$$

Also we obtain from (3.19) for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \leqslant (\lambda_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r). \tag{3.24}$$

Further it follows from (3.20) for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r), \tag{3.25}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r). \tag{3.26}$$

Moreover, we obtain from (3.21) for a sequence of positive numbers of r tending to infinity that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq (\varrho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r). \tag{3.27}$$

Therefore from (3.15) and (3.22), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_2(r)}$$

$$i.e., \ \limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h \cdot \varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.28}$$

Similarly from (3.16) and (3.22), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r)}{(\varrho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_2(r)}$$

$$i.e., \ \liminf_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \varrho_{(\alpha_2,\beta_2)}[g]. \tag{3.29}$$

In the same way also from (3.15) and (3.23), we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h \cdot \lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.30}$$

Similarly from (3.15) and (3.24), we get that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \varrho_{(\alpha_2,\beta_2)}[g]. \tag{3.31}$$

Thus from (3.29), (3.30) and (3.31), it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \min\left\{\varrho_{(\alpha_2,\beta_2)}[g], \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h \cdot \lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}\right\}. \tag{3.32}$$

Further from (3.14) and (3.25), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(r))(\widehat{|h|}}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r)}{(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_2(r)}$$

$$i.e., \liminf_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h \cdot \lambda_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.33}$$

Similarly, from (3.17) and (3.25) we obtain that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \geq \lambda_{(\alpha_2,\beta_2)}[g]. \tag{3.34}$$

Likewise from (3.14) and (3.26), we get that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h \cdot \varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}, \tag{3.35}$$

Similarly from (3.14) and (3.27), we have that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \geq \lambda_{(\alpha_2,\beta_2)}[g]. \tag{3.36}$$

Thus from (3.34), (3.35) and (3.36) it follows that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \geq \max\left\{\lambda_{(\alpha_2,\beta_2)}[g], \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h \cdot \varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_h}\right\}. \tag{3.37}$$

Therefore the first part of the theorem follows from (3.28), (3.32), (3.33) and (3.37).

Case II. Let $\beta_1(\alpha_2^{-1}(r)) \in L^0$.

Now we have from (3.18) for all sufficiently large positive numbers of r that

$$\begin{aligned} \alpha_1(\widehat{|h|}(|f(g)|(r))) &\leqslant (\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}((\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r))) \\ i.e., \ \alpha_1(\widehat{|h|}(|f(g)|(r))) &\leqslant (1 + o(1))(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r))) \end{aligned} \tag{3.38}$$

Further from (3.20), it follows for all sufficiently large positive numbers of r that

$$\alpha_1(\widehat{|h|}(|f(g)|(r))) \geq (1 + o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r))). \tag{3.39}$$

Now from (3.15) and (3.38), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \leqslant \frac{(1 + o(1))(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e., \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \leqslant \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.40}$$

Also from (3.16) and (3.38) it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \leqslant \frac{(1 + o(1))(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\varrho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e., \ \liminf_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \leqslant 1. \tag{3.41}$$

Further from (3.14) and (3.39), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\varrho_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e., \ \liminf_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\varrho_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.42}$$

Also from (3.17) and (3.39) it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \geq \frac{(1+o(1))(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}{(\lambda_{(\alpha_1,\beta_1)}[f]_h + \varepsilon)\beta_1(\alpha_2^{-1}(\beta_2(r)))}$$

$$i.e., \ \limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_1(\widehat{|h|}(|f|(\alpha_2^{-1}(\beta_2(r)))))} \geq 1. \tag{3.43}$$

Hence the second part of the theorem follows from (3.40), (3.41), (3.42) and (3.43).

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L^0$.

Then we have from (3.18) for all sufficiently large positive numbers of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \leqslant (1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r), \tag{3.44}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \leqslant (1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r). \tag{3.45}$$

Further, it follows from (3.20) for all sufficiently large positive numbers of r that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \geq (1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r), \tag{3.46}$$

and for a sequence of positive numbers of r tending to infinity that

$$\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r))))) \geq (1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g] - \varepsilon)\beta_2(r). \tag{3.47}$$

Now from (3.15) and (3.44), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{(1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_2(r)}$$

$$i.e., \ \limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_h}. \tag{3.48}$$

Also from (3.16) and (3.44) it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{(1+o(1))(\varrho_{(\alpha_2,\beta_2)}[g] + \varepsilon)\beta_2(r)}{(\varrho_{(\alpha_1,\beta_1)}[f]_h - \varepsilon)\beta_2(r)}$$

$$i.e., \ \liminf_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))))} \leqslant \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_f}. \tag{3.49}$$

Similarly from (3.15) and (3.45), we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))}\leqslant\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_f}. \tag{3.50}$$

Thus from (3.49) and (3.50) it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))}\leqslant\min\left\{\frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_f},\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_f}\right\}. \tag{3.51}$$

Further from (3.14) and (3.46), we have for all sufficiently large positive numbers of r that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))}\geq\frac{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r)}{(\varrho_{(\alpha_1,\beta_1)}[f]_f+\varepsilon)\beta_2(r)}$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))}\geq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_f}. \tag{3.52}$$

Also from (3.17) and (3.46) it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))}\geq\frac{(1+o(1))(\lambda_{(\alpha_2,\beta_2)}[g]-\varepsilon)\beta_2(r)}{(\lambda_{(\alpha_1,\beta_1)}[f]_f+\varepsilon)\beta_2(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))}\geq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_f}. \tag{3.53}$$

Similarly from (3.14) and (3.47), we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))}\geq\frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_f}. \tag{3.54}$$

Thus from (3.53) and (3.54) it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_1(\widehat{|h|}(|f|(\beta_1^{-1}(\beta_2(r)))}\geq\max\left\{\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]_f},\frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_1,\beta_1)}[f]_f}\right\}. \tag{3.55}$$

Thus the third part of the theorem follows from (3.48), (3.51), (3.52) and (3.55). □

Theorem 3.6. *Let* $f,\ g,\ h,\ k\in\mathcal{A}(\mathbb{K})$ *be such that* $0<\lambda_{(\alpha_1,\beta_1)}[f]_h\leq\varrho_{(\alpha_1,\beta_1)}[f]_h<+\infty$, $0<\lambda_{(\alpha_2,\beta_2)}[g]\leq\varrho_{(\alpha_2,\beta_2)}[g]<+\infty$ *and* $0<\lambda_{(\alpha_2,\beta_2)}[g]_k\leq\varrho_{(\alpha_2,\beta_2)}[g]_k<+\infty$.
(i) If $\beta_1(r)=\alpha_2(r)$, *then*

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_2,\beta_2)}[g]_k} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_2(\widehat{|k|}(|g|(r)))}\\ &\leqslant \min\left\{\frac{\varrho_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k},\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\}\end{aligned}$$

and

$$\begin{aligned}\max\left\{\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h\cdot\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_2,\beta_2)}[g]_k},\frac{\varrho_{(\alpha_1,\beta_1)}[f]_h\cdot\lambda_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_2,\beta_2)}[g]_k}\right\} &\leq \limsup_{r\to+\infty}\frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_2(\widehat{|k|}(|g|(r)))}\\ &\leq \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h\cdot\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.\end{aligned}$$

(ii) *If* $\beta_1(\alpha_2^{-1}(r)) \in L^0$, *then*

$$\frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\varrho_{(\alpha_2,\beta_2)}[g]_k} \leq \liminf_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_2(\widehat{|k|}(|g|(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))} \leqslant$$
$$\min\left\{\frac{\varrho_{(\alpha_1,\beta_1)}[f]_h}{\varrho_{(\alpha_2,\beta_2)}[g]_k}, \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\} \leq$$
$$\max\left\{\frac{\varrho_{(\alpha_1,\beta_1)}[f]_h}{\varrho_{(\alpha_2,\beta_2)}[g]_k}, \frac{\lambda_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\} \leq$$
$$\limsup_{r\to+\infty} \frac{\alpha_1(\widehat{|h|}(|f(g)|(r)))}{\alpha_2(\widehat{|k|}(|g|(\beta_2^{-1}(\beta_1(\alpha_2^{-1}(\beta_2(r)))))))} \leq \frac{\varrho_{(\alpha_1,\beta_1)}[f]_h}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.$$

(iii) *If* $\alpha_2(\beta_1^{-1}(r)) \in L^0$, *then*

$$\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_2,\beta_2)}[g]_k} \leq \liminf_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_2(\widehat{|k|}(|g|(r)))} \leqslant \min\left\{\frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_2,\beta_2)}[g]_k}, \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\} \leqslant$$
$$\max\left\{\frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\varrho_{(\alpha_2,\beta_2)}[g]_k}, \frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}\right\} \leq \limsup_{r\to+\infty} \frac{\alpha_2(\beta_1^{-1}(\alpha_1(\widehat{|h|}(|f(g)|(r)))))}{\alpha_2(\widehat{|k|}(|g|(r)))} \leq \frac{\varrho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_2,\beta_2)}[g]_k}.$$

The proof of Theorem 3.6 is omitted as it can be carried out in the line of Theorem 3.5.

References

[1] L. Bernal, Orden relativo de crecimiento de funciones enteras, *Collect. Math.*, **39**, 209–229 (1988).

[2] T. Biswas and C. Biswas, *On the Growth Properties of Composite p-adic Entire Functions*, Noor Publishing, Chisinau-2068, Republic of Moldova Europe, 133p., (2021).

[3] T. Biswas and C. Biswas, Some remarks on the growth of composite p-adic entire function, *Korean J. Math.* **29**(4), 715–723 (2021).

[4] T. Biswas, Relative (p,q)-φ order oriented some growth properties of p-adic entire functions, *J. Fract. Calc. Appl.*, **11**(1), 161–169 (2020).

[5] T. Biswas and C. Biswas, Generalized (α,β) order based on some growth properties of wronskians, *Mat. Stud.*, **54**(1), 46–55 (2020).

[6] T. Biswas, C. Biswas and R. Biswas, A note on generalized growth analysis of composite entire functions, *Poincare J. Anal. Appl.*, **7**(2), 257–266 (2020).

[7] T. Biswas, Some growth properties of composite p-adic entire functions on the basis of their relative order and relative lower order, *Asian-Eur. J. Math.*, **12**(3) (2019), 1950044, 15p., https://doi.org/10.1142/S179355711950044X.

[8] T. Biswas, Some growth aspects of composite p-adicentire functions in the light of their (p,q)-th relative order and (p,q)-th relative type, *J. Chungcheong Math. Soc.*, **31** (4), 429–460 (2018).

[9] T. Biswas, On some growth analysis of p-adic entire functions on the basis of their (p,q)-th relative order and (p,q)-th relative lower order, *Uzbek Math. J.*, **2018**(4), 160–169 (2018).

[10] T. Biswas, Relative order and relative type based growth properties of iterated p-adic entire functions, *Korean J. Math.*, **26**(4), 629–663 (2018).

[11] T. Biswas, A note on (p,q)-th relative order and (p,q)-th relative type of p-adic entire functions, *Honam Math. J.*, **40**(4), 621–659 (2018).

[12] T. Biswas,(p,q)-th order oriented growth measurement of composite p-adic entire functions, *Carpathian Math. Publ.*, **10**(2), 248–272 (2018).

[13] K. Boussaf, A. Boutabaa and A. Escassut, Order, type and cotype of growth for p-Adic entire functions, A survey with additional properties, p-Adic Numbers, *Ultrametric Anal. Appl.*, **8**(4), 280–297 (2016).

[14] K. Boussaf, A. Boutabaa and A. Escassut, Growth of p-adic entire functions and applications, *Houston J. Math.*, **40**(3), 715–736 (2014).

[15] A. Escassut, K. Boussaf and A. Boutabaa, Order, type and cotype of growth for p-adic entire functions, *Sarajevo J. Math.*, Dedicated to the memory of Professor Marc Krasner, **12(25)**(2), 429–446, (2016) suppl.

[16] A. Escassut, *Value Distribution in p-adic Analysis*, World Scientific Publishing Co. Pte. Ltd., Singapore, (2015).

[17] A. Escassut and J. Ojeda, Exceptional values of p-adic analytic functions and derivative, *Complex Var. Elliptic Equ.*, **56**(1-4), 263–269 (2011).

[18] A. Escassut, *p-adic Value Distribution. Some Topics on Value Distribution and Differentability in Complex and P-adic Analysis*, Math. Monogr., Series 11, Science Press, Beijing, 42–138 (2008).

[19] A. Escassut, *Analytic Elements in p-adic Analysis*, World Scientific Publishing Co. Pte. Ltd. Singapore, (1995).

[20] P. C. Hu and C. C. Yang, *Meromorphic Functions over non-Archimedean Fields*, Kluwer Academic Publishers, Dordrecht, The Netherlands, (2000).

[21] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p, q)-order and lower (p, q)-order of an entire function, *J. Reine Angew. Math.*, **282**, 53–67 (1976).

[22] A. Robert, *A Course in p-adic analysis*, Graduate texts, Springer, New York, (2000).

[23] M. N. Sheremeta, Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion, *Izv. Vyssh. Uchebn. Zaved Mat.*, **2**, 100–108 (1967), (in Russian).

Author information

Tanmay Biswas, Rajbari, Rabindrapally, R. N. Tagore Road, P.O. Krishnagar, P.S. Kotwali, Dist-Nadia, PIN-741101, West Bengal, India.
E-mail: `tanmaybiswas_math@rediffmail.com`

Chinmay Biswas, Department of Mathematics, Nabadwip Vidyasagar College, Nabadwip, Dist.- Nadia, PIN-741302,West Bengal, India.
E-mail: `chinmay.shib@gmail.com`

Received: August 30th, 2021.
Accepted: January 17th, 2022.