Palestine Journal of Mathematics

Vol. 12(1)(2023) , 581–587

CURVES OF CONSTANT BREADTH IN A STRICT WALKER 3-MANIFOLD

Athoumane NIANG, Ameth NDIAYE and Moussa KOIVOGUI

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53A10; Secondary 53C42, 53C50.

Keywords and phrases: Curves, Curvature, Torsion, Frenet frame, Walker manifolds.

The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

Abstract In this paper, we investigate the properties of curves of constant breadth in a strict Walker 3-manifold and we construct examples of curves of constant breadth.

1 Introduction

The study of submanifolds of a given ambiant space is a naturel interesting problem which enriches our knowledge and understanding of the geometry of the space itself. Here the ambiant space we will consider is a Lorentzian three-manifold admitting a parallel null vector field called Walker manifold. It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. For more detail see ([2, 3]).
Recently, Shaikh et. al initiated the study of surface curves in a different way: in [8, 9, 13, 14], they investigate a sufficient condition for which a rectifying curve on a smooth surface remains invariant under isometry of surfaces, and they show that under such an isometry the component of the position vector of a rectifying curve on a smooth surface along the normal to the surface is invariant. And they find the normal and geodesic curvature for a rectifying curve on a smooth surface and they also prove that geodesic curvature is invariant under the isometry of surfaces such that rectifying curves remain. They also find a sufficient condition for which an osculating curve on a smooth surface remains invariant under isometry of surfaces and also prove that the component of the position vector of an osculating curve $\alpha(s)$ on a smooth surface along any tangent vector to the surface at $\alpha(s)$ is invariant under such isometry. In [10, 11], they investigate an osculating curve under the conformal map, and obtain a sufficient condition for the conformal invariance of an osculating curve. They also investigate a sufficient condition for the invariance of a normal curve on a smooth immersed surface under isometry.
The curves of constant breadth were first defined in 1778 by Euler. Then, Solow [15] and Blascke [1] investigated the curves of constant breadth. In Euclidean spaces $\mathbb{E}^3$ and $\mathbb{E}^4$, plane curves of constant breadth were studied by Kose [6].
In the paper [17], some geometric properties of curves of constant breadth in Minkowski 3-space were given. Also, these curves in Minkowski 4-space were obtained by Kazaz, Onder and Kocayigit [5]. A number of authors have, recently, studied the curves of constant breadth under different conditions (see [4, 5, 7, 18]).
In this paper we study curves of contant braith in 3-dimension strict Walker manifold; and we construct examples.

2 Preliminaries

A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker ([16]). Walker has derived adapted coordinates to a parallel plan field. Hence, the metric of a three-dimensional Walker manifold (M, g_f^ϵ) with coordinates (x, y, z) is expressed as

$$g_f^\epsilon = dx \circ dz + \epsilon dy^2 + f(x, y, z)dz^2 \tag{2.1}$$

and its matrix form as

$$g_f^\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix} \quad \text{with inverse} \quad (g_f^\epsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x, y, z)$, where $\epsilon = \pm 1$ and thus $D = \text{Span}\partial_x$ as the parallel degenerate line field. Notice that when $\epsilon = 1$ and $\epsilon = -1$ the Walker manifold has signature $(2, 1)$ and $(1, 2)$ respectively, and therefore is Lorentzian in both cases.

It follows after a straightforward calculation that the Levi-Civita connection of any metric (2.1) is given by:

$$\begin{aligned} \nabla_{\partial_x}\partial z &= \frac{1}{2}f_x\partial_x, \quad \nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \\ \nabla_{\partial_z}\partial z &= \frac{1}{2}(ff_x + f_z)\partial_x + \frac{1}{2}f_y\partial_y - \frac{1}{2}f_x\partial_z \end{aligned} \tag{2.2}$$

where ∂_x, ∂_y and ∂_z are the coordinate vector fields $\frac{\partial}{\partial_x}$, $\frac{\partial}{\partial_y}$ and $\frac{\partial}{\partial_z}$, respectively. Hence, if (M, g_f^ϵ) is a strict Walker manifolds i.e., $f(x, y, z) = f(y, z)$, then the associated Levi-Civita connection satisfies

$$\nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \quad \nabla_{\partial_z}\partial z = \frac{1}{2}f_z\partial_x - \frac{\epsilon}{2}f_y\partial_y. \tag{2.3}$$

Note that the existence of a null parallel vector field (i.e $f = f(y, z)$) simplifies the non-zero components of the Christoffel symbols and the curvature tensor of the metric g_f^ϵ as follows:

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2}f_y, \ \Gamma_{33}^1 = \frac{1}{2}f_z, \ \Gamma_{33}^2 = -\frac{\epsilon}{2}f_y \tag{2.4}$$

Starting from local coordinates (x, y, z) for which (2.1) holds, it is easy to check that

$$e_1 = \partial_y, \ \ e_2 = \frac{2-f}{2\sqrt{2}}\partial_x + \frac{1}{\sqrt{2}}\partial_z, \ \ e_3 = \frac{2+f}{2\sqrt{2}}\partial_x - \frac{1}{\sqrt{2}}\partial_z$$

are local pseudo-orthonormal frame fields on (M, g_f^ϵ), with $g_f^\epsilon(e_1, e_1) = 1$, $g_f^\epsilon(e_2, e_2) = \epsilon$ and $g_f^\epsilon(e_3, e_3) = 1$. Thus the signature of the metric g_f^ϵ is $(1, \epsilon, -1)$.
Let now u and v be two vectors in M. Denoted by $(\vec{i}, \vec{j}, \vec{k})$ the canonical frame in $\mathbb{R}^3$.
The vector product of u and v in (M, g_f^ϵ) with respect to the metric g_f^ϵ is the vector denoted by $u \times v$ in M defined by

$$g_f^\epsilon(u \times v, w) = \det(u, v, w) \tag{2.5}$$

for all vector w in M, where $\det(u, v, w)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then by using (2.5), we have:

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) \vec{i} - \epsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \vec{k} \tag{2.6}$$

Let $\alpha : I \subset \mathbb{R} \longrightarrow (M, g_f^\epsilon)$ be a curve parametrized by its arc-length s.
The Frenet frame of α is the vectors T, N and B along α where T is the tangent, N the principal normal and B the binormal vector. They satisfied the Frenet formulas

$$\left\{ \begin{array}{rcl} \nabla_T T(s) &=& \epsilon_2 \kappa(s) N(s) \\ \nabla_T N(s) &=& -\epsilon_1 \kappa T(s) - \epsilon_3 \tau B(s) \\ \nabla_T B(s) &=& \epsilon_2 \tau(s) N(s) \end{array} \right. \tag{2.7}$$

where κ and τ are respectively the curvature and the torsion of the curve α, with $\epsilon_1 = g_f(T;T)$; $\epsilon_2 = g_f(N;N)$ and $\epsilon_3 = g_f(B, B)$.

3 Space curves of constant breadth in Walker manifold

In this section, we define space curves of constant breadth in the three dimensional Walker manifold.

Definition 3.1. A curve $\alpha : I \to (M, g_f^\epsilon)$ in the three-dimensional Walker manifold (M, g_f^ϵ) is called a curve of constant breadth if there exists a curve $\beta : I \to M_f$ such that, at the corresponding points of curves, the parallel tangent vectors of α and β at $\alpha(s)$ and $\beta(s^\star)$ at $s; s^\star \in I$ are opposite directions and the distance $g_f^\epsilon(\beta - \alpha, \beta - \alpha)$ is constant. In this case, $(\alpha; \beta)$ is called a pair curve of constant breadth.

Let now $(\alpha; \beta)$ be a pair curve of constant breadth and $s, s^\star$ be arc-length of α and β, respectively. Then we may write the following equation:

$$\beta(s^\star) = \alpha(s) + m_1(s)T(s) + m_2(s)N(s) + m_3(s)B(s); \tag{3.1}$$

where $m_i (i = 1, 2, 3)$ are smooth functions of s.
Differentiating (3.1) equation with respect to s and using (2.7) we obtain

$$\begin{aligned} \frac{d\beta}{ds} &= \frac{d\beta}{ds^\star}\frac{ds^\star}{ds} \\ &= T^\star \frac{ds^\star}{ds} = (1 + m_1' - \epsilon_1 m_2 \kappa(s))T \\ &\quad + (m_2' + \epsilon_2 m_1 \kappa(s) + \epsilon_2 m_3 \tau(s))N + (m_3' - \epsilon_3 m_2 \tau(s))B \end{aligned} \tag{3.2}$$

where $T^\star$ denotes the tangent vector of β.
Since $T = -T^*$, from the equation in (3.2) we have

$$\begin{cases} 1 + m_1' - \epsilon_1 m_2 \kappa(s) & = -\frac{ds^\star}{ds} \\ m_2' + \epsilon_2 m_1 \kappa(s) + \epsilon_2 m_3 \tau(s) & = 0 \\ m_3' - \epsilon_3 m_2 \tau(s) & = 0 \end{cases} \tag{3.3}$$

Let us introduce the angle ϕ between the tagent vector of α with a chosen fixed direction. The curvature of α is $\kappa = \frac{d\phi}{ds}$ and the curvature of β is $\kappa^\star = \frac{d\phi}{ds^\star}$. If we denote by $\rho = \frac{1}{\kappa}$ and $\rho^\star = \frac{1}{\kappa^\star}$, the radius of curvature of α and β; and $g(\phi) = \rho + \rho^\star$, then the relation (3.3) can be rewritten as

$$\begin{cases} \frac{dm_1}{d\phi} &= \epsilon_1 m_2 - g(\phi) \\ \frac{dm_2}{d\phi} &= -\epsilon_2(m_1 + m_3\rho\tau) \\ \frac{dm_3}{d\phi} &= \epsilon_3 m_2 \rho\tau. \end{cases} \tag{3.4}$$

Differentiating the second equation of (3.4) with respect to ϕ and using the first and the third equations of (3.4), we obtain the following equation:

$$\frac{d^2 m_2}{d\phi^2} - \frac{1}{\rho\tau}\frac{d(\rho\tau)}{d\phi}\left(\frac{dm_2}{d\phi} + \epsilon_2 m_1\right) + \epsilon_2\epsilon_3 m_2(\rho\tau)^2 = -\epsilon_2(\epsilon_1 m_2 - g(\phi)). \tag{3.5}$$

If the distance between the opposite points of α and β is constant, then we get

$$\epsilon_1 m_1 \frac{dm_1}{d\phi} + \epsilon_2 m_2 \frac{dm_2}{d\phi} + \epsilon_3 m_3 \frac{dm_3}{d\phi} = 0 \tag{3.6}$$

Combining (3.4) and (3.6) we get

$$\epsilon_1 m_1 \frac{dm_1}{d\phi} + \epsilon_2 m_2(-\epsilon_2 m_1 - \epsilon_2 m_3 \rho\tau) + \epsilon_3\epsilon_3 m_3 m_2 \rho\tau = 0.$$

Then we get

$$m_1\left(\epsilon_1 \frac{dm_1}{d\phi} - m_2\right) = 0. \tag{3.7}$$

Case 1: $\epsilon_1 \frac{dm_1}{d\phi} - m_2 = 0$.
If m_1 is non-zero constant, from the first equation of (3.4) we find $m_2 = 0$ and $g(\phi) = 0$. Then we have $\frac{d}{\phi}(\rho\tau) = 0$. Thus $\rho\tau$ is constant, that is, $\tau = \kappa =$constant. Now, we have the following theorem:

Theorem 3.2. *The space curve of constant breadth with the tangent component m_1 is non-zero constant and the principal normal component $m_2 = 0$, is a general helix in the three dimensional Walker manifold (M, g_f^{ϵ}).*

Case 2: $m_1 = 0$.
Then using the first line (3.4)we get $m_2 = \epsilon_1 g(\phi)$.
Thus the equation (3.5) becomes

$$\frac{d^2 m_2}{d\phi^2} - \frac{1}{\rho\tau}\frac{d(\rho\tau)}{d\phi}\left(\frac{dm_2}{d\phi} + \epsilon_2 m_1\right) + \epsilon_2\epsilon_3 m_2(\rho\tau)^2 = 0 \tag{3.8}$$

one can solve the equation (3.8) by considering the new varibale $z = z(\phi)$ defined by $\frac{dz}{d\phi} = (\rho\tau)$. So we get

$$\begin{cases} \frac{dm_2}{d\phi} = \frac{dm_2}{dz}(\rho\tau) \\ \frac{d^2 m_2}{d\phi^2} = \frac{d^2 m_2}{dz^2}(\rho\tau)^2 + \frac{dm_2}{dz}\frac{(\rho\tau)}{d\phi} \end{cases} \tag{3.9}$$

By (3.9) the equation (3.8) becomes

$$\frac{d^2 m_2}{dz^2} + \epsilon_2\epsilon_3 m_2 = 0. \tag{3.10}$$

If $\epsilon_2\epsilon_3 = 1$, then

$$m_2 = c\cos\left(\int_0^{\phi}(\rho\tau)d\phi + b\right); \tag{3.11}$$

which gives, by using (3.4), that

$$m_3 = \epsilon_3 c\sin\left(\int_0^{\phi}(\rho\tau)d\phi\right). \tag{3.12}$$

If $\epsilon_2\epsilon_3 = -1$, then

$$m_2 = ae^{-\int_0^{\phi}(\rho\tau)d\phi} + be^{\int_0^{\phi}(\rho\tau)d\phi}; \tag{3.13}$$

which gives by using (3.4), that

$$m_3 = -\epsilon_3\left(-ae^{-\int_0^{\phi}(\rho\tau)d\phi} + be^{\int_0^{\phi}(\rho\tau)d\phi}\right). \tag{3.14}$$

Theorem 3.3. *Let $(\alpha;\beta)$ be a curve pair of constant breadth in (M, g_f). If α is a curve with $m_1 = 0$, then the curve β have the following form:*

(i) if the normale N and the binormal B have the same sign then

$$\beta(s) = \alpha(s) + c\cos\left(\int_0^{\phi}(\rho\tau)d\phi + b\right)N(s) + \epsilon_3 c\sin\left(\int_0^{\phi}(\rho\tau)d\phi\right)B(s),$$

(ii) if the normale N and the binormal B have not the same sign then

$$\beta(s) = \alpha(s) + \left(ae^{-\int_0^{\phi}(\rho\tau)d\phi} + be^{\int_0^{\phi}(\rho\tau)d\phi}\right)N(s) - \epsilon_3\left(-ae^{-\int_0^{\phi}(\rho\tau)d\phi} + be^{\int_0^{\phi}(\rho\tau)d\phi}\right)B(s).$$

4 Examples

In the case 2 when $m_1 = 0$, the equation (3.8) has very simple solution. The solutions of (3.8) when $\rho\tau$ is assumed to be constant are obtained by the equation

$$\frac{d^2 m_2}{d\phi^2} + \epsilon_2\epsilon_3 m_2(\rho\tau)^2 = 0. \tag{4.1}$$

We will consider that the function $f = f(y, z)$ wich defines the geometry of the strict Walker manifold is given by

$$f(y) = -2ae^{-2y}, \quad a \in \mathbb{R}, \quad -1 < a < 0. \tag{4.2}$$

We consider the curve α given by

$$\alpha(s) = (-ae^{-s}, s, e^s), \quad s \in \mathbb{R}. \tag{4.3}$$

So we have

$$\alpha'(s) = (ae^{-s}, 1, e^s), \quad s \in \mathbb{R}. \tag{4.4}$$

An easy computation show that $g_f^{\epsilon}(\alpha'(s), \alpha'(s)) = \epsilon = \epsilon_1$.
If we denoted the coordonates (x, y, z) of (M, g_f^{ϵ}) by $(x_1, x_2 x_3)$, a vector field

$$Z = Z(s) = \sum_{i=1}^{3} Z^i(s) \frac{\partial}{\partial x_i} \tag{4.5}$$

of α has co-variant derivative $Z'(s)$ given by

$$Z'(s) = \begin{pmatrix} \frac{dZ^1}{ds} + \Gamma_{23}^1 x_2' Z^3 + \Gamma_{32}^1 x_3' Z^2 \\ \frac{dZ^2}{ds} + \Gamma_{33}^2 x_3' Z^3 \\ \frac{dZ^3}{ds} \end{pmatrix} \tag{4.6}$$

where Γ_{ij}^k is given in (2.4).
By using (4.6)with (4.4) we obtain

$$\alpha''(s) = \begin{pmatrix} 3ae^{-s} \\ -a\epsilon \\ e^s \end{pmatrix}. \tag{4.7}$$

NB: We work with $\epsilon = 1$.
Since $\alpha'' = \epsilon_2 \kappa N$ then ϵ is the sign of $g_f^{\epsilon}(\alpha'', \alpha'')$ which equal to $\epsilon_2 \kappa^2$.
By (4.7), an easy computation gives that

$$g_f^{\epsilon}(\alpha'', \alpha'') = 4a(1 + a) < 0. \tag{4.8}$$

So we have

$$N = \frac{1}{\sqrt{-4a(1 + a)}} \begin{pmatrix} 3ae^{-s} \\ -2a \\ e^s \end{pmatrix}. \tag{4.9}$$

We see that $\epsilon_2 = -1$.
Using the vector product (2.4, the binormal vector of α is given by

$$B = T \times N = \frac{1}{\sqrt{-4a(1 + a)}} \begin{pmatrix} a^2 e^{-s} \\ 2a \\ e^s(1 + 2a) \end{pmatrix}, \tag{4.10}$$

and the sign of B is $\epsilon_3 = 1$ because $g_f^{\epsilon}(B, B) = a^2(1 - 2a) > 0$. Then the equation (4.1) becomes

$$\frac{d^2 m_2}{d\phi^2} - m_2(\rho\tau)^2 = 0. \tag{4.11}$$

We know that the curvature of α is the constant $\kappa = \sqrt{-4a(1 + a)}$. We have $\tau = -g_f(N', B)$ where

$$N' = \frac{1}{\sqrt{-4a(1 + a)}} \begin{pmatrix} -3ae^{-s} \\ -2a \\ e^s \end{pmatrix}. \tag{4.12}$$

Using (4.12) and (4.10) we get $\tau = -\frac{1+5a}{4(1+a)}$ is constant so $\rho\tau$ is constant. Then the equation (4.11) gives

$$\begin{cases} m_2 = ae^{\tau s} + be^{-\tau s} \\ m_3 = ae^{\tau s} - be^{-\tau s} \end{cases} \tag{4.13}$$

Thus the pair of curves (α, β) given by

$$\alpha = (-ae^{-s}, s, e^{s}), \quad \beta = \alpha + (ae^{\tau s} + be^{-\tau s})N + (ae^{\tau s} - be^{-\tau s})B,$$

are of constant breadth.

References

[1] W. Blascke, Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts, Math. Ann. **76**, 504-513 (1915).

[2] M. Brozos-Vázquez, E. García-Rio, P. Gilkey, S. Nikević and R. Vázquez-Lorenzo, *The Geometry of Walker Manifolds*, Synthesis Lectures on Mathematics and Statistics, 5. Morgan and Claypool Publishers, Williston, VT, (2009).

[3] A. S. Diallo, A. Ndiaye and A. Niang, *Minimal graphs on three-dimensional Walker manifolds*, Proceedings of the First NLAGA-BIRS Symposium, Dakar, Senegal, Trends Math., Birkhä user/Springer, Cham, 425-438 (2020).

[4] M. Fujivara, *On space curve of constant breadth*,Tohoku Math. J. **5**, 179-184 (1914).

[5] M. Kazaz, M. Onder and H. Kocajigit, *Spacelike curves of constant breadth in Minkowski 4-space*, Int. journal of Math. Analysis2, 1061-1068 (2008).

[6] O. Kose, *Some properties of ovals and curves of constant width in a plane*, Doga Sci. J. Serial B **8** (2), 119-126 (1984).

[7] A. Magden, Ö. Köse, and P. Piu, *On the curves of constant breadth in* $\mathbb{E}^4$, Turk. J. Math. **21**, 277-284 (1997).

[8] A. A. Shaikh, and P. R. Ghosh, *Rectifying curves on a smooth surfaces immersed in the Euclidean space*, Indian J. Pure Appl. Math., **50** (4), 883-890 (2019).

[9] A. A. Shaikh, and P. R. Ghosh, *Rectifying and Osculating curves on a smooth surface*, Indian J. Pure Appl. Math., **51** (1), 67-75 (2020).

[10] A. A. Shaikh, M. S. Lone and P. R. Ghosh, *Conformal image of an osculating curves on a smooth immersed surface*, J. Geom. Phy., **151**, (2020).

[11] A. A. Shaikh, and P. R. Ghosh, *Curves on a smooth surface with position vectors lie in the tangent plane*, Indian J. Pure Appl. Math., **51** (3), 1097-1104 (2020).

[12] A. A. Shaikh, M. S. Lone and P. R. Ghosh, *Normal curves on a smooth immersed surface*, Indian J. Pure Appl. Math., **51** (4), 1343-1355 (2020).

[13] A. A. Shaikh, M. S. Lone and P. R. Ghosh, *Rectifying curves under conformal transformation*, J. Geom. Phy., **163**, (2021).

[14] A.A. Shaikh, Y. H. Kim and P. R. Ghosh, *Some characterizations of rectifying and osculating curves on a smooth immersed surface*, J. Geom. Phy., **171**, (2022).

[15] R. M. Solow, *Quarterly Journal of Economics*, **70**, 65-94, (1956).

[16] A. G. Walker, *Canonical form for a Riemannian space with a parallel field of null planes*, Quart. J. Math. Oxford 1, **2**, 69-79 (1950).

[17] S. Yilmaz and M. Turgut, *Partially null curves of constant breadth in semi-Riemannian space*, Modern Applied Science, **3**, 60-63, (2009).

[18] D. W. Yoon, *Curves of constant breadth in Galilean 3-space*, Applied Mathematical Sciences, **8**, 7013-7018, (2014).

Author information

Athoumane NIANG, Département de Mathématiques et Informatique, FST, Université Cheikh Anta Diop, B.P 5005, Dakar, Sénégal.
E-mail: `athoumane.niang@ucad.edu.sn`

Ameth NDIAYE, Département de Mathématiques, FASTEF, Université Cheikh Anta Diop, B.P 5036, Dakar, Sénégal.
E-mail: ameth1.ndiaye@ucad.edu.sn

Moussa KOIVOGUI, Laboratoire des Sciences et Technologies de l'Information et de la Communication, Ecole Supérieure Africaine des TIC, Abidjan, Côte d'Ivoire.
E-mail: moussa.koivogui@esatic.edu.ci

Received: September 1st, 2021
Accepted: February 27th, 2022