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ON A PROPERTY OF DIAGONAL TERM RATIOS FOR ROOTS OF A 2×2 MATRIX CLASS

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Abstract We state and prove a new result which involves the diagonal entries of any nth root of a general 2×2 matrix belonging to a particular class, and offer supporting examples.

1 Introduction

Let

$$\mathbf{M} = \mathbf{M}(A, B, C, D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

be a general (real) 2×2 matrix. Based on a complete (that is, fully symbolic) diagonalised decomposition of $\mathbf{M}$ then, writing any nth root of $\mathbf{M}$ ($n \geq 2$) as

$$\mathbf{R} = \mathbf{M}^{1/n} = \begin{pmatrix} R_{1,1} & R_{1,2} \\ R_{2,1} & R_{2,2} \end{pmatrix}, \tag{1.2}$$

say, it was shown in previous work by the authors that [1, (P.4), p. 10]

$$\begin{aligned} R_{1,1} &= [\alpha + \beta + (\beta - \alpha)(A - D)/K]/2, \\ R_{1,2} &= B(\beta - \alpha)/K, \\ R_{2,1} &= C(\beta - \alpha)/K, \\ R_{2,2} &= [\alpha + \beta - (\beta - \alpha)(A - D)/K]/2, \end{aligned} \tag{1.3}$$

where α, β are (resp.) nth roots of the eigenvalues

$$\begin{aligned} \lambda &= \lambda(A, B, C, D) &= (A + D - K)/2, \\ \mu &= \mu(A, B, C, D) &= (A + D + K)/2, \end{aligned} \tag{1.4}$$

of $\mathbf{M}(A, B, C, D)$, and (writing the trace of $\mathbf{M}$ as $\mathrm{Tr}\{\mathbf{M}\} = A + D$ and its determinant as $|\mathbf{M}| = AD - BC$)

$$K^2 = K^2(A, B, C, D) = (A - D)^2 + 4BC = \mathrm{Tr}^2\{\mathbf{M}\} - 4|\mathbf{M}|. \tag{1.5}$$

The analysis of [1] was conducted under the assumption $K^2 > 0$ ($A \neq D$), for which the eigenvalues λ, μ of $\mathbf{M}$ are real and distinct. In this paper our matrix class is simply one for which $K^2 < 0$, leading to a different result that, once stated and proved, is supported by some illustrative examples. For additional context the reader is directed to Section 2 of [1] for details of the diagonalising matrix alluded to above.

2 Result and Proof

2.1 Result

Theorem 2.1 (Rawlin's Theorem). *For $K^2 < 0$ ($A \neq D$), every nth root matrix $\mathbf{R} = \mathbf{R}(\alpha, \beta; A, B, C, D)$ of $\mathbf{M}$ has a diagonals term ratio $R_{1,1}/R_{2,2}$ that (where it exists) is real, so that the diagonal terms of any nth root matrix are in real proportion.*

2.2 Proof

Proof. Since $K^2 < 0$, then we will take $K = K'i$ where (real) $K' = \sqrt{|K^2|} > 0$. The eigenvalues λ, μ (1.4) become the complex conjugate pair $\lambda, \mu = \frac{1}{2}(\text{Tr}\{\mathbf{M}\} \mp K'i)$, with root matrix elements (1.3) of $\mathbf{R}$ now

$$\begin{aligned} R_{1,1} &= [\alpha + \beta + (\beta - \alpha)\rho i]/2, \\ R_{1,2} &= -B(\beta - \alpha)i/K', \\ R_{2,1} &= -C(\beta - \alpha)i/K', \\ R_{2,2} &= [\alpha + \beta - (\beta - \alpha)\rho i]/2, \end{aligned} \tag{P.1}$$

where $\rho = -(A - D)/K'$.

With $\alpha^n = \lambda$ and $\beta^n = \mu = \overline{\lambda} = \overline{\alpha^n} = \overline{\alpha}^n$, then $(\beta/\overline{\alpha})^n = 1$ and $\beta/\overline{\alpha}$ is an nth root of unity. Thus, for every $k = 1, 2, \ldots, n$ there exists a $\theta = \theta(k; n) = 2\pi k/n$ for which $\beta/\overline{\alpha} = e^{i\theta}$. We write (for x, y real) α in general complex form

$$\alpha = x + iy, \tag{P.2}$$

so that

$$\beta = \overline{\alpha}e^{i\theta} = (x - iy)[\cos(\theta) + i\sin(\theta)] = T(x, y, \theta) + iU(x, y, \theta), \tag{P.3}$$

where

$$\begin{aligned} T(x, y, \theta) &= x\cos(\theta) + y\sin(\theta), \\ U(x, y, \theta) &= x\sin(\theta) - y\cos(\theta), \end{aligned} \tag{P.4}$$

for which

$$T^2(x, y, \theta) + U^2(x, y, \theta) = x^2 + y^2 \tag{P.5}$$

is independent of θ.[1] In turn, we note that

$$\begin{aligned} \alpha + \beta &= T(x, y, \theta) + x + i[U(x, y, \theta) + y], \\ \beta - \alpha &= T(x, y, \theta) - x + i[U(x, y, \theta) - y], \end{aligned} \tag{P.6}$$

and, after a little algebra,

$$\begin{aligned} &\alpha + \beta \pm (\beta - \alpha)\rho i \\ &\quad = T(x, y, \theta) + x \mp [U(x, y, \theta) - y]\rho + i\{U(x, y, \theta) + y \pm [T(x, y, \theta) - x]\rho\}. \end{aligned} \tag{P.7}$$

Consider now, from (P.1),(P.7),

$$\begin{aligned} 2\text{Re}\{R_{1,1}\} &= \text{Re}\{\alpha + \beta + (\beta - \alpha)\rho i\} = T(x, y, \theta) + x - [U(x, y, \theta) - y]\rho, \\ 2\text{Im}\{R_{1,1}\} &= \text{Im}\{\alpha + \beta + (\beta - \alpha)\rho i\} = U(x, y, \theta) + y + [T(x, y, \theta) - x]\rho. \end{aligned} \tag{P.8}$$

Writing down similar expressions for $\text{Re}\{R_{2,2}\}$ and $\text{Im}\{R_{2,2}\}$ yields, with some work (an elementary reader exercise in cancellation)

$$\begin{aligned} \text{Re}\{R_{1,1}\}\text{Im}\{R_{2,2}\} - \text{Re}\{R_{2,2}\}\text{Im}\{R_{1,1}\} &= -[T^2(x, y, \theta) + U^2(x, y, \theta) - (x^2 + y^2)]\rho/2 \\ &= 0 \end{aligned} \tag{P.9}$$

(by (P.5)), or

$$\frac{\text{Re}\{R_{1,1}\}}{\text{Re}\{R_{2,2}\}} = \frac{\text{Im}\{R_{1,1}\}}{\text{Im}\{R_{2,2}\}} = s, \tag{P.10}$$

say, for some real s. Thus,

$$\text{Re}\{R_{1,1}\} = s\text{Re}\{R_{2,2}\} \qquad \text{and} \qquad \text{Im}\{R_{1,1}\} = s\text{Im}\{R_{2,2}\}, \tag{P.11}$$

[1] Or, more directly, $T^2 + U^2 = |\beta|^2 = |\overline{\alpha}e^{i\theta}|^2 = |\overline{\alpha}|^2|e^{i\theta}|^2 = |\overline{\alpha}|^2 = x^2 + y^2$.

whence

$$\begin{aligned} R_{1,1} &= \mathrm{Re}\{R_{1,1}\} + i\mathrm{Im}\{R_{1,1}\} \\ &= s(\mathrm{Re}\{R_{2,2}\} + i\mathrm{Im}\{R_{2,2}\}) \\ &= sR_{2,2}, \end{aligned} \tag{P.12}$$

completing the proof. □

Remark 2.1. If $A = D$ then $\rho = 0$ and $R_{1,1} = R_{2,2}$ from (P.1), whereupon Theorem 2.1 is satisfied trivially with s, the constant of proportionality, being 1 (root matrix diagonal terms are identical also in this case for $K^2 > 0$, as seen in (1.3)).

3 Some Examples

Let $\mathbf{R}_i(n) = \mathbf{M}^{1/n}$ be an nth root matrix of $\mathbf{M}$ ($i = 1, \ldots, n$). For any given $n \geq 2$, we appeal to a 2004 algorithm of A. Choudhry (see [1, Remark 2.1, p. 9]) to extract n root matrices $\mathbf{R}_1(n), \ldots, \mathbf{R}_n(n)$ in a systematic fashion;[2] where a root has real entries only then Theorem 2.1 holds trivially, so only complex matrix roots are of interest in this examples section.

We begin by presenting the matrix (with $K^2 = -1296$)

$$\mathbf{M} = \begin{pmatrix} -23 & -37 \\ 9 & -29 \end{pmatrix}, \tag{3.1}$$

for which the cube root

$$\mathbf{R}_1(3) = \begin{pmatrix} (-1+\sqrt{3}i)/4 & 37(-1+\sqrt{3}i)/12 \\ 3(1-\sqrt{3}i)/4 & 3(-1+\sqrt{3}i)/4 \end{pmatrix}, \tag{3.2}$$

the fourth root

$$\mathbf{R}_4(4) = \begin{pmatrix} 1.615490335(-1+i) & 0.5198048153(-1+i) \\ 0.1264390091(1-i) & 1.699783008(-1+i) \end{pmatrix}, \tag{3.3}$$

and the fifth root

$$\mathbf{R}_3(5) = \begin{pmatrix} 0.9245817496 - 0.6717479617i & -3.008692302 + 2.185942911i \\ 0.7318440735 - 0.5317158432i & 0.4366857006 - 0.3172707329i) \end{pmatrix}, \tag{3.4}$$

have (resp.) diagonals ratios of $1/3$, 0.95040975 (8 d.p.) and 2.1172705 (7 d.p.), while

$$\mathbf{M} = \begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix} \tag{3.5}$$

(for which $K^2 = -36$) has a cube root

$$\mathbf{R}_2(3) = \begin{pmatrix} 0.399988572 - 0.6928005303i & -0.5013488269 + 0.8683616439i \\ 1.253372068 - 2.170904109i & -0.101360255 + 0.1755611142i \end{pmatrix} \tag{3.6}$$

with a diagonals ratio of -3.946207 (6 d.p.); the corresponding ratio is 0.253983 (6 d.p.) in the fifth root

$$\mathbf{R}_4(5) = \begin{pmatrix} -0.2070284236 + 0.1504149542i & -0.6080975793 + 0.4418087525i \\ 1.520243948 - 1.104521881i & -0.8151260028 + 0.5922237067i \end{pmatrix}. \tag{3.7}$$

[2] Each of these can be multiplied by an nth root of unity to offer a total of n^2 root matrices—there is no need to do this, however, for Theorem 2.1 here remains unaffected; note that these n^2 matrices are also available from combining (pairwise) all possible n values (as nth eigenvalue roots) of each of α and β in the matrix elements (P.1) of any root matrix $\mathbf{R} = \mathbf{R}_i(n)$.

The cube root

$$\mathbf{R}_2(3) = \begin{pmatrix} -1.106553876 - 1.916607874i & 0.3623653656 + 0.6276353378i \\ -7.24730731 - 12.55270677i & 1.79236905 + 3.104474829i \end{pmatrix} \tag{3.8}$$

of

$$\mathbf{M} = \begin{pmatrix} 2 & 1/2 \\ -10 & 6 \end{pmatrix} \tag{3.9}$$

($K^2 = -4$) has a diagonals ratio of -0.617369 (6 d.p.), while the matrix ($K^2 = -10$)

$$\mathbf{M} = \begin{pmatrix} 2\sqrt{2} & -3 \\ 1 & \sqrt{2} \end{pmatrix} \tag{3.10}$$

has tenth roots

$$\mathbf{R}_6(10) = \begin{pmatrix} 0.40073012 - 0.13020511i & -1.9848193 + 0.64490689i \\ 0.66160644 - 0.21496896i & -0.53492269 + 0.17380692i \end{pmatrix} \tag{3.11}$$

and

$$\mathbf{R}_{10}(10) = \begin{pmatrix} -0.89959761 + 0.29229698i & -0.49226746 + 0.15994739i \\ 0.16408915 - 0.053315798i & -1.1316547 + 0.36769691i \end{pmatrix} \tag{3.12}$$

with (resp.) diagonals ratios of -0.74913651 and 0.79494001 (8 d.p.); a sixteenth root

$$\mathbf{R}_{14}(16) = \begin{pmatrix} -0.64585844 + 0.12846923i & -1.0319974 + 0.20527704i \\ 0.34399912 - 0.06842568i & -1.1323467 + 0.22523776i \end{pmatrix} \tag{3.13}$$

has the diagonals ratio 0.57037166 (8 d.p.). The matrix ($K^2 = -32$)

$$\mathbf{M} = \begin{pmatrix} 11 & 3\sqrt{3} \\ -\sqrt{3} & 9 \end{pmatrix} \tag{3.14}$$

has (resp.) diagonals ratios of 1.2801901 and 0.2754113 (7 d.p.) in a twenty-ninth root

$$\mathbf{R}_{26}(29) = \begin{pmatrix} 1.1430777 - 0.12431719i & 0.64998895 - 0.070690558i \\ -0.21666298 + 0.023563519i & 0.89289685 - 0.097108384i \end{pmatrix} \tag{3.15}$$

and a fortieth root

$$\mathbf{R}_{13}(40) = \begin{pmatrix} -0.24119584 + 0.018982524i & 1.6486613 - 0.12975246i \\ -0.54955375 + 0.043250818i & -0.87576586 + 0.068924268i \end{pmatrix}. \tag{3.16}$$

Finally, the one hundredth root

$$\mathbf{R}_{51}(100) = \begin{pmatrix} 0.13180063 - 0.0041420018i & -0.10208718 + 0.0032082188i \\ 10.208718 - 0.32082188i & 0.029713458 - 0.00093378304i \end{pmatrix} \tag{3.17}$$

of the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -1 \\ 100 & 0 \end{pmatrix} \tag{3.18}$$

($K^2 = -399$) has a diagonals ratio of 4.4357218 (7 d.p.), with 0.9444123 (7 d.p.) being that of the root

$$\mathbf{R}_{92}(100) = \begin{pmatrix} -0.86278091 + 0.027113983i & -0.050782873 + 0.0015959161i \\ 5.0782873 - 0.15959161i & -0.91356379 + 0.028709899i \end{pmatrix}. \tag{3.19}$$

Numerous other test cases have been run, and Theorem 2.1 validated (we thank Dr. James Clapperton for undertaking all computations related to this paper using the algebraic software package Maple). After many hours of computations, for instance (the algorithm deployed in this work integrated an extremely high level of necessary precision in Maple's internal floating-point arithmetic so as to preserve the integrity of outputs—the software defaults to 8 digits per calculation, but $8n$ digit accuracy was set for any given value of n which slowed down processing for larger n values), we obtained all of the two hundredth matrix roots of $\mathbf{M}$ (3.18), representative ones being

$$\mathbf{R}_{103}(200) = \begin{pmatrix} 0.090558827 - 0.0014226117i & -0.10119306 + 0.0015896676i \\ 10.119306 - 0.15896676i & -0.010634234 + 0.00016705589i \end{pmatrix} \tag{3.20}$$

and

$$\mathbf{R}_{185}(200) = \begin{pmatrix} 0.99387905 - 0.0156131i & -0.024389604 + 0.00038314251i \\ 2.4389604 - 0.038314251i & 0.96948945 - 0.015229957i \end{pmatrix}, \tag{3.21}$$

with (resp.) diagonals ratios of -8.5157829 and 1.0251572 (7 d.p.).

4 Summary

This paper states and proves (and validates computationally) a new result for a particular class of 2×2 matrices,[3] motivated by a more complicated result for a converse class treated previously [1, Theorem 3.1 (Stanton's Theorem), p. 9]. We assert that neither result is an intuitive one, and that together they add to the body of knowledge on arbitrary roots of general matrices of dimension 2 (in this regard, a good overview of the various methods of 2×2 matrix root extraction has been given not too long ago by Özdemir [2, p. 23]).

References

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[2] M. Özdemir, Finding n-th roots of a 2×2 real matrix using de Moivre's formula, *Adv. Appl. Cliff. Alg.* **29** (Issue 1), 25 pp. (2019).

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[3] The author Lee Rawlin was involved in this work as part of his undergraduate final year dissertation project at the University of Derby (supervised by P.J.L.) during the 2020-21 academic year. The motivation for thinking about properties of matrix roots arose from some extensive computational experiments conducted by James Stanton.