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# Some Statistical properties of Extended Generalized Bessel function

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**Abstract** In the present paper, our aim is to discuss a probability density function involving the Extended Generalized Bessel function, and its properties.

## 1 Introduction

A fairly wide range of important functions in applied sciences are defined via improper integrals or infinite series or infinite products. McNolty [6] showed the applicability of Univariate Bessel Function Distributions in signal output of radar. We give brief introduction of Bessel function of first kind.

The Bessel function of first kind $J_\nu(z)$ [5, P. 109] is represented as,

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+k)k!} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{1.1}$$

where $|z| < \infty$, $|\arg z| < \pi$.

In 2013, Salehbhai et al. [8, P. 2] introduced Extended Generalized Bessel Function in following manner,

$$_hJ_v^m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{\Gamma(1+mv+hk)\, k!}, \tag{1.2}$$

where, $h \in N$, $|z| < \infty$, $|\arg z| < \pi$, $m\ and\, v \in C$.

The well known generalized hypergeometric function [5, P. 73] is defined by

$$_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, ..., \alpha_p\\ \beta_1, \beta_2, ..., \beta_q\end{matrix}\middle| z\right] = 1 + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k...(\alpha_p)_k}{(\beta_1)_k(\beta_2)_k...(\beta_q)_k} \frac{z^k}{k!}, \quad |z| < 1, \tag{1.3}$$

where $p$ and $q$ are nonnegative integers and $\beta_j$ $(j = 1, 2, \ldots, q)$ is zero or a negative integer. Here, $(\alpha)_k$ is a Pochhammer symbol [5, P. 22] and defined as

$$(\alpha)_k := \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} 1 & (k=0;\ \alpha \in \mathbf{C}\setminus\{0\}) \\ \alpha(\alpha+1)\ldots(\alpha+k-1) & (k \in \mathbf{N};\ \alpha \in \mathbf{C}). \end{cases} \tag{1.4}$$

Fox-Wright function [10] defined as following form,

$$_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(\alpha_i, A_i)_{1,p}\\ (\beta_j, B_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1+A_1k)\cdots\Gamma(\alpha_p+A_pk)}{\Gamma(\beta_1+B_1k)\cdots\Gamma(\beta_q+B_qk)} \frac{z^k}{k!}, \tag{1.5}$$

where $i = 1, 2, \cdots, p;\quad j = 1, 2, \cdots, q$ and $z, \alpha_i, \beta_j \in \mathbf{C}$, and the coefficients $A_1, ...., A_p \in \mathbf{R}^+$ and $B_1, ...., B_q \in \mathbf{R}^+$ satisfying the following condition

$$\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1. \tag{1.6}$$

We recall the Hadamard product (or the Hadamard composition) here.

**Definition 1.1.** [6] Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \ \ (|z| < R_f)$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n \ \ (|z| < R_g)$ be two given power series whose radii of convergence are denoted by $R_f$ and $R_g$ respectively. Then their Hadamard product $(f * g)(z)$ is given by the following power series ,

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \quad (|z| < R), \tag{1.7}$$

where

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n b_n}{a_{n+1} b_{n+1}} \right| = \left( \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \right) \cdot \left( \lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| \right) \\ &= R_f . R_g , \end{aligned} \tag{1.8}$$

Some of the following facts are needed in our study.

**Definition 1.2.** [4] Probability Density Functions is one, discrete or continuous and variable are denoted $p(r)$ and $f(x)$, respectively. They are assumed to be properly normalized such that

$$\sum_r p(r) = 1 \quad and \quad \int_{-\infty}^{\infty} f(x)dx = 1, \tag{1.9}$$

where the sum or the integral are taken over all relevant values for which the probability density function is defined.

**Definition 1.3.** [4] Algebraic moments of order $r$ are defined as expectation value

$$E(x^r) = \sum_k k^r p(k) \quad or \quad \int_{-\infty}^{\infty} x^r f(x)\, dx \tag{1.10}$$

**Definition 1.4.** [1] For a distribution in a continuous variable $x$ the Fourier transform of the probability density function

$$\phi(t) = E\left(e^{ixt}\right) = \int_{-\infty}^{\infty} e^{ixt} f(x)dx \tag{1.11}$$

is called the characteristic function. It has the property that $\phi(0) = 1$ and $|\phi(t)| \leqslant 1$ for all $t$.

**Definition 1.5.** [1] For commulative distribution function, $f(x)$ is the probability that the variates takes a value less than or equal to $x$.

$$F(x) = \Pr[X \leqslant x] = \alpha \tag{1.12}$$

$$F(x) = \int_{-\infty}^{x} f(u)du \tag{1.13}$$

## 2 An Extended Generalized Bessel Distribution

In this section, we discuss a general family of statistical probability distributions.

We begin by recalling the celebrated Ramanujan's Master Theorem [2], which was widely used by Srinivasa Ramanujan Iyengar (1887-1920) in order to evaluate definite integrals and infinite series. The proof of Ramanujan's Master Theorem was provided by Godfrey Harold Hardy (1877-1947) by making the use of Cauchy's Residue Theorem as well as the well-known Mellin Inversion Theorem.

**Definition 2.1.** The Extended Generalized Bessel function is defined (1.2) as

$$_{h}J_{v}^{m}\left(z\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}z^{k}}{\Gamma\left(1+mv+hk\right)k!},$$

where, $h\in N$, $|z|<\infty$, $|\arg z|<\pi$, $m\; and\, v\in C$.

One can write the equation (1.2) as,

$$_{h}J_{v}^{m}\left(z\right)=\Gamma\left(1+mv\right)\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}z^{k}}{\left(1+mv\right)_{hk}k!},\tag{2.1}$$

or,

$$_{h}J_{v}^{m}\left(z\right)=\Gamma\left(1+mv\right){}_{0}F_{h}\left[\left.\begin{matrix}-\\ \left(1+mv\right)\end{matrix}\right|-z\right],\tag{2.2}$$

or,

$$_{h}J_{v}^{m}\left(z\right)=\Gamma\left(1+mv\right)\sum_{k=0}^{\infty}\Theta_{n}\frac{\left(-z\right)^{k}}{k!},\tag{2.3}$$

where $\Theta_n=(1+mv)_{hn}^{-1}$.

**Theorem 2.2.** *(Ramanujan's Master Theorem)[2] Assume that the function $f\left(x\right)$ has a power series expansion in the following form .*

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{\phi\left(n\right)}{n!}\left(-x\right)^{n}\tag{2.4}$$

*Then the Mellin transform of the $f(x)$ is given by*

$$M\left[f\left(t\right);s\right]=\int_{0}^{\infty}t^{s-1}f\left(t\right)dt=\Gamma\left(s\right)\phi\left(-s\right),\tag{2.5}$$

*provided the integral in (2.5) exists.*

On employing Theorem 2.2 and Eq.(1.2), we can deduce the following corollary.

**Corollary 2.3.** *Under the conditions for (2.5), the following Mellin transform formula holds true.*

$$M\left[{}_{h}J_{v}^{m}\left(z\right)\right]=\Gamma\left(1+mv\right)\Gamma\left(s\right)\Theta_{-s},\tag{2.6}$$

*where*

$$\Theta_{-s}=\left.\Theta_{n}\right|_{n=-s}\tag{2.7}$$

*in terms of the coefficients $\Theta_n$, given by Eq.(2.3).*

**Definition 2.4.** For a statistical probability distribution of a random variable $X$, let the probability density function

$$B_X\left(X\right)=\begin{cases}\left(\dfrac{1}{\Gamma\left(s\right)\Theta_{-s}}\right)\dfrac{1}{\Gamma\left(1+mv\right)}x^{s-1}{}_1F_h\left[\begin{array}{c|}- \\ (1+mv)\end{array}-x\right], & x>0\\ 0, & elsewhere\end{cases} \tag{2.8}$$

where $\Theta_{-s}$ is given by (2.7) and it is tacitly assumed that the various arguments parameters involved in the definitions (2.2) and (2.3), are restricted as

$$B_X\left(X\right)>0, \tag{2.9}$$

clearly, it follows from corollary (2.3) that

$$\int_0^\infty B_X\left(X\right)dx=1, \tag{2.10}$$

provided that the integral (2.10) exists.

# 3 Properties of Random variable $X$

In this section, we give some properties of the random variable $X$.

## 3.1 The $k^{th}$ Moment

**Theorem 3.1.** *The $k^{th}$ Moment $E\left[x^k\right]$ of the random variable $X$ is given by*

$$E\left[x^k\right]=\frac{\Gamma\left(s+k\right)\Theta_{-s-k}}{\Gamma\left(s\right)\Theta_{-s}}. \tag{3.1}$$

*Proof.* The $k^{th}$ moment of the random variable $X$ is given as,

$$E\left[x^k\right]=\int_0^\infty t^k B_X\left(t\right)dt \tag{3.2}$$

On using (2.8), we obtain,

$$E\left[x^k\right]=\left(\frac{1}{\Gamma\left(s\right)\Theta_{-s}\Gamma\left(1+mv\right)}\right)\int_0^\infty t^{s+k-1}{}_1F_h\left[\begin{array}{c|}- \\ (1+mv)\end{array}-t\right]dt \tag{3.3}$$

From (2.5) and (3.3), we get,

$$E\left[x^k\right]=\frac{\Gamma\left(s+k\right)\Theta_{-s-k}\Gamma\left(1+mv\right)}{\Gamma\left(s\right)\Theta_{-s}\Gamma\left(1+mv\right)}.$$

□

**Corollary 3.2.** *The mean $\mu_x$ is the first moment, the expected value of the random variable $X$ is a special case of the $k^{th}$ moment as defined (3.3) for $k=1$.*

$$\mu_x=E\left[X\right]=\frac{\Gamma\left(s+1\right)\Theta_{-s-1}}{\Gamma\left(s\right)\Theta_{-s}}. \tag{3.4}$$

*This can also be written as,*

$$\mu_x=E\left[X\right]=\frac{s\Gamma\left(s\right)\Theta_{-s-1}}{\Gamma\left(s\right)\Theta_{-s}}.$$

*On further simplification, we arrive at*

$$\mu_x=E\left[X\right]=\frac{s\Theta_{-s-1}}{\Theta_{-s}}. \tag{3.5}$$

**Corollary 3.3.** *On setting $k = 2$, the formula (2.11), this yields the second moment as follows.*

$$E\left[X^2\right] = \frac{\Gamma\left(s+2\right)\Theta_{-s-2}}{\Gamma\left(s\right)\Theta_{-s}}. \tag{3.6}$$

$$E\left[X^2\right] = \frac{s\left(s+1\right)\Theta_{-s-2}}{\Theta_{-s}} \tag{3.7}$$

**Corollary 3.4.** *Making the use of Eq.(3.5) and (3.7), the variance $\sigma_x^2$ of the random variable $X$ can easily be computed,*

$$\sigma_x^2 = E\left[X^2\right] - \left(E\left[X\right]\right)^2, \tag{3.8}$$

*this can also be written as,*

$$\sigma_x^2 = \frac{s\left(s+1\right)\Theta_{-s-2}}{\Theta_{-s}} - \left(\frac{s\Theta_{-s-1}}{\Theta_{-s}}\right)^2. \tag{3.9}$$

### 3.2 The Characteristic Function

**Lemma 3.5.** *The Extended Generalized Bessel function defined as (1.2), the following integral formula holds true.*

$$\begin{aligned}&\int_0^\infty t^{\rho-1}\, e^{-st}\, {}_hJ_v^m\left(zt^\sigma\right)dt \\ &= s^{-\rho}\Gamma\left(1+mv\right)\,{}_0F_h\left[\begin{matrix}- \\ \left(1+mv\right)\end{matrix}\middle|\frac{-z}{s^\sigma}\right] * {}_1\psi_0\left[\begin{matrix}\left(\delta,\sigma\right) \\ -\end{matrix}\middle|\frac{-z}{s^\sigma}\right],\end{aligned} \tag{3.10}$$

*where $*$ is the Hadamard product defined as (1.7) and ${}_p\psi_q$ denotes the Fox-wright generalized hypergeometric function and it is assumed that each member of (3.10) exists.*

*Proof.* The demonstration of lemma is based upon the following well-known result [7].

$$\int_0^\infty t^{\rho-1}e^{-st}dt = \frac{\Gamma\left(\rho\right)}{s^\rho}\quad\left(\min\left\{R\left(s\right),\, R\left(\rho\right)\right\} > 0\right) \tag{3.11}$$

Let us denote L.H.S of (3.10) by I and afterwards using (3.11) and (1.2), we get,

$$I = \int_0^\infty t^{\rho-1}\, e^{-st}\,\Gamma\left(1+mv\right)\,{}_0F_h\left[\begin{matrix}- \\ \left(1+mv\right)\end{matrix}\middle| -zt^\sigma\right]. \tag{3.12}$$

On changing order of integration and summation, this yields

$$I = \sum_{n=0}^\infty \frac{\Theta_n\Gamma\left(1+mv\right)}{n!}\int_0^\infty t^{\rho-1}\, e^{-st}\left(-zt^\sigma\right)^k dt. \tag{3.13}$$

we can also write (3.13) as,

$$I = \sum_{n=0}^\infty \frac{\Theta_n\Gamma\left(1+mv\right)}{n!}\left(-z\right)^k\int_0^\infty t^{\rho+\sigma k-1}\, e^{-st}\, dt \tag{3.14}$$

From (3.10) and (3.14), we get,

$$I = \sum_{n=0}^\infty \frac{\Theta_n\Gamma\left(1+mv\right)}{n!}\left(-z\right)^k\frac{\Gamma\left(\rho+\sigma k\right)}{s^{\rho+\sigma k}}. \tag{3.15}$$

One can easily write Equation (3.15) in the following form

$$I = s^{-\rho}\Gamma(1+mv)\ {}_0F_h\left[\begin{matrix}- \\ (1+mv)\end{matrix}\middle|\frac{-z}{s^\sigma}\right] * {}_1\psi_0\left[\begin{matrix}(\delta,\sigma) \\ -\end{matrix}\middle|\frac{-z}{s^\sigma}\right] \tag{3.16}$$

□

**Theorem 3.6.** *The Characteristic function $\phi_x(t)$ of $B_X(t)$ associated with the random variable $X$ is given as*

$$\phi_x(t) = E\left[e^{itx}\right] = \int_0^\infty e^{itx}\, B_X(t)\, dt, \tag{3.17}$$

*where $E$ denotes the mathematical expectation.*

*Proof.* Let us write left hand side of (3.17) in following form by using (2.8) as,

$$\phi_x(t) = \int_0^\infty e^{itx}\left(\frac{1}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)}\right) x^{s-1}{}_1F_h\left[\begin{matrix}- \\ (1+mv)\end{matrix}\middle| -x\right] dx. \tag{3.18}$$

On changing the order of integration, we arrive at

$$\phi_x(t) = \sum_{n=0}^{\infty}\frac{(-1)^n\,\Theta_n}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)}\int_0^\infty e^{itx}\,x^{s+n-1}\,dx. \tag{3.19}$$

From (3.19) and (3.11), we obtain,

$$\phi_x(t) = \sum_{n=0}^{\infty}\frac{(-1)^n\,\Theta_n}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)}\,\frac{\Gamma(s+n)}{(-it)^{s+n}}. \tag{3.20}$$

On using (1.7), one can get following form of (3.20), i.e.,

$$\phi_x(t) = \frac{(-it)^s}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)}\ {}_1F_h\left[\begin{matrix}- \\ (1+mv)\end{matrix}\middle|\frac{1}{-it}\right] * {}_1\psi_0\left[\begin{matrix}(s,1) \\ -\end{matrix}\middle|\frac{1}{-it}\right] \tag{3.21}$$

This completes the proof. □

### 3.3 Moment Generating Function

**Theorem 3.7.** *The moment generating function $M_x(t)$ [9, P. 12] of the random variable $X$ is defined as,*

$$M_x(t) = \left(\frac{(-t)^{-s}}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)}\right){}_1F_h\left[\begin{matrix}- \\ (1+mv)\end{matrix}\middle|\frac{1}{-t}\right] * {}_1\psi_0\left[\begin{matrix}(s,1) \\ -\end{matrix}\middle|\frac{1}{-t}\right] \tag{3.22}$$

*Proof.* The moment generating function $M_x(t)$ of the random variable $X$ is defined by,

$$M_x(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx}\, B_X(X)\, dx \tag{3.23}$$

Therefore, we can obtain following result by using (3.21),

$$M_x(t) = E\left[e^{tx}\right] = \phi_x(-it) \tag{3.24}$$

where $\phi_x(t)$ is the characteristic function of $B_X(X)$ and associated with the random variable $x$, which is given by (3.12), can be expressed as,

$$M_x(t) = \left(\frac{(-t)^{-s}}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)}\right){}_1F_h\left[\begin{matrix}- \\ (1+mv)\end{matrix}\middle|\frac{1}{-t}\right] * {}_1\psi_0\left[\begin{matrix}(s,1) \\ -\end{matrix}\middle|\frac{1}{-t}\right] \tag{3.25}$$

Hence the proof. □

### 3.4 The Cumulative Distribution Function

**Theorem 3.8.** *From the definition of 1.5, for $x > 0$ the cumulative distribution function $F_x(X)$ is given by,*

$$F_x(X) = \left( \frac{x^s}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)} \right) {}_1F_h \left[ \begin{matrix} - \\ (1+mv) \end{matrix} \middle| -x \right] * {}_1\psi_1 \left[ \begin{matrix} (s,1) \\ (s+1,1) \end{matrix} \middle| -x \right] \tag{3.26}$$

*Proof.* For $x > 0$ the cumulative distribution function $F_x(X)$ is given by,

$$F_x(X) = \int_0^x B_X(t)\, dt \tag{3.27}$$

On writing (3.27) in following form by using (2.8),

$$F_x(X) = \frac{1}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)} \sum_{n=0}^{\infty} \frac{(-1)^n\,\Theta_n}{n!} \int_0^x t^{s+n-1}\, dt. \tag{3.28}$$

After simplification, we find that

$$F_x(X) = \frac{1}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)} \sum_{n=0}^{\infty} \frac{(-1)^n\,\Theta_n}{n!} \frac{x^{s+n}\Gamma(s+n)}{\Gamma(s+n+1)}. \tag{3.29}$$

Use of (1.3) and (1.7) gives,

$$F_x(X) = \left( \frac{x^s}{\Gamma(s)\,\Theta_{-s}\Gamma(1+mv)} \right) {}_1F_h \left[ \begin{matrix} - \\ (1+mv) \end{matrix} \middle| -x \right] * {}_1\psi_1 \left[ \begin{matrix} (s,1) \\ (s+1,1) \end{matrix} \middle| -x \right]. \tag{3.30}$$

This completes the proof. □

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