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Congruences for $[j,k]-$overpartition pairs with even parts distinct

M. S. Mahadeva Naika, Harishkumar T and Y. Veeranna

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Abstract. Let $\overline{bped}_{j,k}(n)$ denote the number of $[j,k]-$overpartition pairs of a positive integer n with even parts distinct in which the first occurrence of each distinct part congruent to j modulo k may be overlined. In this work, we establish many infinite families of congruences modulo powers of 2 for $\overline{bped}_{3,3}(n)$ and $\overline{bped}_{3,6}(n)$. For example, for any $n\geq 0$ and $\alpha,\beta\geq 0$,

$$\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+j)+7\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}\right)\equiv 0 \pmod{32},$$

where $j=0,1,3,4$.

1 Introduction

Corteel and Lovejoy [4] introduced overpartitions. An overpartition of a positive integer n is a partition in which the first occurrence of each distinct part may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n with $\overline{p}(0)=1$. The generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}\left(n\right)q^{n}=\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}=\frac{f_2}{f_1^2}, \tag{1.1}$$

where

$$f_1:=(q;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}$$

and

$$f_\ell:=(q^\ell;q^\ell)_\infty=\prod_{m=1}^{\infty}(1-q^{m\ell}).$$

Many authors proved many Ramanujan-type identities and congruences for $\overline{p}(n)$, including [1, 3, 6, 9, 10, 14, 15, 16, 17].

For positive integers j and k such that $k>j\geq 1$, an $[j,k]-$overpartitions of n is a partition in which the first occurrence of each distinct parts are congruent to $j \pmod{k}$ may be overlined. Let $\overline{p}_{j,k}(n)$ denote the number of such overpartitions of n with $\overline{p}_{j,k}(0)=1$. The generating function for $\overline{p}_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}_{j,k}(n)q^n=\frac{(-q^j;q^k)_\infty}{(q;q)_\infty}. \tag{1.2}$$

For example, the $[5,10]$-overpartitions of 5 are

$$5,\ \overline{5},\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

Mahadeva Naika et al. [12] obtained many infinite families of congruences modulo powers of 2 for $\overline{p}_{j,k}(n)$, the number of $[j,k]-$regular overpartitions of n in which none of the parts are congruent to $j \pmod{k}$. For example, for all $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{p}_{9,18}\left(3^{4\alpha+1} \cdot 5^{2\beta+1}(24(5n+i)+23)\right) \equiv 0 \pmod{64},$$

where $i = 0, 1, 2, 4$.

In [11, 13], the authors proved many infinite families of congruences modulo powers of 2 for $\overline{ped}_{3,3}(n)$, $\overline{ped}_{3,6}(n)$ and $\overline{ped}_{9,9}(n)$, congruences modulo powers of 2 and 3 for $\overline{ped}_{9,18}(n)$. For example, for any $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{ped}_{3,6}\left(8 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+2}n + c_1 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{64},$$

where $c_1 \in \{23, 47, 71, 119\}$.

By the motivation of the above work, in this paper, we define $\overline{bped}_{j,k}(n)$, the number of $[j,k]-$overpartition pairs of a positive integer n with even parts distinct in which the first occurrence of each distinct parts are congruent to j modulo k may be overlined. The generating function for $\overline{bped}_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{bped}_{j,k}(n)\, q^n = \frac{(q^4;q^4)_\infty^2(-q^j;q^k)_\infty^2}{(q;q)_\infty^2}. \tag{1.3}$$

Also, we establish many infinite families of congruences modulo powers of 2 for $\overline{bped}_{3,3}(n)$ and $\overline{bped}_{3,6}(n)$. For example, for any $n \geq 0$ and $\alpha, \beta \geq 0$,

$$\overline{bped}_{3,6}\left(8 \cdot 3^{4\alpha+3} \cdot 5^{2\beta+1}(5n+j) + 7 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32},$$

where $j = 0, 1, 3, 4$.

2 Preliminary results

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1. *The following* 2*-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8} \tag{2.1}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.2}$$

The identity (2.1) is the 2-dissection of $\phi(q)$ [5, (1.9.4)]. The identity (2.2) is the 2-dissection of $\phi(q)^2$ [5, (1.10.1)]. Also, one can see [2, p.40].

Lemma 2.2. *The following* 2*-dissections hold:*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.3}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \tag{2.4}$$

and

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{2.5}$$

The equations (2.3) and (2.4) are the same as (30.12.3) and (30.10.4) in [5]. The equation (2.5) is the same as (22.1.14) in [5] (after using 22.1.6 and 22.1.7).

Lemma 2.3. *The following* 3*-dissections hold:*

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}, \tag{2.6}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{2.7}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{2.8}$$

Lemma 2.3 was proved by Hirschhorn and Sellers [7].

Lemma 2.4. *The following* 3*-dissections hold:*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3} \tag{2.9}$$

and

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{2.10}$$

The equation (2.9) is the same as $(14.8.5)$ in [5]. Also, see [2, p.345]. For a proof of (2.10), we can see [8].

Lemma 2.5. *We have*

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2R(q^5)), \tag{2.11}$$

where

$$R(q) = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}$$

and

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The identity (2.11) is essentially $(8.1.1)$ in [5].

Lemma 2.6. *We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.12}$$

where $A(q) = f(-q^3,-q^4)$, $B(q) = f(-q^2,-q^5)$ *and* $C(q) = f(-q,-q^6)$.

Lemma 2.6 is an exercise in [5], see [5, (10.5.1)]. Also, one can see [2, p.303, Entry 17(v)].

3 Congruences for $\overline{bped}_{3,3}(n)$

Theorem 3.1. *For all* $n \geq 0$ *and* $\alpha \geq 0$, *we have*

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+1}n + \frac{3^{4\alpha+1}-1}{2}\right)q^n \equiv 2\frac{f_2^6}{f_1^6} + 24q f_1 f_2 f_3^3 f_6^3 \pmod{32}, \tag{3.1}$$

$$\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+4}n + \frac{5\cdot 3^{4\alpha+3}-1}{2}\right) \equiv \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+2}n + \frac{5\cdot 3^{4\alpha+1}-1}{2}\right) \pmod{32}, \tag{3.2}$$

$$\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+5}n + \frac{7\cdot 3^{4\alpha+4}-1}{2}\right) \equiv \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+3}n + \frac{7\cdot 3^{4\alpha+2}-1}{2}\right) \pmod{32}, \tag{3.3}$$

$$\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+5}n + \frac{11\cdot 3^{4\alpha+4}-1}{2}\right) \equiv \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+3}n + \frac{11\cdot 3^{4\alpha+2}-1}{2}\right) \pmod{32}. \tag{3.4}$$

Proof. Setting $j=k=3$ in (1.3), we find that

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}(n)\,q^n=\frac{f_4^2f_6^2}{f_1^2f_3^2}. \tag{3.5}$$

Employing (2.3) in (3.5) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}(2n+1)\,q^n=\frac{f_2^6f_6^4}{f_1^6f_3^4}. \tag{3.6}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2^m}\equiv f_{2k}^{2^{m-1}}\pmod{2^m}. \tag{3.7}$$

Using (2.6) and (2.8) in (3.6) and invoking (3.7), we arrive at

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}(6n+1)\,q^n\equiv 2\frac{f_2^6}{f_1^6}+24qf_1f_2f_3^3f_6^3\pmod{32}, \tag{3.8}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}(6n+3)\,q^n\equiv 4\frac{f_2^3f_3^3}{f_1}\pmod{8} \tag{3.9}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}(6n+5)\,q^n\equiv 2f_4f_6^3\pmod{4}. \tag{3.10}$$

The equation (3.8) is $\alpha=0$ case of (3.1). Suppose that the congruence (3.1) is true for $\alpha\geq 0$. Substituting (2.6), (2.8) and (2.10) in (3.1) and then collecting the coefficients of q^{3n},q^{3n+1} and q^{3n+2} from the resultant equation, we have

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+2}n+\frac{3^{4\alpha+1}-1}{2}\right)q^n\equiv 2\frac{f_2^2}{f_1^2}+16qf_6^3f_8+24q\frac{f_2f_3^3f_6^3}{f_1^3}\pmod{32}, \tag{3.11}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+2}n+\frac{5\cdot 3^{4\alpha+1}-1}{2}\right)q^n\equiv 12\frac{f_2f_3^5}{f_1f_6}+24f_1^2f_2^4+24q\frac{f_6^6}{f_1^2}\pmod{32} \tag{3.12}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+2}n+\frac{3^{4\alpha+3}-1}{2}\right)q^n\equiv 8f_1^3f_2^3f_3f_6+10\frac{f_6^6}{f_3^6}\pmod{32}. \tag{3.13}$$

Using (2.10) in (3.13), we obtain

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+3}n+\frac{3^{4\alpha+3}-1}{2}\right)q^n\equiv 18\frac{f_2^6}{f_1^6}+24qf_1f_2f_3^3f_6^3\pmod{32}, \tag{3.14}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+2}-1}{2}\right)q^n\equiv 8f_1^3f_2f_3f_6+16qf_2f_6^6\pmod{32} \tag{3.15}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+3}n+\frac{11\cdot 3^{4\alpha+2}-1}{2}\right)q^n\equiv 8f_2^2f_3^2f_6^2\pmod{32}. \tag{3.16}$$

Utilizing (2.6), (2.8) and (2.10) in (3.14), we get

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+4}n+\frac{3^{4\alpha+3}-1}{2}\right)q^n \equiv 18\frac{f_2^2}{f_1^2}+16qf_6^3f_8+24q\frac{f_2f_3^3f_6^3}{f_1^3} \pmod{32}, \quad (3.17)$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+4}n+\frac{5\cdot 3^{4\alpha+3}-1}{2}\right)q^n \equiv 12\frac{f_2f_3^5}{f_1f_6}+24f_1^2f_2^4+24q\frac{f_6^6}{f_1^2} \pmod{32} \quad (3.18)$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+4}n+\frac{3^{4\alpha+5}-1}{2}\right)q^n \equiv 8f_1^3f_2^3f_3f_6+26\frac{f_6^6}{f_3^6} \pmod{32}. \quad (3.19)$$

From the equations (3.12) and (3.18), we obtain (3.2).

Employing (2.10) in (3.19), we have

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+5}n+\frac{3^{4\alpha+5}-1}{2}\right)q^n \equiv 2\frac{f_2^6}{f_1^6}+24qf_1f_2f_3^3f_6^3 \pmod{32}, \quad (3.20)$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+5}n+\frac{7\cdot 3^{4\alpha+4}-1}{2}\right)q^n \equiv 8f_1^3f_2f_3f_6+16qf_2f_6^6 \pmod{32} \quad (3.21)$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+5}n+\frac{11\cdot 3^{4\alpha+4}-1}{2}\right)q^n \equiv 8f_2^2f_3^2f_6^2 \pmod{32}. \quad (3.22)$$

The equation (3.20) is $\alpha+1$ case of (3.1). By induction, the congruence (3.1) holds for all integer $\alpha \geq 0$.

From the equations (3.15) and (3.21), we get (3.3).

From the congruences (3.16) and (3.22), we obtain (3.4). □

Theorem 3.2. *For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have*

$$\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+4}n+\frac{3^{4\alpha+3}-1}{2}\right) \equiv \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+2}n+\frac{3^{4\alpha+1}-1}{2}\right) \pmod{16}, \quad (3.23)$$

$$\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}n+\frac{23\cdot 3^{4\alpha+2}-1}{2}\right) \equiv 0 \pmod{16}, \quad (3.24)$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+\frac{11\cdot 3^{4\alpha+2}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 8f_2f_3^3 \pmod{16}, \quad (3.25)$$

$$\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+i)+\frac{7\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}-1}{2}\right) \equiv 0 \pmod{16}, \quad (3.26)$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+\frac{7\cdot 3^{4\alpha+2}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 8f_1^7 \pmod{16}, \quad (3.27)$$

$$\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+j)+\frac{11\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}-1}{2}\right) \equiv 0 \pmod{16}, \quad (3.28)$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+\frac{19\cdot 3^{4\alpha+2}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 8f_1f_6^3 \pmod{16}, \quad (3.29)$$

$$\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+k)+\frac{23\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}-1}{2}\right) \equiv 0 \pmod{16}, \quad (3.30)$$

where $i = 0,1,3,4$, $j = 0,2,3,4$ and $k = 0,1,2,4$.

Proof. In view of the equations (3.11) and (3.17), we arrive at (3.23).

The equation (3.16) becomes

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+3}n+\frac{11\cdot 3^{4\alpha+2}-1}{2}\right)q^n \equiv 8f_4f_6^3 \pmod{16}, \tag{3.31}$$

which implies (3.24) and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}n+\frac{11\cdot 3^{4\alpha+2}-1}{2}\right)q^n \equiv 8f_2f_3^3 \pmod{16}, \tag{3.32}$$

which is $\beta = 0$ case of (3.25). Consider the congruence (3.25) is true for $\beta \geq 0$. Employing (2.11) in (3.25) and then collecting the coefficients of q^{5n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}n+\frac{7\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}-1}{2}\right)q^n \equiv 8q^2f_{10}f_{15}^3 \pmod{16}, \tag{3.33}$$

which implies (3.26) and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}\cdot 5^{2\beta+2}n+\frac{11\cdot 3^{4\alpha+2}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 8f_2f_3^3 \pmod{16}, \tag{3.34}$$

which implies that the congruence (3.25) is true for $\beta+1$. By induction, the congruence (3.25) holds for all integers $\alpha, \beta \geq 0$.

The equation (3.15) becomes

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+2}-1}{2}\right)q^n \equiv 8\frac{f_2^3f_3^3}{f_1} \pmod{16}. \tag{3.35}$$

Using (2.5) in (3.35) and then comparing the coefficients of q^{2n} and q^{2n+1}, we find that

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}n+\frac{7\cdot 3^{4\alpha+2}-1}{2}\right)q^n \equiv 8f_1^7 \pmod{16} \tag{3.36}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+3}n+\frac{19\cdot 3^{4\alpha+2}-1}{2}\right)q^n \equiv 8f_1f_6^3 \pmod{16}. \tag{3.37}$$

The remaining proofs of the congruences (3.27)-(3.30) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details. □

Theorem 3.3. *For all* $n \geq 0$ *and* $\alpha, \beta, \gamma \geq 0$, *we have*

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}\cdot 7^{2\gamma}n+\frac{5\cdot 3^{4\alpha+1}\cdot 7^{2\gamma}-1}{2}\right)q^n \equiv 4f_1^5 \pmod{8}, \tag{3.38}$$

$$\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}\cdot 7^{2\gamma+1}(7n+i)+\frac{11\cdot 3^{4\alpha+1}\cdot 7^{2\gamma+1}-1}{2}\right) \equiv 0 \pmod{8}, \tag{3.39}$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}\cdot 5^{2\beta}n+\frac{13\cdot 3^{4\alpha+1}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1^{13} \pmod{8}, \tag{3.40}$$

$$\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}(5n+j)+\frac{17\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}-1}{2}\right) \equiv 0 \pmod{8}, \tag{3.41}$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(12\cdot 5^{2\beta}n+\frac{7\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1^7 \pmod{8}, \tag{3.42}$$

$$\overline{bped}_{3,3}\left(12\cdot 5^{2\beta+1}(5n+k)+\frac{11\cdot 5^{2\beta+1}-1}{2}\right)\equiv 0 \pmod 8, \tag{3.43}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(12\cdot 5^{2\beta}n+\frac{19\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_6^3 \pmod 8, \tag{3.44}$$

$$\overline{bped}_{3,3}\left(12\cdot 5^{2\beta+1}(5n+l)+\frac{23\cdot 5^{2\beta+1}-1}{2}\right)\equiv 0 \pmod 8, \tag{3.45}$$

where $i=0,2,3,4,5,6$, $j=0,1,3,4$, $k=0,2,3,4$ *and* $l=0,1,2,4$.

Proof. The equation (3.12) becomes

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+2}n+\frac{5\cdot 3^{4\alpha+1}-1}{2}\right)q^n\equiv 4\frac{f_2f_3^3}{f_1} \pmod 8. \tag{3.46}$$

Utilizing (2.5) in (3.46) and then extracting the coefficients of q^{2n} from the resultant equation, we get

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}n+\frac{5\cdot 3^{4\alpha+1}-1}{2}\right)q^n\equiv 4f_1^5 \pmod 8, \tag{3.47}$$

which is $\gamma=0$ case of (3.38). Suppose that the congruence (3.38) is true for $\gamma\geq 0$. Substituting (2.12) in (3.38) and then collecting the coefficients of q^{7n+3} from the resultant equation, we have

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}\cdot 7^{2\gamma+1}n+\frac{11\cdot 3^{4\alpha+1}\cdot 7^{2\gamma+1}-1}{2}\right)q^n\equiv 4qf_7^5 \pmod 8, \tag{3.48}$$

which implies (3.39) and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}\cdot 7^{2\gamma+2}n+\frac{5\cdot 3^{4\alpha+1}\cdot 7^{2\gamma+2}-1}{2}\right)q^n\equiv 4f_1^5 \pmod 8, \tag{3.49}$$

which implies that the congruence (3.38) is true for $\gamma+1$. By induction, the congruence (3.38) holds for all integers $\alpha,\gamma\geq 0$.

The equation (3.11) becomes

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(2\cdot 3^{4\alpha+2}n+\frac{3^{4\alpha+1}-1}{2}\right)q^n\equiv 2\frac{f_2^2}{f_1^2} \pmod 8. \tag{3.50}$$

Using (2.1) in (3.50), we obtain

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}n+\frac{3^{4\alpha+1}-1}{2}\right)q^n\equiv 2\frac{f_4}{f_1^3} \pmod 8 \tag{3.51}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}n+\frac{13\cdot 3^{4\alpha+1}-1}{2}\right)q^n\equiv 4f_1^{13} \pmod 8. \tag{3.52}$$

The equation (3.52) is $\beta=0$ case of (3.40). The rest of the proofs of the congruences (3.40) and (3.41) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details.

Utilizing (2.5) in (3.9), we arrive at

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}(12n+3)\,q^n\equiv 4f_1^7 \pmod 8 \tag{3.53}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,3}(12n+9)\,q^n\equiv 4f_1f_6^3 \pmod 8. \tag{3.54}$$

The remaining proofs of the congruences (3.42)-(3.45) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details. □

Theorem 3.4. *For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have*

$$\overline{bped}_{3,3}\left(12n+11\right) \equiv 0 \pmod 4, \tag{3.55}$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(12\cdot 5^{2\beta}n + \frac{11\cdot 5^{2\beta}-1}{2}\right) q^n \equiv 2f_2f_3^3 \pmod 4, \tag{3.56}$$

$$\overline{bped}_{3,3}\left(12\cdot 5^{2\beta+1}(5n+i) + \frac{7\cdot 5^{2\beta+1}-1}{2}\right) \equiv 0 \pmod 4, \tag{3.57}$$

$$\overline{bped}_{3,3}\left(4\cdot 3^{4\alpha+2}n + \frac{3^{4\alpha+1}-1}{2}\right) \equiv \begin{cases} 2 \pmod 4 & \text{if } n \text{ is a pentagonal number,} \\ 0 \pmod 4 & \text{otherwise,} \end{cases} \tag{3.58}$$

where $i = 0, 1, 3, 4$.

Proof. The equation (3.10) implies (3.55) and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,3}\left(12n+5\right) q^n \equiv 2f_2f_3^3 \pmod 4, \tag{3.59}$$

which is $\beta = 0$ case of (3.56). The rest of the proofs of the congruences (3.56) and (3.57) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details.

From the equation (3.51), we arrive at (3.58). □

4 Congruences for $\overline{bped}_{3,6}(n)$

Theorem 4.1. *For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+2}n + 3^{4\alpha+2}\right) q^n \equiv 36\frac{f_1^5f_3^5}{f_2f_6} + 56f_1^4f_2^4 \pmod{64}, \tag{4.1}$$

$$\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+5}n + 5\cdot 3^{4\alpha+4}\right) \equiv \overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+3}n + 5\cdot 3^{4\alpha+2}\right) \pmod{64}, \tag{4.2}$$

$$\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+6}n + 5\cdot 3^{4\alpha+5}\right) \equiv \overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+4}n + 5\cdot 3^{4\alpha+3}\right) \pmod{64}. \tag{4.3}$$

Proof. Setting $j = 3$ and $k = 6$ in (1.3), we find that

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(n\right) q^n = \frac{f_4^2f_6^4}{f_1^2f_3^2f_{12}^2}. \tag{4.4}$$

Using (2.3) in (4.4) and then extracting the coefficients of q^{2n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(2n+1\right) q^n = 2\frac{f_2^6f_6^2}{f_1^6f_3^2}. \tag{4.5}$$

Employing (2.6) and (2.8) in (4.5) and invoking (3.7) we arrive at

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(6n+3\right) q^n \equiv 12\frac{f_2^3f_6^3}{f_1^3f_3^3} + 56qf_3^4f_6^4 \pmod{64}. \tag{4.6}$$

Substituting (2.6) and (2.8) in (4.6) and then comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2} on both sides of the resultant equation, we have

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(18n+3\right) q^n \equiv 12\frac{f_2^4}{f_1^4} + 16q\frac{f_2f_3^3f_6^3}{f_1} \pmod{32}, \tag{4.7}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}(18n+9)\,q^n\equiv 36\frac{f_1^5f_3^5}{f_2f_6}+56f_1^4f_2^4 \pmod{64} \tag{4.8}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}(18n+15)\,q^n\equiv 8\frac{f_2^2f_3^2f_6^2}{f_1^2} \pmod{32}. \tag{4.9}$$

The equation (4.8) is $\alpha=0$ case of (4.1). Suppose that the congruence (4.1) is true for $\alpha\geq 0$. Utilizing (2.7), (2.9) and (2.10) in (4.1) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+3}n+3^{4\alpha+2}\right)q^n\equiv 28\frac{f_1^4f_3^8}{f_6^4}+48q\frac{f_2f_3^3f_6^3}{f_1} \pmod{64}, \tag{4.10}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+3}n+3^{4\alpha+3}\right)q^n\equiv 44\frac{f_1^5f_3^5}{f_2f_6}+32qf_6^6+56qf_3^4f_6^4 \pmod{64} \tag{4.11}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+3}n+5\cdot 3^{4\alpha+2}\right)q^n\equiv 40\frac{f_2^2f_3^2f_6^2}{f_1^2} \pmod{64}. \tag{4.12}$$

Substituting (2.7) and (2.9) in (4.11), we get

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+4}n+3^{4\alpha+3}\right)q^n\equiv 44\frac{f_1^4f_3^8}{f_6^4}+48q\frac{f_2f_3^3f_6^3}{f_1} \pmod{64}, \tag{4.13}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+4}n+3^{4\alpha+4}\right)q^n\equiv 36\frac{f_1^5f_3^5}{f_2f_6}+32f_2^6+32qf_6^6+56f_1^4f_2^4 \pmod{64} \tag{4.14}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+4}n+5\cdot 3^{4\alpha+3}\right)q^n\equiv 8\frac{f_2^2f_3^2f_6^2}{f_1^2} \pmod{64}. \tag{4.15}$$

Using (2.7), (2.9) and (2.10) in (4.14), we arrive at

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+5}n+3^{4\alpha+4}\right)q^n\equiv 28\frac{f_1^4f_3^8}{f_6^4}+32f_4+48q\frac{f_2f_3^3f_6^3}{f_1} \pmod{64}, \tag{4.16}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+5}n+3^{4\alpha+5}\right)q^n\equiv 44\frac{f_1^5f_3^5}{f_2f_6}+32f_2^6+56qf_3^4f_6^4 \pmod{64} \tag{4.17}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+5}n+5\cdot 3^{4\alpha+4}\right)q^n\equiv 40\frac{f_2^2f_3^2f_6^2}{f_1^2} \pmod{64}. \tag{4.18}$$

In view of (4.12) and (4.18), we obtain (4.2).

Employing (2.7) and (2.9) in (4.17), we find that

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+6}n+3^{4\alpha+5}\right)q^n\equiv 32f_4+44\frac{f_1^4f_3^8}{f_6^4}+48q\frac{f_2f_3^3f_6^3}{f_1} \pmod{64}, \tag{4.19}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+6}n+3^{4\alpha+6}\right)q^n\equiv 36\frac{f_1^5f_3^5}{f_2f_6}+56f_1^4f_2^4 \pmod{64} \tag{4.20}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+6}n+5\cdot 3^{4\alpha+5}\right)q^n\equiv 8\frac{f_2^2f_3^2f_6^2}{f_1^2} \pmod{64}. \tag{4.21}$$

The equation (4.20) implies that the congruence (4.1) is true for $\alpha+1$. Hence, by induction, the congruence (4.1) holds for all integer $\alpha \geq 0$.

In view of (4.15) and (4.21), we obtain (4.3). □

Theorem 4.2. *For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have*

$$\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+5}n+3^{4\alpha+4}\right) \equiv \overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+3}n+3^{4\alpha+2}\right) \pmod{32}, \tag{4.22}$$

$$\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+6}n+3^{4\alpha+5}\right) \equiv \overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+4}n+3^{4\alpha+3}\right) \pmod{32}, \tag{4.23}$$

$$\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}n+7\cdot 3^{4\alpha+2}\right) \equiv 0 \pmod{32}, \tag{4.24}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+19\cdot 3^{4\alpha+2}\cdot 5^{2\beta}\right)q^n \equiv 16f_1f_6^3 \pmod{32}, \tag{4.25}$$

$$\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+i)+23\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{4.26}$$

$$\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}n+23\cdot 3^{4\alpha+2}\right) \equiv 0 \pmod{32}, \tag{4.27}$$

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+11\cdot 3^{4\alpha+2}\cdot 5^{2\beta}\right)q^n \equiv 16f_2f_3^3 \pmod{32}, \tag{4.28}$$

$$\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+j)+7\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{4.29}$$

where $i = 0, 1, 2, 4$ and $j = 0, 1, 3, 4$.

Proof. From the equations (4.10) and (4.16), we get (4.22).

In view of (4.13) and (4.19), we obtain (4.23).

The equation (4.10) reduces to

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+3}n+3^{4\alpha+2}\right)q^n \equiv 28\frac{f_2^4}{f_1^4}+16q\frac{f_2f_3^3f_6^3}{f_1} \pmod{32}. \tag{4.30}$$

Using (2.2) and (2.5) in (4.30), we get

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(4\cdot 3^{4\alpha+3}n+3^{4\alpha+2}\right)q^n \equiv 28\frac{f_2^2}{f_1^2}+16q\frac{f_3^3f_6^3}{f_1} \pmod{32} \tag{4.31}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(4\cdot 3^{4\alpha+3}n+7\cdot 3^{4\alpha+2}\right)q^n \equiv 16\frac{f_2^3f_3^3}{f_1}+16f_2^7 \pmod{32}. \tag{4.32}$$

Substituting (2.5) in (4.32), we obtain (4.24) and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}n+19\cdot 3^{4\alpha+2}\right)q^n \equiv 16f_1f_6^3 \pmod{32}, \tag{4.33}$$

which is $\beta=0$ case of (4.25). The rest of the proofs of the congruences (4.25) and (4.26) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details.

The equation (4.12) becomes

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(2\cdot 3^{4\alpha+3}n+5\cdot 3^{4\alpha+2}\right)q^n \equiv 8\frac{f_2^2f_3^2f_6^2}{f_1^2} \pmod{32}. \tag{4.34}$$

Utilizing (2.4) in (4.34), we have

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(4\cdot 3^{4\alpha+3}n+5\cdot 3^{4\alpha+2}\right)q^n \equiv 8\frac{f_3^3 f_4 f_6^2}{f_1^3 f_{12}} \pmod{32} \tag{4.35}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(4\cdot 3^{4\alpha+3}n+11\cdot 3^{4\alpha+2}\right)q^n \equiv 16 f_4 f_6^3 \pmod{32}. \tag{4.36}$$

The equation (4.36) implies (4.27) and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}n+11\cdot 3^{4\alpha+2}\right)q^n \equiv 16 f_2 f_3^3 \pmod{32}, \tag{4.37}$$

which is $\beta=0$ case of (4.28). The rest of the proofs of the congruences (4.28) and (4.29) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details. □

Theorem 4.3. *For all $n\geq 0$ and $\beta\geq 0$, we have*

$$\overline{bped}_{3,6}(72n+21) \equiv 0 \pmod{32}, \tag{4.38}$$

$$\overline{bped}_{3,6}(72n+69) \equiv 0 \pmod{32}, \tag{4.39}$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(72\cdot 5^{2\beta}n+57\cdot 5^{2\beta}\right)q^n \equiv 16 f_1 f_6^3 \pmod{32}, \tag{4.40}$$

$$\overline{bped}_{3,6}\left(72\cdot 5^{2\beta+1}(5n+i)+69\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{4.41}$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(72\cdot 5^{2\beta}n+33\cdot 5^{2\beta}\right)q^n \equiv 16 f_2 f_3^3 \pmod{32}, \tag{4.42}$$

$$\overline{bped}_{3,6}\left(72\cdot 5^{2\beta+1}(5n+j)+21\cdot 5^{2\beta+1}\right) \equiv 0 \pmod{32}, \tag{4.43}$$

where $i=0,1,2,4$ and $j=0,1,3,4$.

Proof. Utilizing (2.2) and (2.5) in (4.7), we get

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}(36n+3)q^n \equiv 12\frac{f_2^2}{f_1^2}+16q\frac{f_3^3 f_6^3}{f_1} \pmod{32} \tag{4.44}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}(36n+21)q^n \equiv 16 f_2^7+16\frac{f_2^3 f_3^3}{f_1} \pmod{32}. \tag{4.45}$$

Using (2.5) in (4.45), we obtain (4.38) and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}(72n+57)q^n \equiv 16 f_1 f_6^3 \pmod{32}, \tag{4.46}$$

which is $\beta=0$ case of (4.40). The rest of the proofs of the congruences (4.40) and (4.41) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details.

Employing (2.4) in (4.9), we find that

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}(36n+15)q^n \equiv 8\frac{f_3^3 f_4 f_6^2}{f_1^3 f_{12}} \pmod{32} \tag{4.47}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}(36n+33)\,q^n \equiv 16 f_4 f_6^3 \pmod{32}. \tag{4.48}$$

The equation (4.48) implies (4.39) and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}(72n+33)\,q^n \equiv 16 f_2 f_3^3 \pmod{32}, \tag{4.49}$$

which is $\beta = 0$ case of (4.42). The rest of the proofs of the congruences (4.42) and (4.43) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details. □

Theorem 4.4. *For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(8 \cdot 3^{4\alpha+3} \cdot 5^{2\beta} n + 13 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}\right) q^n \equiv 8 f_1^{13} \pmod{16}, \tag{4.50}$$

$$\overline{bped}_{3,6}\left(8 \cdot 3^{4\alpha+3} \cdot 5^{2\beta}(5n+i) + 17 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{16}, \tag{4.51}$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(8 \cdot 3^{4\alpha+3} \cdot 7^{2\gamma} n + 5 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma}\right) q^n \equiv 8 f_1^{5} \pmod{16}, \tag{4.52}$$

$$\overline{bped}_{3,6}\left(8 \cdot 3^{4\alpha+3} \cdot 7^{2\gamma+1}(7n+j) + 11 \cdot 3^{4\alpha+2} \cdot 7^{2\gamma+1}\right) \equiv 0 \pmod{16}, \tag{4.53}$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(72 \cdot 5^{2\beta} n + 39 \cdot 5^{2\beta}\right) q^n \equiv 8 f_1^{13} \pmod{16}, \tag{4.54}$$

$$\overline{bped}_{3,6}\left(72 \cdot 5^{2\beta+1}(5n+i) + 51 \cdot 5^{2\beta+1}\right) \equiv 0 \pmod{16}, \tag{4.55}$$

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(72 \cdot 7^{2\gamma} n + 15 \cdot 7^{2\gamma}\right) q^n \equiv 8 f_1^{5} \pmod{16}, \tag{4.56}$$

$$\overline{bped}_{3,6}\left(72 \cdot 7^{2\gamma+1}(7n+j) + 33 \cdot 7^{2\gamma+1}\right) \equiv 0 \pmod{16}, \tag{4.57}$$

where $i = 0, 1, 3, 4$ and $j = 0, 2, 3, 4, 5, 6$.

Proof. The equation (4.31) becomes

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(4 \cdot 3^{4\alpha+3} n + 3^{4\alpha+2}\right) q^n \equiv 12 \frac{f_2^2}{f_1^2} \pmod{16}. \tag{4.58}$$

Employing (2.1) in (4.58), we arrive at

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(8 \cdot 3^{4\alpha+3} n + 3^{4\alpha+2}\right) q^n \equiv 12 \frac{f_4}{f_1^3} \pmod{16} \tag{4.59}$$

and

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(8 \cdot 3^{4\alpha+3} n + 13 \cdot 3^{4\alpha+2}\right) q^n \equiv 8 f_1^{13} \pmod{16}. \tag{4.60}$$

The equation (4.60) is $\beta = 0$ case of (4.50). The rest of the proofs of the congruences (4.50) and (4.51) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details.

The eqution (4.35) reduces to

$$\sum_{n=0}^{\infty} \overline{bped}_{3,6}\left(4 \cdot 3^{4\alpha+3} n + 5 \cdot 3^{4\alpha+2}\right) q^n \equiv 8 \frac{f_2 f_3^3}{f_1} \pmod{16}. \tag{4.61}$$

Using (2.5) in (4.61) and then extracting the coefficients of q^{2n} from the resultant equation, we find that

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}n+5\cdot 3^{4\alpha+2}\right)q^n\equiv 8f_1^5 \pmod{16}, \tag{4.62}$$

which is $\gamma=0$ case of (4.52). The rest of the proofs of the congruences (4.52) and (4.53) are similar to the proofs of the congruences (3.38) and (3.39). So, we omit the details.

The equation (4.44) becomes

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}(36n+3)\,q^n\equiv 12\frac{f_2^2}{f_1^2} \pmod{16}. \tag{4.63}$$

Substituting (2.1) in (4.63), we have

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}(72n+3)\,q^n\equiv 12\frac{f_4}{f_1^3} \pmod{16} \tag{4.64}$$

and

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}(72n+39)\,q^n\equiv 8f_1^{13} \pmod{16}. \tag{4.65}$$

The equation (4.65) is $\beta=0$ case of (4.54). The rest of the proofs of the congruences (4.54) and (4.55) are similar to the proofs of the congruences (3.25) and (3.26). So, we omit the details.

The equation (4.47) becomes

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}(36n+15)\,q^n\equiv 8\frac{f_2f_3^3}{f_1} \pmod{16}. \tag{4.66}$$

Using (2.5) in (4.66), we get

$$\sum_{n=0}^{\infty}\overline{bped}_{3,6}(72n+15)\,q^n\equiv 8f_1^5 \pmod{16}, \tag{4.67}$$

which is $\gamma=0$ case of (4.56). The rest of the proofs of the congruences (4.56) and (4.57) are similar to the proofs of the congruences (3.38) and (3.39). So, we omit the details. □

Theorem 4.5. *For all $n\geq 0$ and $\alpha\geq 0$, we have*

$$\overline{bped}_{3,6}(72n+3)\equiv\begin{cases}4 \pmod 8 & \text{if } n \text{ is a pentagonal number,}\\ 0 \pmod 8 & \text{otherwise,}\end{cases} \tag{4.68}$$

$$\overline{bped}_{3,6}\left(8\cdot 3^{4\alpha+3}n+3^{4\alpha+2}\right)\equiv\begin{cases}4 \pmod 8 & \text{if } n \text{ is a pentagonal number,}\\ 0 \pmod 8 & \text{otherwise.}\end{cases} \tag{4.69}$$

Proof. From (4.64), we arrive at (4.68).

From (4.59), we obtain (4.69). □

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Author information

M. S. Mahadeva Naika, Department of Mathematics, Central College Campus, Bengaluru City University, Bengaluru-560 001, Karnataka, India.
E-mail: msmnaika@rediffmail.com

Harishkumar T, Department of Mathematics, Central College Campus, Bengaluru City University, Bengaluru-560 001, Karnataka, India.
E-mail: harishhaf@gmail.com

Y. Veeranna, Department of Mathematics, Government Science College, Bengaluru-560 001, Karnataka, India.
E-mail: veerannahariyabbe@gmail.com

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