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Quaternion Linear Canonical Curvelet Transform

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Abstract In this paper, we generalize linear canonical curvelet transform to quaternion - valued signals, known as quaternion linear canonical curvelet transform (QLCCT). Firstly, we investigate Parseval's formula, Inversion formula and characterization of range. Finally, we have formulated a couple of Heisenberg's Uncertainty principle's associated with quaternion linear canonical curvelet transform.

1 Introduction

In the early 1970's, a promising linear integral transform namely linear canonical transform was independently introduced by Collins [17] in paraxial optics, and Moshinsky and Quesne [29] in quantum mechanics to study the conservation of information and uncertainty under linear maps of phase space. The LCT can be regarded as generalization of many mathematical transforms such as the classical Fourier transform, Fresnel Transform and the fractional Fourier transform [31, 30]. It plays an important role in many fields like signal processing and optics and serves as a magnanimous analysing tool [7, 32, 29]. For more about LCT and its application we refer to [1, 8, 2, 22, 17, 29, 34, 20, 35, 36].

In the domain of higher dimensional signal processing the quality of wavelet transform tends to decrease because of the fact that the wavelet transform uses isotropic scaling in dimension $n \geq 2$. These isotropic scalings are rather weak and incompetent to capture the edges and corners in higher dimensional signals appearing due to its spatial occlusion between different objects, as for example, in medical imaging curves separate bones and various other soft tissues. Therefore, the key problem in multidimensional signal analysis is to extract and characterize the relevant and directional information regarding the occurrence of boundaries and curves in signals. To address these limitations of wavelet transform, some off-shoots of the wavelet transform, like ridgelet transform [11], curvelet transform [14, 15], contourlet transform [19] and shearlet transform [27], have been introduced.

Inspired by the results of two- dimensional wavelet transform and Stockwell transform, a relatively new two dimensional multi scale integral transform, which is named as the curvelet transform, has appeared in time frequency analysis was introduced by Candes and Donoho [14, 15]. This transform is a higher dimensional generalization of the wavelet transform designed to represent images at different scales and different angles. This transform is widely applied in image processing such as image denoising, imaging in astrophysics, morphological component analysis and seismic imaging. Like the wavelet transform and Stockwell transform , the translations, dilations and rotations are built into the genesis of the curvelet transform. The important difference of curvelet transform from the wavelet transform and Stockwell transform lies in the fact that non isotropic instead of isotropic dilations are used. For more about curvelets, we refer to [16, 28, 12, 13].

The article is organised as follows: In section 2 we present basic notions and preliminaries for quaternion linear canonical curvelet transform. Section 3 initiates the notion of quaternion linear canonical curvelet transform and deciphers some results comprising Parseval's formula, Inversion formula and characterization of range. Towards the culmination of the paper, we have formulated a couple of Heisenberg's uncertainty principle's for linear canonical curvelet transform in quaternion domain.

2 Basic notions and preliminaries

It was in 1843, the theory of quaternion algebra was initiated by the Irish mathematician W. R. Hamilton and is denoted by $\mathbb{H}$ in his honour. The quaternion algebra is an associative, non commutative four dimensional algebra that serves as an extension of the ordinary complex number system. The quaternion algebra $\mathbb{H}$ over $\mathbb{R}$ is given by

$$\mathbb{H} = \Big\{\mathbf{h} = a_0 + i\,a_1 + j\,a_2 + k\,a_3\ :\ a_0, a_1, a_2, a_3 \in \mathbb{R}\Big\},$$

where i, j, k are the three imaginary units, follow the Hamilton's multiplication rules

$$ij = -ji = k,\ jk = -kj = i\,,\ ki = -ik = j\,, \text{ and } i^2 = j^2 = k^2 = ijk = -1.$$

For a quaternion $\mathbf{h} = a_0 + i\,a_1 + j\,a_2 + k\,a_3 \in \mathbb{H}$, a_0 is called scalar part of $\mathbf{h}$ denoted by $Sc(\mathbf{h})$

and a pure quaternion $\mathbf{h}$ denoted by $Vec(\mathbf{h}) = i\,a_1 + j\,a_2 + k\,a_3$.
For quaternions

$$\mathbf{h}_1 = a_0 + i\,a_1 + j\,a_2 + k\,a_3$$

and

$$\mathbf{h}_2 = b_0 + i\,b_1 + j\,b_2 + k\,b_3,$$

the addition is defined component wise as

$$\mathbf{h}_1 + \mathbf{h}_2 = (a_0 + b_0) + (a_1 + b_1)\,i + (a_2 + b_2)\,j + (a_3 + b_3)\,k$$

Also the multiplication is defined as

$$\begin{aligned}\mathbf{h}_1\mathbf{h}_2 &= (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + i\,(a_1b_0 + a_0b_1 + a_2b_3 - a_3b_2)\\ &\quad + j(a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3) + k\,(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1).\end{aligned}$$

For a quaternion $\mathbf{h} = a_0 + i\,a_1 + j\,a_2 + k\,a_3$, the conjugate and norm are respectively given by $\overline{\mathbf{h}} = a_0 - i\,a_1 - j\,a_2 - k\,a_3$ and $\|\mathbf{h}\|_{\mathbb{H}} = \mathbf{h}\overline{\mathbf{h}} = \sqrt{a_0{}^2 + a_1{}^2 + a_2{}^2 + a_3{}^2}$. Also the arbitrary quaternion $\mathbf{h}$ can be represented by two complex numbers as $\mathbf{h} = (a_0 + i\,a_1 + j\,(a_2 - i\,a_3) = f_1 + j\,f_2$, where $f_1, f_2 \in \mathbb{C}$ and hence $\overline{\mathbf{h}} = \overline{f_1} - j\,f_2$, where $\overline{f_1}$ denoting the complex conjugate of f_1. The inner product of any two quaternions $f = f_1 + j\,f_2$ and $g = g_1 + j\,g_2$ in $\mathbb{H}$ is defined by

$$\langle f, g\rangle_{\mathbb{H}} = \big(f_1\overline{g_1} + \overline{f_2}g_2\big) + j\,\big(f_2\overline{g_1} - \overline{f_1}g_2\big).$$

By virtue of Cayley's-Dickson representation a quaternion valued function $f : \mathbb{R}^2 \to \mathbb{H}$ can be decomposed as $f(x) = f_1 + j\,f_2$, where f_1, f_2 are both complex valued functions. The quaternion Fourier transform is defined in a similar way as the classical Fourier transform of the two dimensional functions. The non commutativity property of the quaternion multiplication allows us to have three different definitions of quaternion Fourier transform. Here we only introduce two sided QFT. For more details, we refer to [6, 18, 23].

Definition 2.1. [24] Let $f \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$ be any quaternion valued function, the two sided quaternion Fourier transform (QFT) is denoted by $\mathcal{F}_q$ and is given by

$$\mathcal{F}_q\big[f(\mathbf{x})\big](\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-ix_1\omega_1} f(\mathbf{x})\, e^{-jx_2\omega_2}\, d\mathbf{x}. \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2)$, $\boldsymbol{\omega} = (\omega_1, \omega_2)$, $d\mathbf{x} = dx_1dx_2$ and the quaternion exponential $e^{-ix_1\omega_1}$ and $e^{-jx_2\omega_2}$ are the quaternion Fourier kernels. The correlated inversion formula is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{ix_1\omega_1} f(\boldsymbol{\omega})\, e^{jx_2\omega_2}\, d\boldsymbol{\omega}. \tag{2.2}$$

Now we recall the definition of quaternion linear canonical transform (QLCT). Due to the non-commutativity property of quaternion multiplication, quaternion linear canonical transform are classified into three different types as; left-sided QLCT, right-sided QLCT and two sided QLCT. In this article we will use two sided QLCT [4, 26] .

Definition 2.2. Let $M_s = \begin{bmatrix} a_s & b_s \\ c_s & d_s \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a real matrix parameter such that det $(M_s) = 1$, for $s = 1, 2$. The two sided QLCT of $f \in \mathbb{L}^2(\mathbb{R}^2, \mathbb{H})$ is defined by

$$\mathcal{L}^{\mathbb{H}}_{M_1,M_2}\big[f(\mathbf{x})\big](\boldsymbol{\omega}) = \begin{cases} \int_{\mathbb{R}^2} K^i_{M_1}(x_1,\omega_1)\, f(\mathbf{x})\, K^j_{M_2}(x_2,\omega_2) d\mathbf{x}, & b_1 b_2 \neq 0 \\ \\ \sqrt{d_1 d_2} e^{\frac{i c_1 d_1}{2}\omega_1^2} f(d_1\omega_1, d_2\omega_2) e^{\frac{j c_2 d_2}{2}\omega_2^2}, & b_1 b_2 = 0 \end{cases}$$

where $\mathbf{x} = (x_1, x_2)$, $\boldsymbol{\omega} = (\omega_1, \omega_2)$. The quaternion kernels $K^i_{M_1}(x_1, \omega_1), K^j_{M_2}(x_2, \omega_2)$ are respectively given by

$$K^i_{M_1}(x_1,\omega_1) = \frac{1}{\sqrt{2\pi b_1}} \exp\left\{\frac{i}{2b_1}\left[a_1 x_1^2 - 2x_1\omega_1 + d_1\omega_1^2\right]\right\}$$

and

$$K^j_{M_2}(x_2,\omega_2) = \frac{1}{\sqrt{2\pi b_2}} \exp\left\{\frac{j}{2b_2}\left[a_2 x_2^2 - 2x_2\omega_2 + d_2\omega_2^2\right]\right\}.$$

For the case $b_1 b_2 = 0$, the QLCT of a signal is essentially a quaternion chirp multiplication. So in this paper, we only consider the case $b_1 b_2 \neq 0$.
The correlated inversion formula for the two sided QLCT is given by

$$f(\mathbf{x}) = \int_{\mathbb{R}^2} \overline{K^i_{M_1}(x_1,\omega_1)}\, \mathcal{L}^{\mathbb{H}}_{M_1,M_2}\big[f\big](\boldsymbol{\omega})\, \overline{K^j_{M_2}(x_2,\omega_2)}\, d\boldsymbol{\omega}. \tag{2.3}$$

Also the Parseval's formula for the QLCT is given by [4]

$$\left\langle \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f],\, \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g] \right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \langle f, g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} \tag{2.4}$$

We now recall the definition of linear canonical curvelet transform [25].

Definition 2.3. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix with parameters satisfying $det(M) = ad - bc = 1$. Then linear canonical curvelet transform of a signal $f \in L^2(\mathbb{R}^2)$, is defined as the integral transform

$$(\Gamma^M f)(\alpha, \boldsymbol{\beta}, \theta) = \int_{\mathbb{R}^2} f(\mathbf{x}) \overline{\Gamma^M_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})} d\mathbf{x}, \tag{2.5}$$

where $\mathbf{x} = (x_1, x_2), \alpha \in (0, a_0), \boldsymbol{\beta} \in \mathbb{R}^2, \theta \in (-\pi, \pi)$ and $\Gamma^M_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})$ is given by

$$\Gamma^M_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x}) = \Gamma_{\alpha,0,0}(R_\theta(\mathbf{x} - \boldsymbol{\beta})) \exp\left\{\frac{ia}{b}\mathbf{x}^T(\boldsymbol{\beta} - \mathbf{x})\right\}.$$

3 Quaternion Linear Canonical Curvelet Transform (QLCCT)

In this section, we generalize linear canonical curvelet transform [25], to quaternion- valued signals , known as quaternion linear canonical curvelet transform.

Definition 3.1. Let $M_1 = (a_1, b_1, c_1, d_1)$ and $M_2 = (a_2, b_2, c_2, d_2)$ be two parameters satisfying $det(M_s) = a_s d_s - b_s c_s = 1, s = 1, 2$ then the quaternion linear canonical curvelet transform of a signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined by

$$(\Gamma_{\mathbb{H}}^{M_s} f)(\alpha, \boldsymbol{\beta}, \theta) = \int_{\mathbb{R}^2} f(\mathbf{x}) \overline{\psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x})} d\mathbf{x}. \tag{3.1}$$

where

$$\psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) = \exp\left\{\frac{ia_1}{b_1} x_1(\beta_1 - x_1)\right\} \psi_{\alpha,0,0}(R_\theta(\mathbf{x} - \boldsymbol{\beta})) \exp\left\{\frac{ja_2}{b_2} x_2(\beta_2 - x_2)\right\}. \tag{3.2}$$

Definition 3.2. (Admissibility Condition). A function $\psi \in L^2\left(\mathbb{R}^2, \mathbb{H}\right)$ is said to be admissible if

$$C_\psi = \int_0^{a_o} \int_{-\pi}^{\pi} \left| \mathcal{L}_{M_1,M_2}^{\mathbb{H}} \left[\psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x})\right](\boldsymbol{\omega}) \right|^2 d\theta d\alpha. \tag{3.3}$$

Lemma 3.3. *The two sided quaternion LCT of $\psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x})$, defined in (3.2) is given by*

$$\begin{aligned}
\mathcal{L}_{M_1,M_2}^{\mathbb{H}}[\psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x})](\boldsymbol{\omega}) &= \frac{1}{2\pi\sqrt{b_1 b_2}} \exp\left\{\frac{i}{2b_1}[d_1\omega_1^2 - 2\beta_1(\omega_1 - a_1\beta_1) - a_1\beta_1^2]\right\} \\
&\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\} \psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega} - \mathbf{a}\boldsymbol{\beta}}{\mathbf{b}}\right)\right) \exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right] \\
&\times \exp\left\{\frac{j}{2b_2}[d_2\omega_2^2 - 2\beta_2(\omega_2 - a_2\beta_2) - a_2\beta_2^2]\right\}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\mathcal{L}_{M_1,M_2}^{\mathbb{H}}[\psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x})](\boldsymbol{\omega}) \\
&= \int_{\mathbb{R}^2} K_{M_1}^i(x_1, \omega_1) \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) K_{M_2}^j(x_2, \omega_2) d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} \exp\left\{\frac{i}{2b_1}\left[a_1 x_1^2 - 2x_1\omega_1 + d_1\omega_1^2\right]\right\} \\
&\qquad \times \exp\left\{\frac{ia_1}{b_1} x_1(\beta_1 - x_1)\right\} \psi_{\alpha,0,0}\big(R_\theta(\mathbf{x} - \boldsymbol{\beta})\big) \exp\left\{\frac{ja_2}{b_2} x_2(\beta_2 - x_2)\right\} \\
&\qquad \times \frac{1}{\sqrt{2\pi b_2}} \exp\left\{\frac{j}{2b_2}\left[a_2 x_2^2 - 2x_2\omega_2 + d_2\omega_2^2\right]\right\} dx_1 dx_2 \\
&= \frac{1}{2\pi\sqrt{b_1 b_2}} \int_{\mathbb{R}^2} \exp\left\{\frac{i}{2b_1}\left[a_1 x_1^2 - 2x_1\omega_1 + d_1\omega_1^2\right]\right\} \\
&\qquad \times \exp\left\{\frac{ia_1}{b_1} x_1(\beta_1 - x_1)\right\} \psi_{\alpha,0,0}\big(R_\theta(\mathbf{x} - \boldsymbol{\beta})\big) \exp\left\{\frac{ja_2}{b_2} x_2(\beta_2 - x_2)\right\} \\
&\qquad \times \exp\left\{\frac{j}{2b_2}\left[a_2 x_2^2 - 2x_2\omega_2 + d_2\omega_2^2\right]\right\} dx_1 dx_2 \\
&= \frac{1}{2\pi\sqrt{b_1 b_2}} \int_{\mathbb{R}^2} \exp\left\{\frac{i}{2b_1}\left[-(2x_1\omega_1 - 2a_1 x_1\beta_1) + d_1\omega_1^2\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\times \exp\left\{\frac{-ia_1}{2b_1}x_1^2\right\}\psi_{\alpha,0,0}\big(R_\theta(\mathbf{x}-\boldsymbol{\beta})\big)\exp\left\{\frac{-ja_2}{2b_2}x_2^2\right\}\\
&\times \exp\left\{\frac{j}{2b_2}\big[-(2x_2\omega_2-2a_2x_2\beta_2)+d_2\omega_2^2\big]\right\}dx_1dx_2\\
=&\ \frac{1}{2\pi\sqrt{b_1b_2}}\exp\left\{\frac{i}{2b_1}(d_1\omega_1^2)\right\}\int_{\mathbb{R}^2}\exp\left\{\frac{i}{b_1}\big[-x_1(\omega_1-a_1\beta_1)\big]\right\}\\
&\times \exp\left\{\frac{-ia_1}{2b_1}x_1^2\right\}\psi_{\alpha,0,0}\big(R_\theta(\mathbf{x}-\boldsymbol{\beta})\big)\exp\left\{\frac{-ja_2}{2b_2}x_2^2\right\}\\
&\times \exp\left\{\frac{j}{b_2}\big[-x_2(\omega_2-a_2\beta_2)\big]\right\}dx_1dx_2\exp\left\{\frac{j}{2b_2}(d_2\omega_2^2)\right\}\\
=&\ \frac{1}{2\pi\sqrt{b_1b_2}}\exp\left\{\frac{i}{2b_1}(d_1\omega_1^2)\right\}\int_{\mathbb{R}^2}\exp\left\{-ix_1\left(\frac{\omega_1-a_1\beta_1}{b_1}\right)\right\}\\
&\times \exp\left\{\frac{-ia_1}{2b_1}x_1^2\right\}\psi_{\alpha,0,0}\big(R_\theta(\mathbf{x}-\boldsymbol{\beta})\big)\exp\left\{\frac{-ja_2}{2b_2}x_2^2\right\}\\
&\times \exp\left\{-jx_2\left(\frac{\omega_2-a_2\beta_2}{b_2}\right)\right\}dx_1dx_2\exp\left\{\frac{j}{2b_2}(d_2\omega_2^2)\right\}\\
=&\ \frac{1}{2\pi\sqrt{b_1b_2}}\exp\left\{\frac{i}{2b_1}(d_1\omega_1^2)\right\}\int_{\mathbb{R}^2}\exp\left\{-i(y_1+\beta_1)\left(\frac{\omega_1-a_1\beta_1}{b_1}\right)\right\}\\
&\times \exp\left\{\frac{-ia_1}{2b_1}(y_1+\beta_1)^2\right\}\psi_{\alpha,0,0}\big(R_\theta\mathbf{y}\big)\exp\left\{\frac{-ja_2}{2b_2}(y_2+\beta_2)^2\right\}\\
&\times \exp\left\{-j(y_2+\beta_2)\left(\frac{\omega_2-a_2\beta_2}{b_2}\right)\right\}dy_1dy_2\exp\left\{\frac{j}{2b_2}(d_2\omega_2^2)\right\}\\
=&\ \frac{1}{2\pi\sqrt{b_1b_2}}\exp\left\{\frac{i}{2b_1}d_1\omega_1^2-\frac{i}{b_1}\beta_1(\omega_1-a_1\beta_1)\right\}\\
&\times\int_{\mathbb{R}^2}\exp\left\{-iy_1\left(\frac{\omega_1-a_1\beta_1}{b_1}\right)\right\}\exp\left\{\frac{-ia_1}{2b_1}(y_1+\beta_1)^2\right\}\psi_{\alpha,0,0}\big(R_\theta\mathbf{y}\big)\\
&\times \exp\left\{\frac{-ja_2}{2b_2}(y_2+\beta_2)^2\right\}\exp\left\{-jy_2\left(\frac{\omega_2-a_2\beta_2}{b_2}\right)\right\}\\
&\times dy_1dy_2\exp\left\{\frac{j}{2b_2}d_2\omega_2^2-\frac{j}{b_2}\beta_2(\omega_2-a_2\beta_2)\right\}\\
=&\ \frac{1}{2\pi\sqrt{b_1b_2}}\exp\left\{\frac{i}{2b_1}[d_1\omega_1^2-2\beta_1(\omega_1-a_1\beta_1)-a_1\beta_1^2]\right\}\\
&\times\mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}-\mathbf{a}\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\right.\\
&\left.\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]\\
&\times \exp\left\{\frac{j}{2b_2}[d_2\omega_2^2-2\beta_2(\omega_2-a_2\beta_2)-a_2\beta_2^2]\right\}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.4. *(* **Parseval's Formula)** *Suppose that $\psi\in L^2(\mathbb{R}^2,\mathbb{H})$ be admissible , then for every $f,g\in L^2(\mathbb{R}^2,\mathbb{H})$, we have*

$$
\left\langle(\Gamma_{\mathbb{H}}^{M_s}f),(\Gamma_{\mathbb{H}}^{M_s}g)\right\rangle_{L^2(\mathcal{G},\mathbb{H})}=b_1b_2\,C_\psi\langle f,\,g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{3.4}
$$

Where $\mathcal{G} = (0, a_0) \times \mathbb{R}^2 \times (-\pi, \pi)$ *and* C_ψ *is admissibility given by (3.3).*

Proof. By Parseval's formula for the two sided quaternion linear canonical transform and using lemma (3.3), we have

$$
\begin{aligned}
&\left\langle (\Gamma_{\mathbb{H}}^{M_s} f),\ (\Gamma_{\mathbb{H}}^{M_s} g)\right\rangle_{L^2(\mathcal{G},\mathbb{H})}\\
&= \int\limits_{\mathcal{G}} [\Gamma_{\mathbb{H}}^{M_s} f](\alpha,\boldsymbol{\beta},\theta)\overline{[\Gamma_{\mathbb{H}}^{M_s} g](\alpha,\boldsymbol{\beta},\theta)}\,d\alpha d\boldsymbol{\beta} d\theta\\
&= \int\limits_{\mathcal{G}} \left\langle f, \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}\right\rangle_{L^2(\mathbb{R}^2,\,\mathbb{H})} \overline{\left\langle g, \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}\right\rangle_{L^2(\mathbb{R}^2,\,\mathbb{H})}}\,d\alpha d\boldsymbol{\beta} d\theta\\
&= \int\limits_{\mathcal{G}} \left\langle \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f], \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})]\right\rangle_{L^2(\mathbb{R}^2,\,\mathbb{H})}\\
&\qquad\qquad \times \overline{\left\langle \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g], \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})]\right\rangle_{L^2(\mathbb{R}^2,\,\mathbb{H})}}\,d\alpha d\boldsymbol{\beta} d\theta\\
&= \int\limits_{\mathcal{G}} \left\{ \int\limits_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})]}\,d\boldsymbol{\omega} \overline{\int\limits_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]\overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})]}\,d\boldsymbol{\omega}'} \right\} d\alpha d\boldsymbol{\beta} d\theta\\
&= \int\limits_{\mathcal{G}} \left\{ \int\limits_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})]}\,d\boldsymbol{\omega} \int\limits_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})]\overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}\,d\boldsymbol{\omega}' \right\} d\alpha d\boldsymbol{\beta} d\theta\\
&= \frac{1}{4\pi^2 b_1 b_2} \int\limits_{\mathcal{G}} \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\overline{\exp\left\{\frac{i}{2b_1}[d_1\omega_1^2 - 2\beta_1(\omega_1 - a_1\beta_1) - a_1\beta_1^2]\right\}}\\
&\quad \times \overline{\mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega} - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]}\\
&\quad \times \overline{\exp\left\{\frac{j}{2b_2}[d_2\omega_2^2 - 2\beta_2(\omega_2 - a_2\beta_2) - a_2\beta_2^2]\right\}}\\
&\quad \times \exp\left\{\frac{i}{2b_1}[d_1\omega_1'^2 - 2\beta_1(\omega_1' - a_1\beta_1) - a_1\beta_1^2]\right\}\\
&\quad \times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}' - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]\\
&\quad \times \exp\left\{\frac{j}{2b_2}[d_2\omega_2'^2 - 2\beta_2(\omega_2' - a_2\beta_2) - a_2\beta_2^2]\right\}\\
&\qquad\qquad \times \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}\,d\boldsymbol{\omega} d\boldsymbol{\omega}' d\alpha d\boldsymbol{\beta} d\theta\\
&= \frac{1}{4\pi^2 b_1 b_2} \int\limits_{\mathcal{G}} \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\exp\left\{\frac{-j}{2b_2}[d_2\omega_2^2 - 2\beta_2(\omega_2 - a_2\beta_2) - a_2\beta_2^2]\right\}\\
&\quad \times \overline{\mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega} - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]}\\
&\quad \times \exp\left\{\frac{-i}{2b_1}[d_1\omega_1^2 - 2\beta_1(\omega_1 - a_1\beta_1) - a_1\beta_1^2 - d_1\omega_1'^2 + 2\beta_1(\omega_1' - a_1\beta_1) + a_1\beta_1^2]\right\}\\
&\quad \times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}' - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]
\end{aligned}
$$

$$\times \exp\left\{\frac{j}{2b_2}[d_2\omega_2'^2 - 2\beta_2(\omega_2' - a_2\beta_2) - a_2\beta_2^2]\right\} \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]} d\boldsymbol{\omega} d\boldsymbol{\omega}' d\alpha d\boldsymbol{\beta} d\theta$$

$$= \frac{1}{4\pi^2 b_1 b_2} \int_{\mathcal{G}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f] \exp\left\{\frac{-j}{2b_2}[d_2\omega_2^2 - 2\beta_2(\omega_2 - a_2\beta_2) - a_2\beta_2^2]\right\}$$

$$\overline{\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\} \psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega} - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right) \exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]}$$

$$\times \exp\left\{\frac{-i}{2b_1}[d_1(\omega_1^2 - \omega_1'^2) - 2\beta_1(\omega_1 - \omega_1']\right\}$$

$$\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\} \psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}' - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right) \exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]$$

$$\times \exp\left\{\frac{j}{2b_2}[d_2\omega_2'^2 - 2\beta_2(\omega_2' - a_2\beta_2) - a_2\beta_2^2]\right\} \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]} d\boldsymbol{\omega} d\boldsymbol{\omega}' d\alpha d\boldsymbol{\beta} d\theta$$

$$= \frac{1}{4\pi^2 b_1 b_2} \int_0^{a_o} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f] \exp\left\{\frac{-j}{2b_2}[d_2\omega_2^2 - 2\beta_2(\omega_2 - a_2\beta_2) - a_2\beta_2^2]\right\}$$

$$\overline{\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\} \psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega} - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right) \exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]}$$

$$\times \exp\left\{\frac{-id_1}{2b_1}(\omega_1^2 - \omega_1'^2)\right\} \int_{\mathbb{R}} \exp\left\{\frac{i\beta_1}{b_1}(\omega_1 - \omega_1')\right\} d\beta_1$$

$$\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\} \psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}' - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right) \exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]$$

$$\times \exp\left\{\frac{j}{2b_2}[d_2\omega_2'^2 - 2\beta_2(\omega_2' - a_2\beta_2) - a_2\beta_2^2]\right\} \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]} d\boldsymbol{\omega} d\boldsymbol{\omega}' d\beta_2 d\theta d\alpha$$

$$= \frac{1}{4\pi^2 b_1 b_2} \int_0^{a_o} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f] \exp\left\{\frac{-j}{2b_2}[d_2\omega_2^2 - 2\beta_2(\omega_2 - a_2\beta_2) - a_2\beta_2^2]\right\}$$

$$\overline{\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\} \psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega} - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right) \exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]}$$

$$\times \exp\left\{\frac{-id_1}{2b_1}(\omega_1^2 - \omega_1'^2)\right\} b_1 \delta(\omega_1 - \omega_1')$$

$$\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\} \psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}' - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right) \exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]$$

$$\times \exp\left\{\frac{j}{2b_2}[d_2\omega_2'^2 - 2\beta_2(\omega_2' - a_2\beta_2) - a_2\beta_2^2]\right\} \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]} d\boldsymbol{\omega} d\boldsymbol{\omega}' d\beta_2 d\theta d\alpha$$

$$= \frac{1}{4\pi^2 b_2} \int_0^{a_o} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f] \exp\left\{\frac{-j}{2b_2}[d_2\omega_2^2 - 2\beta_2(\omega_2 - a_2\beta_2) - a_2\beta_2^2]\right\}$$

$$\overline{\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2 + 2y_1\beta_1)\right\} \psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega} - a\boldsymbol{\beta}}{\mathbf{b}}\right)\right) \exp\left\{\frac{-ja_2}{b_2}(y_2^2 + 2y_2\beta_2)\right\}\right]}$$

$$\times \int_{\mathbb{R}} \exp\left\{\frac{-id_1}{2b_1}(\omega_1^2 - \omega_1'^2)\right\} \delta(\omega_1 - \omega_1') d\omega_1'$$

$$\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}'-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]$$

$$\times \exp\left\{\frac{j}{2b_2}[d_2\omega_2'^2-2\beta_2(\omega_2'-a_2\beta_2)-a_2\beta_2^2]\right\}\overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}\,d\boldsymbol{\omega}d\omega_2'd\beta_2d\theta d\alpha$$

$$=\frac{1}{4\pi^2b_2}\int_0^{a_o}\int_{-\pi}^{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\exp\left\{\frac{-j}{2b_2}[d_2\omega_2^2-2\beta_2(\omega_2-a_2\beta_2)-a_2\beta_2^2]\right\}$$

$$\overline{\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]}$$

$$\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}'-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]$$

$$\times \exp\left\{\frac{j}{2b_2}[d_2\omega_2'^2-2\beta_2(\omega_2'-a_2\beta_2)-a_2\beta_2^2]\right\}\overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}\,d\boldsymbol{\omega}d\omega_2'd\beta_2d\theta d\alpha$$

$$=\frac{1}{4\pi^2b_2}\int_0^{a_o}\int_{-\pi}^{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]$$

$$\times \exp\left\{\frac{-j}{2b_2}[d_2\omega_2^2-2\beta_2(\omega_2-a_2\beta_2)-a_2\beta_2^2-d_2\omega_2'^2+2\beta_2(\omega_2'-a_2\beta_2)+a_2\beta_2^2]\right\}$$

$$\overline{\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]}$$

$$\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}'-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]$$

$$\times \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}\,d\boldsymbol{\omega}d\omega_2'd\beta_2d\theta d\alpha$$

$$=\frac{1}{4\pi^2b_2}\int_0^{a_o}\int_{-\pi}^{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\exp\left\{\frac{-j}{2b_2}\left[d_2(\omega_2^2-\omega_2'^2)-2\beta_2(\omega_2-\omega_2')\right]\right\}$$

$$\overline{\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]}$$

$$\times \mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}'-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]$$

$$\times \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}\,d\boldsymbol{\omega}d\omega_2'd\beta_2d\theta d\alpha$$

$$=\frac{1}{4\pi^2b_2}\int_0^{a_o}\int_{-\pi}^{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\exp\left\{\frac{-jd_2}{2b_2}(\omega_2^2-\omega_2'^2)\right\}\int_{\mathbb{R}}\exp\left\{\frac{j\beta_2}{b_2}(\omega_2-\omega_2')\right\}d\beta_2$$

$$\times\left|\mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]\right|^2$$

$$\times \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}\,d\boldsymbol{\omega}d\omega_2'd\theta d\alpha$$

$$=\frac{1}{4\pi^2b_2}\int_0^{a_o}\int_{-\pi}^{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\exp\left\{\frac{-jd_2}{2b_2}(\omega_2^2-\omega_2'^2)\right\}b_2\delta(\omega_2-\omega_2')$$

$$\times \left|\mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]\right|^2$$

$$\times \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}d\boldsymbol{\omega}d\omega_2'd\theta d\alpha$$

$$= \frac{1}{4\pi^2}\int_0^{a_o}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\int_{\mathbb{R}}\exp\left\{\frac{-jd_2}{2b_2}(\omega_2^2-\omega_2'^2)\right\}\delta(\omega_2-\omega_2')d\omega_2'$$

$$\times \left|\mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]\right|^2$$

$$\times \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}d\boldsymbol{\omega}d\theta d\alpha$$

$$= \frac{1}{4\pi^2}\int_0^{a_o}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]$$

$$\times \left|\mathcal{F}_q\left[\exp\left\{\frac{-ia_1}{b_1}(y_1^2+2y_1\beta_1)\right\}\psi_{\alpha,0,0}\left(R_\theta\left(\frac{\boldsymbol{\omega}-a\boldsymbol{\beta}}{\mathbf{b}}\right)\right)\exp\left\{\frac{-ja_2}{b_2}(y_2^2+2y_2\beta_2)\right\}\right]\right|^2$$

$$\times \overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}d\boldsymbol{\omega}d\theta d\alpha$$

$$= \frac{1}{4\pi^2}(2\pi\sqrt{b_1b_2})^2\int_{\mathbb{R}^2}\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f]\int_0^{a_o}\int_{-\pi}^{\pi}\left|\mathcal{L}^{\mathbb{H}}_{M_1,M_2}\left[\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})\right](\boldsymbol{\omega})\right|^2 d\theta d\alpha\overline{\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]}d\boldsymbol{\omega}$$

$$= b_1b_2C_\psi\left\langle\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f],\mathcal{L}^{\mathbb{H}}_{M_1,M_2}[g]\right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}$$

$$= b_1b_2C_\psi\langle f,g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}.$$

This completes the proof. □

Corollary 3.5. *For $f=g$, we have the following identity:*

$$\int_{\mathcal{G}}\left|\left[\Gamma^{M_s}_{\mathbb{H}}f\right](\alpha,\boldsymbol{\beta},\theta)\right|^2 d\alpha d\boldsymbol{\beta}d\theta = b_1b_2C_\psi||f||^2_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{3.5}$$

Now we prove the inversion formula for the quaternion linear canonical curvelet transform using parseval's formula.

Theorem 3.6. ***(*Inversion Formula*)*.** *Suppose that $\psi\in L^2(\mathbb{R}^2,\mathbb{H})$, then any quaternion signal $f\in L^2(\mathbb{R}^2,\mathbb{H})$ can be reconstructed from the quaternion linear canonical transform $\left[\Gamma^{M_s}_{\mathbb{H}}f\right](\alpha,\boldsymbol{\beta},\theta)$ by the following formula:*

$$f(\mathbf{x}) = \frac{1}{b_1b_2C_\psi}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2}\int_0^{a_o}\left[\Gamma^{M_s}_{\mathbb{H}}f\right](\alpha,\boldsymbol{\beta},\theta)\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})d\alpha d\boldsymbol{\beta}d\theta. \tag{3.6}$$

Proof. By parseval's formula for QLCCT, for any arbitrary $g\in L^2(\mathbb{R}^2,\mathbb{H})$, we have

$$b_1b_2C_\psi\langle f,g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int_{\mathcal{G}}[\Gamma^{M_s}_{\mathbb{H}}f](\alpha,\boldsymbol{\beta},\theta)\overline{[\Gamma^{M_s}_{\mathbb{H}}g](\alpha,\boldsymbol{\beta},\theta)}d\alpha d\boldsymbol{\beta}d\theta$$

$$= \int_{\mathcal{G}}[\Gamma^{M_s}_{\mathbb{H}}f](\alpha,\boldsymbol{\beta},\theta)\overline{\left\langle g,\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x})\right\rangle_{L^2(\mathbb{R}^2,\,\mathbb{H})}}d\alpha d\boldsymbol{\beta}d\theta$$

$$= \int_{\mathcal{G}}[\Gamma^{M_s}_{\mathbb{H}}f](\alpha,\boldsymbol{\beta},\theta)\left\langle\psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}(\mathbf{x}),g\right\rangle_{L^2(\mathbb{R}^2,\,\mathbb{H})}d\alpha d\boldsymbol{\beta}d\theta$$

$$
\begin{aligned}
&= \int\limits_{\mathcal{G}} [\Gamma_{\mathbb{H}}^{M_s} f](\alpha, \boldsymbol{\beta}, \theta) \int\limits_{\mathbb{R}^2} \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) \overline{g(\mathbf{x})} d\mathbf{x} d\alpha d\boldsymbol{\beta} d\theta \\
&= \int\limits_{\mathbb{R}^2} \int\limits_{\mathcal{G}} [\Gamma_{\mathbb{H}}^{M_s} f](\alpha, \boldsymbol{\beta}, \theta) \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) d\alpha d\boldsymbol{\beta} d\theta \overline{g(\mathbf{x})} d\mathbf{x} \\
&= \left\langle \int\limits_{\mathcal{G}} [\Gamma_{\mathbb{H}}^{M_s} f](\alpha, \boldsymbol{\beta}, \theta) \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) d\alpha d\boldsymbol{\beta} d\theta, g(\mathbf{x}) \right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}.
\end{aligned}
$$

where we use Fubini's theorem to obtain second last inequality. Therefore, we have

$$
b_1 b_2 C_\psi f(\mathbf{x}) = \int\limits_{\mathcal{G}} [\Gamma_{\mathbb{H}}^{M_s} f](\alpha, \boldsymbol{\beta}, \theta) \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) d\alpha d\boldsymbol{\beta} d\theta
$$

$$
\text{or} \quad f(\mathbf{x}) = \frac{1}{b_1 b_2 C_\psi} \int\limits_{-\pi}^{\pi} \int\limits_{\mathbb{R}^2} \int\limits_{0}^{a_o} \left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha, \boldsymbol{\beta}, \theta) \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) d\alpha d\boldsymbol{\beta} d\theta.
$$

This completes the proof. □

In the next theorem we shall present a complete **"characterization of range"**of the proposed quaternion linear canonical curvelet transform.

Theorem 3.7. (Characterization of Range) *If* $f \in L^2\left(\mathbb{R}^2 \times (0, a_0) \times [-\pi, \pi]\right)$, *then f is the QLCCT of a certain square integrable function if and only if*

$$
f(\alpha', \boldsymbol{\beta}', \theta') = \frac{1}{b_1 b_2 \, C_\psi} \int\limits_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \left\langle \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}, \psi_{\alpha',\boldsymbol{\beta}',\theta'}^{M_s} \right\rangle d\alpha d\boldsymbol{\beta} d\theta. \tag{3.7}
$$

Proof. Let f belongs to the range of proposed transform $\Gamma_{\mathbb{H}}^{M_s}$. Then there exists a square integrable function g such that $\left[\Gamma_{\mathbb{H}}^{M_s} g\right] = f$. In order to show that f satisfies (3.7), we go forward by using inversion formula as

$$
\begin{aligned}
f(\alpha', \boldsymbol{\beta}', \theta') &= \left[\Gamma_{\mathbb{H}}^{M_s} g\right](\alpha', \boldsymbol{\beta}', \theta') \\
&= \int\limits_{\mathbb{R}^2} g(\mathbf{x}) \overline{\psi_{\alpha',\boldsymbol{\beta}',\theta'}^{M_s}(\mathbf{x})} d\mathbf{x} \\
&= \int\limits_{\mathbb{R}^2} \frac{1}{b_1 b_2 C_\psi} \int\limits_{\mathcal{G}} \left[\Gamma_{\mathbb{H}}^{M_s} g\right](\alpha, \boldsymbol{\beta}, \theta) \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) d\alpha d\boldsymbol{\beta} d\theta \overline{\psi_{\alpha',\boldsymbol{\beta}',\theta'}^{M_s}(\mathbf{x})} d\mathbf{x} \\
&= \frac{1}{b_1 b_2 C_\psi} \int\limits_{\mathcal{G}} \int\limits_{\mathbb{R}^2} \left[\Gamma_{\mathbb{H}}^{M_s} g\right](\alpha, \boldsymbol{\beta}, \theta) \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) \overline{\psi_{\alpha',\boldsymbol{\beta}',\theta'}^{M_s}(\mathbf{x})} d\mathbf{x} d\alpha d\boldsymbol{\beta} d\theta \\
&= \frac{1}{b_1 b_2 C_\psi} \int\limits_{\mathcal{G}} \left[\Gamma_{\mathbb{H}}^{M_s} g\right](\alpha, \boldsymbol{\beta}, \theta) \left\{ \int\limits_{\mathbb{R}^2} \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}(\mathbf{x}) \overline{\psi_{\alpha',\boldsymbol{\beta}',\theta'}^{M_s}(\mathbf{x})} d\mathbf{x} \right\} d\alpha d\boldsymbol{\beta} d\theta \\
&= \frac{1}{b_1 b_2 C_\psi} \int\limits_{\mathcal{G}} \left[\Gamma_{\mathbb{H}}^{M_s} g\right](\alpha, \boldsymbol{\beta}, \theta) \left\langle \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}, \psi_{\alpha',\boldsymbol{\beta}',\theta'}^{M_s} \right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} d\alpha d\boldsymbol{\beta} d\theta \\
&= \frac{1}{b_1 b_2 C_\psi} \int\limits_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \left\langle \psi_{\alpha,\boldsymbol{\beta},\theta}^{M_s}, \psi_{\alpha',\boldsymbol{\beta}',\theta'}^{M_s} \right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} d\alpha d\boldsymbol{\beta} d\theta.
\end{aligned}
$$

Conversely suppose that f satisfies (3.7). Then we show that there exists a function $g \in L^2(\mathbb{R}^2, \mathbb{H})$ satisfying $\left[\Gamma_{\mathbb{H}}^{M_s} g\right] = f$. Consider

$$g(\mathbf{x}) = \frac{1}{b_1 b_2 C_\psi} \int_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta} d\alpha d\boldsymbol{\beta} d\theta.$$

First, we show that $g \in L^2(\mathbb{R}^2, \mathbb{H})$.Therefore, we have

$$\begin{aligned}
||g||^2_{L^2(\mathbb{R}^2,\mathbb{H})} &= \int_{\mathbb{R}^2} g(\mathbf{x})\overline{g(\mathbf{x})} d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \left\{ \frac{1}{b_1 b_2 C_\psi} \int_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta} d\alpha d\boldsymbol{\beta} d\theta \right\} \\
&\times \left\{ \overline{\frac{1}{b_1 b_2 C_\psi} \int_{\mathcal{G}} f(\alpha', \boldsymbol{\beta}', \theta') \psi^{M_s}_{\alpha',\boldsymbol{\beta}',\theta'} d\alpha' d\boldsymbol{\beta}' d\theta'} \right\} d\mathbf{x} \\
&= \frac{1}{b_1 b_2 C_\psi \overline{b_1 b_2 C_\psi}} \int_{\mathcal{G}} \int_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \left(\int_{\mathbb{R}^2} \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta} \overline{\psi^{M_s}_{\alpha',\boldsymbol{\beta}',\theta'}} d\mathbf{x} \right) d\alpha d\boldsymbol{\beta} d\theta \\
&\qquad \times \overline{f(\alpha', \boldsymbol{\beta}', \theta')} d\alpha' d\boldsymbol{\beta}' d\theta' \\
&= \frac{1}{b_1 b_2 C_\psi \overline{b_1 b_2 C_\psi}} \int_{\mathcal{G}} \int_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \left\langle \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}, \psi^{M_s}_{\alpha',\boldsymbol{\beta}',\theta'} \right\rangle d\alpha d\boldsymbol{\beta} d\theta \\
&\qquad \times \overline{f(\alpha', \boldsymbol{\beta}', \theta')} d\alpha' d\boldsymbol{\beta}' d\theta' \\
&= \frac{1}{b_1 b_2 \overline{C_\psi}} \int_{\mathcal{G}} f(\alpha', \boldsymbol{\beta}', \theta') \overline{f(\alpha', \boldsymbol{\beta}', \theta')} d\alpha' d\boldsymbol{\beta}' d\theta' \\
&= \frac{1}{b_1 b_2 \overline{C_\psi}} ||f||^2_{L^2(\mathbb{R}^2,\mathbb{H})}.
\end{aligned}$$

Besides , as a result of the renowned Fubini-theorem, we get

$$\begin{aligned}
\left[\Gamma_{\mathbb{H}}^{M_s} g\right](\alpha', \boldsymbol{\beta}', \theta') &= \int_{\mathbb{R}^2} g(\mathbf{x}) \overline{\psi^{M_s}_{\alpha',\boldsymbol{\beta}',\theta'}(\mathbf{x})} d\mathbf{x} \\
&= \frac{1}{b_1 b_2 C_\psi} \int_{\mathbb{R}^2} \left\{ \int_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta} d\alpha d\boldsymbol{\beta} d\theta \right\} \overline{\psi^{M_s}_{\alpha',\boldsymbol{\beta}',\theta'}(\mathbf{x})} d\mathbf{x} \\
&= \frac{1}{b_1 b_2 C_\psi} \int_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \int_{\mathbb{R}^2} \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta} \overline{\psi^{M_s}_{\alpha',\boldsymbol{\beta}',\theta'}} d\mathbf{x} d\alpha d\boldsymbol{\beta} d\theta \\
&= \frac{1}{b_1 b_2 C_\psi} \int_{\mathcal{G}} f(\alpha, \boldsymbol{\beta}, \theta) \left\langle \psi^{M_s}_{\alpha,\boldsymbol{\beta},\theta}, \psi^{M_s}_{\alpha',\boldsymbol{\beta}',\theta'} \right\rangle d\alpha d\boldsymbol{\beta} d\theta \\
&= f(\alpha', \boldsymbol{\beta}', \theta').
\end{aligned}$$

This completes the proof. □

4 Uncertainty Principles for Quaternion Linear Canonical Curvelet Transform

Uncertainty Principle is one of the fundamentals of harmonic analysis and signal processing. In harmonic analysis, the classical Heisenberg's uncertainty principle gives information about the spread of a signal and its Fourier transform by asserting that a signal cannot be sharply localized in both the time and frequency domain [21]. That is, if we restrict the behaviour of one, we lose control over the other. Heisenberg's Uncertainty Principle in harmonic analysis bears a vital significance in time frequency analysis for it supplies a lower bound for optimal resolution of a signal in both the time and frequency domain. Different settings witness the enlargement of the principle and many analogues have appeared in the literature [3, 9, 21, 10, 33]. In this section, we will establish the proof of Heisenberg's uncertainty principle and logarithm uncertainty principle associated with the proposed transform (3.1). In a first, we demonstrate the following lemma.

Lemma 4.1. *Let $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$, be any admissible quaternion curvelet , then for every $f \in L^2(\mathbb{R}^2, \mathbb{H})$, we have*

$$b_1 b_2 C_\psi \int\limits_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](\boldsymbol{\omega})|_{\mathbb{H}}^2 \, d\boldsymbol{\omega} = \int\limits_{-\pi}^{\pi} \int\limits_{0}^{a_0} \int\limits_{\mathbb{R}^2} \left| w_k \mathcal{L}^{\mathbb{H}}_{M_1,M_2} \left[\Gamma^{M_s}_{\mathbb{H}} f\right](\boldsymbol{\omega}) \right|_{\mathbb{H}}^2 d\boldsymbol{\omega} d\alpha d\theta.$$

Proof. From Parseval's formula of quaternion linear canonical curvelet transform, we have

$$b_1 b_2 C_\psi \langle f, g \rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int\limits_{-\pi}^{\pi} \int\limits_{0}^{a_0} \int\limits_{\mathbb{R}^2} \left[\Gamma^{M_s}_{\mathbb{H}} f\right](\alpha, \boldsymbol{\beta}, \theta) \overline{\left[\Gamma^{M_s}_{\mathbb{H}} g\right](\alpha, \boldsymbol{\beta}, \theta)} d\boldsymbol{\beta} d\alpha d\theta.$$

Now applying the Plancheral theorem of two sided QFT on L.H.S and plancheral's theorem of QLCT to the $\boldsymbol{\beta}$-integral on R.H.S of above equation, we obtain

$$b_1 b_2 C_\psi \langle \mathcal{F}_q[f], \mathcal{F}_q[g] \rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int\limits_{-\pi}^{\pi} \int\limits_{0}^{a_0} \left\langle \mathcal{L}^{\mathbb{H}}_{M_1,M_2} \left[\Gamma^{M_s}_{\mathbb{H}}[f], \mathcal{L}^{\mathbb{H}}_{M_1,M_2} \left[\Gamma^{M_s}_{\mathbb{H}}[g]\right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} d\alpha d\theta.$$

On multiplying both sides by $|w_k|$, we get

$$b_1 b_2 C_\psi \langle w_k \mathcal{F}_q[f], w_k \mathcal{F}_q[g] \rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \int\limits_{-\pi}^{\pi} \int\limits_{0}^{a_0} \left\langle w_k \mathcal{L}^{\mathbb{H}}_{M_1,M_2} \left[\Gamma^{M_s}_{\mathbb{H}}[f], w_k \mathcal{L}^{\mathbb{H}}_{M_1,M_2} \left[\Gamma^{M_s}_{\mathbb{H}}[g]\right\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} d\alpha d\theta.$$

For $f = g$, we have

$$b_1 b_2 C_\psi \int\limits_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](\boldsymbol{\omega})|_{\mathbb{H}}^2 \, d\boldsymbol{\omega} = \int\limits_{-\pi}^{\pi} \int\limits_{0}^{a_0} \int\limits_{\mathbb{R}^2} \left| w_k \mathcal{L}^{\mathbb{H}}_{M_1,M_2} \left[\Gamma^{M_s}_{\mathbb{H}} f\right](\boldsymbol{\omega}) \right|_{\mathbb{H}}^2 d\boldsymbol{\omega} d\alpha d\theta.$$

This completes the proof. □

Theorem 4.2. *Let $\psi \in L^2(\mathbb{R}^2, \mathbb{H})$, be any admissible quaternion curvelet , then the quaternion LCCT $\Gamma^{M_s}_{\mathbb{H}}[f](\alpha, \boldsymbol{\beta}, \theta)$ satisfies the following uncertainty inequality:*

$$\int\limits_{0}^{a_0} \int\limits_{-\pi}^{\pi} \int\limits_{\mathbb{R}^2} {\beta_k}^2 \left| \left[\Gamma^{M_s}_{\mathbb{H}} f\right](\alpha, \boldsymbol{\beta}, \theta) \right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha \int\limits_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](\boldsymbol{\omega})|_{\mathbb{H}}^2 \, d\boldsymbol{\omega} \geq \frac{b_k^3}{4} b_1 b_2 C_\psi ||f||^4_{L^2(\mathbb{R}^2,\mathbb{H})}.$$

Proof. Using the Heisenberg's inequality for the QLCT [26], we can write

$$\left\{ \int\limits_{\mathbb{R}^2} {x_k}^2 \left| f(\mathbf{x}) \right|_{\mathbb{H}}^2 d\mathbf{x} \right\}^{1/2} \left\{ \int_{\mathbb{R}^2} w_k^2 \left| \mathcal{L}^{\mathbb{H}}_{M_1,M_2}[f](w) \right|_{\mathbb{H}}^2 d\boldsymbol{\omega} \right\}^{1/2} \geq \frac{b_k}{2} \int_{\mathbb{R}^2} \left| f(\mathbf{x}) \right|_{\mathbb{H}}^2 d\mathbf{x}. \tag{4.1}$$

Where k=1,2
Considering $\Gamma_{\mathbb{H}}^{M_s}[f](\alpha,\boldsymbol{\beta},\theta)$ as a function of $\boldsymbol{\beta}$ and replacing f by $\Gamma_{\mathbb{H}}^{M_s}[f](\alpha,\boldsymbol{\beta},\theta)$ in (4.1), we obtain

$$\left\{\int\limits_{\mathbb{R}^2} {\beta_k}^2 \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta}\right\}^{1/2} \left\{\int_{\mathbb{R}^2} w_k^2 \left|\mathcal{L}_{M_1,M_2}^{\mathbb{H}}\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)(w)\right|_{\mathbb{H}}^2 d\boldsymbol{\omega}\right\}^{1/2}$$
$$\geq \frac{b_k}{2}\int_{\mathbb{R}^2}\left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta}. \qquad (4.2)$$

Integrating (4.2) with respect to measure $d\alpha d\theta$ and using (3.5), we have

$$\int\limits_0^{a_0}\int\limits_{-\pi}^{\pi}\left\{\int\limits_{\mathbb{R}^2} {\beta_k}^2 \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta}\right\}^{1/2} \left\{\int_{\mathbb{R}^2} w_k^2 \left|\mathcal{L}_{M_1,M_2}^{\mathbb{H}}\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)(w)\right|_{\mathbb{H}}^2 d\boldsymbol{\omega}\right\}^{1/2} d\theta d\alpha$$
$$\geq \frac{b_k}{2}\int\limits_0^{a_0}\int\limits_{-\pi}^{\pi}\int_{\mathbb{R}^2}\left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha$$
$$= \frac{b_k}{2} b_1 b_2 C_\psi ||f||_{L^2(\mathbb{R}^2,\mathbb{H})}^2$$

Now applying Cauchy-Schwartz inequality and Fubini theorem ,we obtain

$$\left\{\int\limits_0^{a_0}\int\limits_{-\pi}^{\pi}\int\limits_{\mathbb{R}^2} {\beta_k}^2 \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha\right\}^{1/2}$$
$$\times\left\{\int\limits_0^{a_0}\int\limits_{-\pi}^{\pi}\int_{\mathbb{R}^2} w_k^2 \left|\mathcal{L}_{M_1,M_2}^{\mathbb{H}}\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)(w)\right|_{\mathbb{H}}^2 d\boldsymbol{\omega} d\theta d\alpha\right\}^{1/2}$$
$$\geq \frac{b_k}{2} b_1 b_2 C_\psi ||f||_{L^2(\mathbb{R}^2,\mathbb{H})}^2.$$

Now applying lemma (4.1) into L.H.S of above equation, we get

$$\left\{\int\limits_0^{a_0}\int\limits_{-\pi}^{\pi}\int\limits_{\mathbb{R}^2} {\beta_k}^2 \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha\right\}^{1/2} \left\{b_1 b_2 C_\psi \int\limits_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](\boldsymbol{\omega})|_{\mathbb{H}}^2 \, d\boldsymbol{\omega}\right\}^{\frac{1}{2}}$$
$$\geq \frac{b_k}{2} b_1 b_2 C_\psi ||f||_{L^2(\mathbb{R}^2,\mathbb{H})}^2$$

or,

$$\left\{\int\limits_0^{a_0}\int\limits_{-\pi}^{\pi}\int\limits_{\mathbb{R}^2} {\beta_k}^2 \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha\right\}^{1/2} \left\{\int\limits_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](\boldsymbol{\omega})|_{\mathbb{H}}^2 \, d\boldsymbol{\omega}\right\}^{\frac{1}{2}}$$
$$\geq \frac{b_k}{2\sqrt{b_1 b_2}\sqrt{C_\psi}} b_1 b_2 C_\psi ||f||_{L^2(\mathbb{R}^2,\mathbb{H})}^2$$

Squaring both sides we get

$$\int\limits_0^{a_0}\int\limits_{-\pi}^{\pi}\int\limits_{\mathbb{R}^2} {\beta_k}^2 \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha \int\limits_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](\boldsymbol{\omega})|_{\mathbb{H}}^2 \, d\boldsymbol{\omega}$$
$$\geq \frac{b_k^3}{4} C_\psi ||f||_{L^2(\mathbb{R}^2,\mathbb{H})}^4.$$

This completes the proof. □

Next, we will prove the logarithm uncertainty principle associated with QLCCT.

Theorem 4.3. *Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$, be the quaternion function , then the quaternion LCCT $\Gamma_{\mathbb{H}}^{M_s}[f](\alpha,\boldsymbol{\beta},\theta)$ satisfies the following uncertainty principle:*

$$\int_0^{a_0}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \ln|\boldsymbol{\beta}| \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha + b_1 b_2 C_\psi \int_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](\boldsymbol{\omega})|_{\mathbb{H}}^2 \, d\boldsymbol{\omega}$$
$$\geq \Big(D + \ln|b|\Big) b_1 b_2 C_\psi ||f||^2_{L^2(\mathbb{R}^2,\mathbb{H})},$$

where $D = \left(\dfrac{\Gamma'(1/2)}{\Gamma(1/2)} - \ln\pi\right)$ *and* Γ *denotes the Gamma function.*

Proof. The logarithm Uncertainty principle for the quaternion linear canonical transform is given by [5]

$$\int_{\mathbb{R}^2} \ln|\mathbf{x}|\, |f(\mathbf{x})|_{\mathbb{H}}^2 \, d\mathbf{x} + \int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}| \left|\mathcal{L}_{M_1,M_2}^{\mathbb{H}}[f](\boldsymbol{\omega})\right|_{\mathbb{H}}^2 d\boldsymbol{\omega} \geq \Big(D + \ln|b|\Big) \int_{\mathbb{R}^2} |f(\mathbf{x})|_{\mathbb{H}}^2 d\mathbf{x}. \tag{4.3}$$

Now we shall identify $\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)$ as a function of $\boldsymbol{\beta}$ and replacing $f(\mathbf{x})$ by $\Gamma_{\mathbb{H}}^{M_s}[f](\alpha,\boldsymbol{\beta},\theta)$ in (4.3), we obtain

$$\int_{\mathbb{R}^2} \ln|\boldsymbol{\beta}| \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} + \int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}| \left|\mathcal{L}_{M_1,M_2}^{\mathbb{H}}\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)(\boldsymbol{\omega})\right|_{\mathbb{H}}^2 d\boldsymbol{\omega}$$
$$\geq \Big(D + \ln|b|\Big) \int_{\mathbb{R}^2} \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta}.$$

Now integrating above equation with respect to $d\alpha d\theta$ and using fubini's theorem, we obtain

$$\int_0^{a_0}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \ln|\boldsymbol{\beta}| \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha$$
$$+ \int_0^{a_0}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}| \left|\mathcal{L}_{M_1,M_2}^{\mathbb{H}}\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)(\boldsymbol{\omega})\right|_{\mathbb{H}}^2 d\boldsymbol{\omega} d\theta d\alpha$$
$$\geq \Big(D + \ln|b|\Big) \int_0^{a_0}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha. \tag{4.4}$$

Now using lemma (4.1) for ${w_k}^2 = \ln|\boldsymbol{\omega}|$ in L.H.S of (4.4) and corollary (3.5) on R.H.S of (4.4), we get

$$\int_0^{a_0}\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} \ln|\boldsymbol{\beta}| \left|\left[\Gamma_{\mathbb{H}}^{M_s} f\right](\alpha,\boldsymbol{\beta},\theta)\right|_{\mathbb{H}}^2 d\boldsymbol{\beta} d\theta d\alpha + b_1 b_2 C_\psi \int_{\mathbb{R}^2} |w_k \mathcal{F}_q[f](\boldsymbol{\omega})|_{\mathbb{H}}^2 \, d\boldsymbol{\omega}$$
$$\geq \Big(D + \ln|b|\Big) b_1 b_2 C_\psi ||f||^2_{L^2(\mathbb{R}^2,\mathbb{H})}.$$

This completes the proof. □

References

[1] O. Ahmad, N. A. Sheikh and F. A. Shah, Fractional Multiresolution Analysis and associated scaling functions in $L^2(\mathbb{R})$, *Analysis and Mathematical Physics* **11**, 1–20 (2021).

[2] O. Ahmad, N. A. Sheikh and F. A. Shah, Fractional biorthogonal wavelets in $L^2(\mathbb{R})$, *Applicable Analysis*, (2021).

[3] O. Ahmad and N. A. Sheikh, Novel special affine wavelet transform and associated uncertainty principles, *International Journal of Geometric Methods in Modern Physics* **18** (2021).

[4] M. Bahri and R. Ashino, A simplified proof of uncertainty principle for quaternion linear canonical transform, *Abstract and Applied Analysis*, Hindawi (2016).

[5] M. Bahri and R. Ashino, Logarithmic uncertainty principle for quaternion linear canonical transform, *International Conference on Wavelet Analysis and Pattern Recognition (ICWAPR)*, (140–145) (2016).

[6] M. Bahri, E. Hitzer, R. Ashino and A. Hayashi, An uncertainty principle for quaternion Fourier transform, *Computers and Mathematics with Applications* **56**, 2398–2410 (2008).

[7] B. Barshan, M. A. Kutay and H. M. Ozaktas, Optimal filtering with linear canonical transformations, *Optics communications* **135**, 32–36 (1997).

[8] M. Y. Bhat and A. H. Dar, Multiresolution analysis for linear canonical S transform, *Advances in Operator Theory* **6**, 1–11 (2021).

[9] M. Y. Bhat and A. H. Dar, The algebra of 2D Gabor quaternionic offset linear canonical transform and uncertainty principles, *The Journal of Analysis*, (2021).

[10] M. Y. Bhat and A. H. Dar, Donoho-Stark's and Hardy's Uncertainty Principles for the Short-time Quaternion Offset Linear Canonical Transform, arXiv:2110.02754 (2021).

[11] E. J. Candés and D. L. Donoho, Ridgelets: a key to higher dimensional intermittency, *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* **357**, 2495–2509 (1999).

[12] E. J. Candés and D. L. Donoho, Recovering edges in ill-posed inverse problems: Optimality of curvelet frames, *The Annals of Statistics* **30**, 784–842 (2002).

[13] E. J. Candés and D. L. Donoho, New tight frames of curvelets and optimal representations of objects with piecewise C^2 singularities, *Communications on Pure and Applied Mathematics* **57**, 219–266 (2004).

[14] E. J. Candés and D. L. Donoho, Continuous curvelet transform: I. Resolution of the wavefront set, *Appl. Comput. Harmon. Anal.* **19**, 162–197 (2005).

[15] E. J. Candés and D. L. Donoho, Continuous curvelet transform: II. Discretization and frames, *Appl. Comput. Harmon. Anal.* **19**, 198–222 (2005).

[16] E. J. Candés and L. Demanet, The curvelet representation of wave propagators is optimally sparse, *Communications on Pure and Applied Mathematics* **58**, 1472–1528 (2005).

[17] S. A. Collins, Lens-system diffraction integral written in terms of matrix optics, *Journal of the Optical Society of America* **60**, 1168–1177 (1970).

[18] H. De Bie , New techniques for the two-sided quaternion Fourier transform, *proceedings of Applied Geometric Algebras in Computer Science* (2012).

[19] M. N. Do and M. Vetterli, The contourlet transform: an efficient directional multiresolution image representation, *IEEE Transactions on Image Processing* **14**, 2091–2106 (2005).

[20] Q. Feng and B. Z. Li Convolution and correlation theorems for the two-dimensional linear canonical transform and its applications, *IET Signal Processing* **10**, (2016) 125–132.

[21] G. B. Folland and A. Sitaram, The uncertainty principle: a mathematical survey *Journal of Fourier analysis and applications* **3**, 207–238 (1997).

[22] J. J. Healy, M. A. Kutay, H. M. Ozaktas and J.T. Sheridan, *Linear Canonical Transforms*, Springer, New York (2016).

[23] E. Hitzer, Quaternion Fourier transform on quaternion fields and generalizations, *Advances in Applied Clifford Algebras* **17**, 497–517 (2007).

[24] E. Hitzer and S. J. Sangwine, *Quaternion and Clifford Fourier transforms and wavelets*, Birkhäuser, Basel (2013).

[25] A. A. Khan and K. Ravikumar, Linear canonical curvelet transform and the associated Heisenberg-type inequalities, *International Journal of Geometric Methods in Modern Physics* **18**, (2021).

[26] K. I. Kou, J. Y. Ou and J. Morais, On uncertainty principle for quaternionic linear canonical transform, *Abstract and Applied Analysis*, (2013).

[27] G. Kutyniok and D. Labate, Resolution of the wavefront set using continuous shearlets, *Transactions of the American Mathematical Society* **361**, 2719–2754 (2009).

[28] J. Li and M. W. Wong, Localization operators for curvelet transforms, *Journal of Pseudo-Differential Operators and Applications* **3**, 121–143 (2012).

[29] M. Moshinsky and C. Quesne, Linear canonical transformations and their unitary representations, *Journal of Mathematical Physics* **12**, 1772–1780 (1971).

[30] D. Mustard, Uncertainty principles invariant under the fractional Fourier Transform, *J. Aust. Math. Soc., ser. B* **33**, 180–191 (1991).

[31] H. M. Ozaktas, Z. Zalevsky and M. A. Kutay, *The fractional Fourier Transform with Applications in Optics and Signal Processing*, Wiley, New York (2000).

[32] S. C. Pei and J. J. Ding, Eigen functions of the offset Fourier, fractional Fourier and linear canonical transform, *Journal of the Optical Society of America* **20**, 522–532 (2003).

[33] E. Wilczok, New uncertainty principles for the continuous Gabor transform and the continuous wavelet transform, *Documenta Mathematica* **5**, 201–226 (2000).

[34] T. Z. Xu and B. Z. Li, *Linear Canonical Transform and Its Applications*, Science Press, Beijing, China (2013).

[35] A. I. Zayed, Sampling of signals bandlimited to a disc in the linear canonical transform domain, *IEEE signal processing letters* **25**,1765–1769 (2018).

[36] Q. Zhang , Zak transform and uncertainty principles associated with linear canonical transform, *IET signal process.* **10**, 791–797 (2016).

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