Palestine Journal of Mathematics

Vol. 12(1)(2023) , 64–74

SWITCHING CONTROLS FOR PARABOLIC SYSTEMS

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 93B05, 35K10; Secondary 93C20.

Keywords and phrases: controllability, parabolic systems, switching controls.

The authors were partially supported by the Agence Nationale de la Recherche, Project TRECOS, ANR-20-CE40-0009.

Abstract We consider the controllability of an abstract parabolic system by using switching controls. More precisely, we show that, under general hypotheses, if a parabolic system is null-controllable for any positive time with N controls, then it is also null-controllable with the property that at each time, only one of these controls is active. The main difference with previous results in the literature is that we can handle the case where the main operator of the system is not self-adjoint. We give several examples to illustrate our result: coupled heat equations with terms of orders 0 and 1, the Oseen system or the Boussinesq system.

1 Introduction and main result

Let us consider the following system

$$\begin{cases} z' + Az = \sum_{j=1}^{N} B_j u_j \quad \text{in } (0,T), \\ z(0) = z^0, \end{cases} \tag{1.1}$$

where z is the state of the system and u_j are controls. We are interested by the problem proposed in [23] where the idea is to control z by using "switching" controls so that only one control u_j is active at the same time. We consider here parabolic systems in the sense defined below. Our aim is to generalize the results already obtained in [23] and in [6]. In [6], the authors manage to prove the null-controllability of the system (1.1) with switching controls in the case where A is a self-adjoint operator in a Hilbert space and if (1.1) is null-controllable for all $T > 0$ without the switching conditions on the controls. They also obtain this result in the case where the state space is finite-dimensional and they conclude with a result for non self-adjoint operator in a Hilbert space but with conditions on A that can be quite restrictive (see the discussion after Theorem 5.1 in [6]). Here, we show a more general result with less restrictive conditions on A and we give at the end of this article several examples showing that we can handle many important situations. We also state a corresponding result for the approximate controllability with switching controls.

We assume that, in (1.1), $A : \mathcal{D}(A) \to \mathcal{H}$ is an unbounded operator in a Hilbert space $\mathcal{H}$, and B_j, $j = 1, \dots, N$ are control operators satisfying $B_j \in \mathcal{L}(\mathcal{U}_j, \mathcal{D}((A^*)^k)')$ for some $k \geqslant 0$, where $\mathcal{U}_j$ is a Hilbert space. Here and in what follows, we identify $\mathcal{H}$ with its dual and for a Hilbert space $\mathcal{V} \subset \mathcal{H}$ such that $\mathcal{V}$ is dense in $\mathcal{H}$, $\mathcal{V}'$ stands for the dual space of $\mathcal{V}$ with respect to the pivot space $\mathcal{H}$.

In what follows, we assume that

$$-A \text{ is the infinitesimal generator of an analytic semigroup } \left(e^{-tA}\right)_{t\geqslant 0}. \tag{H1}$$

This implies that the spectrum $\sigma(A)$ of A is contained in a sector of $\mathbb{C}$ (see, for instance, [3, Theorem 2.11, p.112]). We assume moreover that

$$\sigma(A) \text{ is only composed by eigenvalues } (\lambda_j) \text{ with finite algebraic multiplicity.} \tag{H2}$$

We recall that one can define the algebraic multiplicity of λ_j by using a projection operator

$$\mathbb{P}(\lambda_j) = \frac{1}{2\pi\imath} \oint_\Gamma (\lambda I - A)^{-1} d\lambda, \tag{1.2}$$

where Γ is a positively oriented simple closed curve enclosing λ_j but no other point of $\sigma(A)$ (see [17, pp. 180-181]). Then, the algebraic multiplicity of λ_j is the dimension of $\operatorname{Im}\mathbb{P}(\lambda_j)$. There exists $n \in \mathbb{N}^*$ such that $\operatorname{Im}\mathbb{P}(\lambda_j) = \operatorname{Ker}(\lambda_j I - A)^n$ and an element of $\operatorname{Im}\mathbb{P}(\lambda_j)$ is called a root vector of A. We denote by m_j the smallest n satisfying the above equality (the index of λ_j). We have similar definitions for A^* and we assume that

$$\text{The root vectors of } A \text{ are complete in } \mathcal{H}. \tag{H3}$$

Our main result is

Theorem 1.1. *Assume* (H1), (H2), *and* (H3). *If* (1.1) *is null-controllable in any time* $T > 0$, *then it is null-controllable in any time* $T > 0$ *with the following constraint on the controls* $(u_k)_{k=1,\dots,N}$:

$$\prod_{j=1}^{N} \left(\sum_{k \neq j} \|u_k(t)\|_{\mathcal{U}_k} \right) = 0 \text{ a.e. } t \in (0,T). \tag{1.3}$$

The condition (1.3) means that at most one control u_k is active (not null) at the same time. In the case $N = 2$, (1.1) writes

$$\begin{cases} z' + Az = B_1 u_1 + B_2 u_2 & \text{in } (0,T), \\ z(0) = z^0, \end{cases} \tag{1.4}$$

and the condition (1.3) reduces to the condition

$$\|u_1(t)\|_{\mathcal{U}_1} \|u_2(t)\|_{\mathcal{U}_2} = 0 \text{ a.e. } t \in (0,T). \tag{1.5}$$

Remark 1.2. To be more precise in the statement of Theorem 1.1, let us define for $r = k + \alpha$, $k \in \mathbb{N}$, $\alpha \in [0,1]$, the spaces:

$$\mathcal{H}_r \stackrel{\text{def}}{=} \left[\mathcal{D}(A^{k+1}), \mathcal{D}(A^k)\right]_{1-\alpha}, \quad \mathcal{H}_{-r} \stackrel{\text{def}}{=} \left[\mathcal{D}((A^*)^{k+1}), \mathcal{D}((A^*)^k)\right]'_{1-\alpha}, \tag{1.6}$$

where $[\cdot,\cdot]_\cdot$ denotes the interpolation space obtained with the complex interpolation method. Then, there exists $\gamma \geqslant 0$ such that

$$B_j \in \mathcal{L}(\mathcal{U}_j, \mathcal{H}_{-\gamma}) \quad (j = 1, \dots, N). \tag{1.7}$$

From standard result on parabolic systems, if $u_j \in L^2(0,T;\mathcal{U}_i)$ and if $z^0 \in [\mathcal{H}_{1-\gamma}, \mathcal{H}_{-\gamma}]_{1/2}$, then the solution z of (1.1) satisfies

$$z \in H^1(0,T;\mathcal{H}_{-\gamma}) \cap L^2(0,T;\mathcal{H}_{1-\gamma}) \cap C^0([0,T];[\mathcal{H}_{1-\gamma}, \mathcal{H}_{-\gamma}]_{1/2}) \tag{1.8}$$

and Theorem 1.1 states that if (1.1) is null-controllable for any time $T > 0$ in $[\mathcal{H}_{1-\gamma}, \mathcal{H}_{-\gamma}]_{1/2}$ with controls in $L^2(0,T;\mathcal{U}_j)$ then for any $z^0 \in [\mathcal{H}_{1-\gamma}, \mathcal{H}_{-\gamma}]_{1/2}$, there exist $u_j \in L^2(0,T;\mathcal{U}_j)$ satisfying (1.3) and such that $z(T) = 0$.

Note that, using the parabolic regularity, if z^0 is in a larger space $\mathcal{H}_{-\gamma'}$, with $\gamma' > \gamma$, then taking $u_j \equiv 0$ in $(0,\varepsilon)$ with $\varepsilon \in (0,T)$ arbitrarily small, we have $z(\varepsilon) \in [\mathcal{H}_{1-\gamma}, \mathcal{H}_{-\gamma}]_{1/2}$ and we deduce the null-controllability with switching controls.

Remark 1.3. With respect to the literature, we generalize here the results obtained in [23] and in [6]. In [6], the authors assume that the operator A is self-adjoint or that $\mathcal{H}$ is of finite dimension. They have also obtained an extension in the case of a parabolic system (Theorem 5.1 in [6]) but they need a hypothesis on the semigroup $\left(e^{-tA^*}\right)_{t\geqslant 0}$ and they explain that this hypothesis is difficult to check in practice with some examples. Here we have more general hypotheses on

A and there is a typical situation where (H1), (H2) and (H3) hold: if A can be decomposed as $A = A_0 + A_1$ with A_0 a self-adjoint operator in $\mathcal{H}$ with compact resolvent, $\mathcal{D}(A_1) \subset \mathcal{D}(A_0^\alpha)$ for some $\alpha \in [0,1)$ and the eigenvalues μ_j of A_0 satisfy for some $1 \leqslant p < \infty$ and for some $J \geqslant 1$,

$$\sum_{j=J}^{\infty} \frac{1}{|\mu_j|^p} < \infty, \tag{1.9}$$

then from a classical result for the perturbation of analytic semigroup (see, for instance, [19, Theorem 2.1, p. 80]), (H1) holds. Moreover, one can use the Keldy Theorem (see [15, Thm.10.1, p.276] or [21, Theorem 3, p.394] combined with [21, Relation (6), p.393 and Lemma 2, p.395])) that yields (H2) and (H3).

With a similar proof as Theorem 1.1, one can also obtain the following result:

Theorem 1.4. *Assume* (H1), (H2) *and* (H3). *If* (1.1) *is approximately controllable for some time* $T > 0$, *then it is approximately controllable in time* $T > 0$ *with the constraint* (1.3).

Remark 1.5. We can make the above statement more precise, using Remark 1.2. If (1.1) is approximately controllable for some time $T > 0$, and if B_j satisfies (1.7), then for any $z^0, z^1 \in [\mathcal{H}_{1-\gamma}, \mathcal{H}_{-\gamma}]_{1/2}$ and for any $\varepsilon > 0$, there exist $u_j \in L^2(0,T;\mathcal{U}_i)$ satisfying (1.3) so that the solution z of (1.1) (satisfying (1.8)) verifies $\left\|z(T) - z^1\right\|_{[\mathcal{H}_{1-\gamma},\mathcal{H}_{-\gamma}]_{1/2}} \leqslant \varepsilon$.

The outline of this article is as follows: in the next section, we show Theorem 1.1 and give the main ideas to adapt the proof to Theorem 1.4. Then in Section 3, we present some examples to illustrate this result.

2 Proof of the main result

The proof of Theorem 1.1 follows the same scheme as in [23] and in [6]. The new part of this proof corresponds to Lemma 2.2 . We only repeat all the proof for sake of clarity. To simplify, we only show Theorem 1.1 for $N = 2$. The details of the proof for $N > 2$ can be found in [6]. Finally, we can also assume that $B_j \in \mathcal{L}(U_j, \mathcal{H})$ (the case of "bounded" control operators). Indeed, taking $\mu_0 > -\inf_{j \geqslant 1} \operatorname{Re} \lambda_j$, one can apply $(\mu_0 + A)^{-k}$ to (1.4) and we obtain the control system

$$\left\{\begin{array}{ll} \widetilde{z}' + A\widetilde{z} = \widetilde{B}_1 u_1 + \widetilde{B}_2 u_2 & \text{in } (0,T), \\ \widetilde{z}(0) = \widetilde{z}^0, & \end{array}\right. \tag{2.1}$$

where

$$\widetilde{z} \stackrel{\text{def}}{=} (\mu_0 + A)^{-k} z, \quad \widetilde{z}^0 \stackrel{\text{def}}{=} (\mu_0 + A)^{-k} z^0, \quad \widetilde{B}_j \stackrel{\text{def}}{=} (\mu_0 + A)^{-k} B_j \in \mathcal{L}(U_j, \mathcal{H}).$$

If there exist controls u_j satisfying (1.5) and such that $\widetilde{z}(T) = 0$ then it implies Theorem 1.1.

First, we consider the adjoint system of (1.1)

$$\left\{\begin{array}{ll} \varphi' + A^* \varphi = 0 & \text{in } (0,T), \\ \varphi(0) = \varphi^0, & \end{array}\right. \tag{2.2}$$

that is

$$\varphi(t) = e^{-tA^*} \varphi^0 \quad (t \geqslant 0).$$

Since (1.1) is null-controllable in any time $\tau > 0$, we have by a standard duality argument (see, for instance, [22, Theorem 11.2.1, p.357]) the following observability inequality:

$$\left\|e^{-\tau A^*} \varphi^0\right\|_{\mathcal{H}} \leqslant C(\tau) \left(\int_0^\tau \left\{ \left\|B_1^* e^{-tA^*} \varphi^0\right\|_{\mathcal{U}_1}^2 + \left\|B_2^* e^{-tA^*} \varphi^0\right\|_{\mathcal{U}_2}^2 \right\} dt \right)^{1/2} \quad (\varphi^0 \in \mathcal{H}). \tag{2.3}$$

Following the proof of [6], we introduce the function

$$\alpha(t) = 1 + \frac{1}{2} \sin(\omega t), \tag{2.4}$$

with $\omega \in \mathbb{R}$ that will be fixed further. As explained in [6], where the authors introduce such a function, many other choices are possible for α. We fix $T > 0$ and we define the following function

$$\|\varphi^0\|_{\mathcal{X}} \stackrel{\text{def}}{=} \left(\int_0^T \max \left\{ \left\| B_1^* e^{-tA^*} \varphi^0 \right\|_{\mathcal{U}_1}^2, \alpha(t) \left\| B_2^* e^{-tA^*} \varphi^0 \right\|_{\mathcal{U}_2}^2 \right\} dt \right)^{1/2} \tag{2.5}$$

and due to the bounds of α, we deduce from (2.3) that

$$\left\| e^{-TA^*} \varphi^0 \right\|_{\mathcal{H}} \leqslant C \|\varphi^0\|_{\mathcal{X}} \quad (\varphi^0 \in \mathcal{H}). \tag{2.6}$$

Using that $\left(e^{-tA^*}\right)_{t\geqslant 0}$ is an analytic semigroup, the above relation implies that $\|\cdot\|_{\mathcal{X}}$ is a norm. We consider the closure of $\mathcal{H}$ with respect to this norm:

$$\mathcal{X} \stackrel{\text{def}}{=} \text{Clos}_{\|\cdot\|_{\mathcal{X}}} \mathcal{H}. \tag{2.7}$$

Remark 2.1. Note that from (2.6), we can extend the application $\varphi^0 \mapsto e^{-TA^*}\varphi^0$ into a bounded map of $\mathcal{L}(\mathcal{X}, \mathcal{H})$. Using (2.3), for any $\tau \in (0, T)$, we can also extend the application $\varphi^0 \mapsto e^{-\tau A^*}\varphi^0$ into a bounded map of $\mathcal{L}(\mathcal{X}, \mathcal{H})$. Finally, for $j = 1, 2$, we extend

$$\varphi^0 \mapsto \left[t \mapsto B_j^* e^{-tA^*} \varphi^0 \right]$$

into a bounded map of $\mathcal{L}(\mathcal{X}, L^2(0, T; \mathcal{U}_j))$.

We can now define

$$J(\varphi^0) \stackrel{\text{def}}{=} \frac{1}{2} \|\varphi^0\|_{\mathcal{X}}^2 + \left(z^0, e^{-TA^*} \varphi^0 \right)_{\mathcal{H}} \quad (\varphi^0 \in \mathcal{X}).$$

The function J is convex, continuous and coercive: there exists a minimizer $\phi^0 \in \mathcal{X}$ of J. First, if $\phi^0 = 0$, then let us consider $\varphi^0 \in \mathcal{H}$. For any $\varepsilon > 0$,

$$J(\varepsilon\varphi^0) = \frac{1}{2}\varepsilon^2 \|\varphi^0\|_{\mathcal{X}}^2 + \varepsilon \left(z^0, e^{-TA^*} \varphi^0 \right)_{\mathcal{H}} \geqslant 0 = J(\phi^0)$$

and similarly,

$$\frac{1}{2}\varepsilon^2 \|\varphi^0\|_{\mathcal{X}}^2 - \varepsilon \left(z^0, e^{-TA^*} \varphi^0 \right)_{\mathcal{H}} \geqslant 0.$$

Dividing by ε and taking $\varepsilon \to 0$ in the two above relations, we deduce that $\left(e^{-TA} z^0, \varphi^0\right) = 0$ for any $\varphi^0 \in \mathcal{H}$ and thus $e^{-TA} z^0 = 0$. In particular, the controls $u_1 = 0$ and $u_2 = 0$ in $(0, T)$ lead the system to rest and satisfy the switching condition. Second, assume $\phi^0 \neq 0$. Then, we apply Lemma 2.2 below: it states that there exists ω such that the set

$$\mathcal{I}_0 \stackrel{\text{def}}{=} \left\{ t \in (0, T) \; ; \; \left\| B_1^* e^{-tA^*} \phi^0 \right\|_{\mathcal{U}_1}^2 = \alpha(t) \left\| B_2^* e^{-tA^*} \phi^0 \right\|_{\mathcal{U}_2}^2 \right\} \tag{2.8}$$

is of Lebesgue measure equal to 0. We then define

$$\mathcal{I}_1 \stackrel{\text{def}}{=} \left\{ t \in (0, T) \; ; \; \left\| B_1^* e^{-(T-t)A^*} \phi^0 \right\|_{\mathcal{U}_1}^2 > \alpha(T-t) \left\| B_2^* e^{-(T-t)A^*} \phi^0 \right\|_{\mathcal{U}_2}^2 \right\},$$

and

$$\mathcal{I}_2 \stackrel{\text{def}}{=} \left\{ t \in (0, T) \; ; \; \left\| B_1^* e^{-(T-t)A^*} \phi^0 \right\|_{\mathcal{U}_1}^2 < \alpha(T-t) \left\| B_2^* e^{-(T-t)A^*} \phi^0 \right\|_{\mathcal{U}_2}^2 \right\}.$$

We also define

$$j(t, \varphi^0) \stackrel{\text{def}}{=} \max \left\{ \left\| B_1^* e^{-(T-t)A^*} \varphi^0 \right\|_{\mathcal{U}_1}^2, \alpha(T-t) \left\| B_2^* e^{-(T-t)A^*} \varphi^0 \right\|_{\mathcal{U}_2}^2 \right\}$$

and we note that the differentiate of $j(t,\cdot)$ at the point ϕ^0 in the direction φ^0 satisfies

$$D_{\phi^0} j(t,\varphi^0) = \begin{cases} 2\left(B_1^* e^{-(T-t)A^*}\phi^0, B_1^* e^{-(T-t)A^*}\varphi^0\right)_{\mathcal{U}_1} & \text{if } t \in \mathcal{I}_1, \\ 2\alpha(T-t)\left(B_2^* e^{-(T-t)A^*}\phi^0, B_2^* e^{-(T-t)A^*}\varphi^0\right)_{\mathcal{U}_2} & \text{if } t \in \mathcal{I}_2. \end{cases}$$

By considering all the possibilities for the maximum, one can show (see the appendix A of [6] for more details) the existence of $C > 0$ such that

$$\left|\frac{1}{h}\left(j(t,\phi^0 + h\varphi^0) - j(t,\phi^0)\right)\right| \leqslant C\left(\left\|B_1^* e^{-(T-t)A^*}\phi^0\right\|_{\mathcal{U}_1}^2 + \left\|B_2^* e^{-(T-t)A^*}\phi^0\right\|_{\mathcal{U}_2}^2 \right.$$
$$\left. + \left\|B_1^* e^{-(T-t)A^*}\varphi^0\right\|_{\mathcal{U}_1}^2 + \left\|B_2^* e^{-(T-t)A^*}\varphi^0\right\|_{\mathcal{U}_2}^2\right) \quad (h \in (0,1)).$$

Using the Lebesgue theorem and the fact that $|\mathcal{I}_0| = 0$, we deduce that

$$D_{\phi^0} J(\varphi^0) = \int_{\mathcal{I}_1} \left(B_1^* e^{-(T-t)A^*}\phi^0, B_1^* e^{-(T-t)A^*}\varphi^0\right)_{\mathcal{U}_1} dt$$
$$+ \int_{\mathcal{I}_2} \alpha(T-t)\left(B_2^* e^{-(T-t)A^*}\phi^0, B_2^* e^{-(T-t)A^*}\varphi^0\right)_{\mathcal{U}_2} dt + \left(z^0, e^{-TA^*}\varphi^0\right)_{\mathcal{H}}. \quad (2.9)$$

Setting

$$u_1(t) \stackrel{\text{def}}{=} 1_{\mathcal{I}_1}(t) B_1^* e^{-(T-t)A^*}\phi^0, \quad u_2(t) \stackrel{\text{def}}{=} 1_{\mathcal{I}_2}(t)\alpha(T-t) B_2^* e^{-(T-t)A^*}\phi^0 \quad (2.10)$$

and taking $\varphi^0 \in \mathcal{H}$, we deduce that

$$D_{\phi^0} J = e^{-TA} z^0 + \int_0^T e^{-(T-t)A}\left(B_1 u_1(t) + B_2 u_2(t)\right)\, dt.$$

Using that ϕ^0 is a minimizer we deduce that the solution z of (1.4) with u_1 and u_2 defined by (2.10) satisfies $z(T) = 0$ whereas u_1 and u_2 satisfy (1.5).

This ends the proof of Theorem 1.1, provided we can prove that the set $\mathcal{I}_0$ defined by (2.8) is of Lebesgue measure equal to 0. This is the new part of this proof since it is done without assuming that $\mathcal{H}$ is finite dimensional or that A is self-adjoint.

Lemma 2.2. *Assume that*

$$\omega \notin \left\{\operatorname{Im}(\lambda_k - \lambda_{k'}), \frac{1}{2}\operatorname{Im}(\lambda_k - \lambda_{k'}) \textit{ for all } (k,k') \in (\mathbb{N}^*)^2 \textit{ such that } \operatorname{Re}(\lambda_k) = \operatorname{Re}(\lambda_{k'})\right\}. \quad (2.11)$$

If $\phi^0 \neq 0$ *then* $|\mathcal{I}_0| = 0$.

Before proving the above result, let us recall a result proved in [6] (that we have slightly adapted):

Lemma 2.3. *Assume* $(\mu_j)_{j=1}^N$ *is family of* N *distinct real numbers and* $(p_j)_{j=1}^N$ *a family of* N *polynomial functions. Then*

$$\lim_{t\to\infty} \sum_{j=1}^N p_j(t) e^{i\mu_j t} = 0 \implies \forall j \in \{1,\dots,N\}, \quad p_j = 0.$$

Proof. Lemma 2.5 in [6] shows this result if p_j are polynomials of degree $\leqslant 0$. To prove Lemma 2.3, we consider

$$d \stackrel{\text{def}}{=} \max_{j\in\{1,\dots,N\}} \deg p_j$$

that is $\geqslant 0$ if the p_j are not all null. Then by taking the limit of the above sum divided by t^d, Lemma 2.5 in [6] yields that all the monomials of p_j of degree d are 0 which leads to a contradiction. □

We can now prove Lemma 2.2.

Proof of Lemma 2.2. Assume that

$$|\mathcal{I}_0| \neq 0. \tag{2.12}$$

Then, there exists $n \in \mathbb{N}^*$ such that

$$\left|\mathcal{I}_0 \cap \left(\frac{T}{n}, T\right)\right| \neq 0. \tag{2.13}$$

On the other hand, using Remark 2.1, $e^{-\frac{T}{n}A^*}\phi^0 \in \mathcal{H}$. This fact and (2.13) allow us to reduce the proof of Lemma 2.2 to the case $\phi^0 \in \mathcal{H}$.

Using (2.12), (2.8) and the analyticity of the semigroup, we deduce that

$$\left\|B_1^* e^{-tA^*}\phi^0\right\|_{\mathcal{U}_1}^2 = \alpha(t)\left\|B_2^* e^{-tA^*}\phi^0\right\|_{\mathcal{U}_2}^2 \quad (t \geqslant 0). \tag{2.14}$$

From the eigenvalues of A, we can define an increasing sequence (σ_j) of $\mathbb{R}$ such that

$$\{\sigma_j,\ j \in \mathbb{N}^*\} = \{\operatorname{Re}\lambda_k,\ k \in \mathbb{N}^*\}.$$

Let us define

$$\Sigma_1 \overset{\text{def}}{=} \left\{\overline{\lambda_k}\ ;\ \operatorname{Re}\lambda_k = \sigma_1\right\}, \quad K_1 \overset{\text{def}}{=} \{k \in \mathbb{N}^*\ ;\ \operatorname{Re}\lambda_k = \sigma_1\}.$$

From (H1) and (H2) there exists a positively oriented simple closed curve Γ_1 enclosing Σ_1 but no other point of $\sigma(A^*)$. Then, we consider the projection operator (see [17, Theorem 6.17, p.178])

$$\mathbb{P}_1^* \overset{\text{def}}{=} \frac{1}{2\pi\imath}\oint_{\Gamma_1} (\lambda I - A^*)^{-1}\, d\lambda \tag{2.15}$$

and we define

$$\mathcal{H}_+ \overset{\text{def}}{=} \mathbb{P}_1^*(\mathcal{H}), \quad \mathcal{H}_- \overset{\text{def}}{=} (I - \mathbb{P}_1^*)(\mathcal{H}).$$

We have that

$$\mathcal{H}_+ = \bigoplus_{k \in K_1} \operatorname{Ker}\left(A^* - \overline{\lambda_k} I\right)^{m_k} \subset \mathcal{D}(A^*),$$

where m_k is the index of λ_k. We also define

$$A_+^* \in \mathcal{L}(\mathcal{H}_+), \quad A_+^* z \overset{\text{def}}{=} A^* z \quad (z \in \mathcal{H}_+),$$

$$\mathcal{D}(A_-^*) \overset{\text{def}}{=} \mathcal{H}_- \cap \mathcal{D}(A^*), \quad A_-^* z \overset{\text{def}}{=} A^* z \quad (z \in \mathcal{D}(A_-^*)).$$

Then, from Theorem 6.17 in [17, p.178], the spectra of A_+^* and of A_-^* are respectively Σ_1 and $\sigma(A^*) \setminus \Sigma_1$. In particular, there exists $\varepsilon > 0$ such that

$$\inf\left\{\operatorname{Re}\lambda, \lambda \in \sigma(A_-^*)\right\} > \sigma_1 + \varepsilon. \tag{2.16}$$

Then, we set

$$\phi_+(t) \overset{\text{def}}{=} \left(\mathbb{P}_1^* e^{-tA^*}\phi^0\right) e^{\sigma_1 t} = \left(e^{-tA_+^*}\mathbb{P}_1^*\phi^0\right) e^{\sigma_1 t}, \tag{2.17}$$

$$\phi_-(t) \overset{\text{def}}{=} \left((I - \mathbb{P}_1^*)\, e^{-tA^*}\phi^0\right) e^{\sigma_1 t} = \left(e^{-tA_-^*}(I - \mathbb{P}_1^*)\,\phi^0\right) e^{\sigma_1 t}. \tag{2.18}$$

Using that A_-^* is the infinitesimal generator of an analytic semigroup and (2.16) (see, for instance, [3, Proposition 2.9, p. 120], we deduce that for some constant $C > 0$,

$$\|B_1^*\phi_-(t)\|_{\mathcal{U}_1} + \|B_2^*\phi_-(t)\|_{\mathcal{U}_2} \leqslant C\left\|\phi^0\right\|_{\mathcal{H}} e^{-\varepsilon t} \quad (t \geqslant 0). \tag{2.19}$$

Moreover, there exist $\phi_{k,\ell} \in \mathcal{H}_+ \subset \mathcal{D}(A^*)$, $k \in K_1$ and $\ell \in \{0, \ldots, m_k\}$ such that

$$\phi_+(t) = \sum_{k \in K_1} e^{\imath \operatorname{Im}\lambda_k t} \sum_{\ell=0}^{m_k} t^\ell \phi_{k,\ell}. \tag{2.20}$$

Thus, we deduce from (2.14) that

$$\|B_1^*\phi_+(t)\|_{\mathcal{U}_1}^2 - \alpha(t)\,\|B_2^*\phi_+(t)\|_{\mathcal{U}_2}^2 \to 0 \quad \text{if } t \to \infty, \tag{2.21}$$

and using (2.4), we can develop the above expression as in [6]:

$$\begin{aligned}
&\|B_1^*\phi_+(t)\|_{\mathcal{U}_1}^2 - \alpha(t)\,\|B_2^*\phi_+(t)\|_{\mathcal{U}_2}^2 \\
&= \sum_{k,k'\in K_1} e^{\imath(\operatorname{Im}\lambda_k - \operatorname{Im}\lambda_{k'})t} \sum_{\ell=0}^{m_k}\sum_{\ell'=0}^{m_{k'}} t^{\ell+\ell'} \left[(B_1^*\phi_{k,\ell}, B_1^*\phi_{k',\ell'})_{\mathcal{U}_1} - (B_2^*\phi_{k,\ell}, B_2^*\phi_{k',\ell'})_{\mathcal{U}_2}\right] \\
&\qquad - \frac{1}{4\imath} \sum_{k,k'\in K_1} e^{\imath(\operatorname{Im}\lambda_k - \operatorname{Im}\lambda_{k'}+\omega)t} \sum_{\ell=0}^{m_k}\sum_{\ell'=0}^{m_{k'}} t^{\ell+\ell'} (B_2^*\phi_{k,\ell}, B_2^*\phi_{k',\ell'})_{\mathcal{U}_2} \\
&\qquad\qquad + \frac{1}{4\imath} \sum_{k,k'\in K_1} e^{\imath(\operatorname{Im}\lambda_k - \operatorname{Im}\lambda_{k'}-\omega)t} \sum_{\ell=0}^{m_k}\sum_{\ell'=0}^{m_{k'}} t^{\ell+\ell'} (B_2^*\phi_{k,\ell}, B_2^*\phi_{k',\ell'})_{\mathcal{U}_2}.
\end{aligned}$$

This implies that $\|B_1^*\phi_+(t)\|_{\mathcal{U}_1}^2 - \alpha(t)\,\|B_2^*\phi_+(t)\|_{\mathcal{U}_2}^2$ has the form $\sum_{j=1}^{N} p_j(t)e^{i\mu_j t}$ of Lemma 2.3. From this lemma, we deduce that all the polynomials $p_j = 0$ and in particular the ones associated with $\mu_j = 0$ and $\mu_j = \omega$. From (2.11), these polynomials are respectively

$$\sum_{k\in K_1} \left\|\sum_{\ell=0}^{m_k} t^\ell B_1^*\phi_{k,\ell}\right\|_{\mathcal{U}_1}^2 - \left\|\sum_{\ell=0}^{m_k} t^\ell B_2^*\phi_{k,\ell}\right\|_{\mathcal{U}_2}^2 = 0$$

and

$$-\frac{1}{4\imath} \sum_{k\in K_1} \left\|\sum_{\ell=0}^{m_k} t^\ell B_2^*\phi_{k,\ell}\right\|_{\mathcal{U}_2}^2 = 0$$

and we deduce that

$$\forall k \in K_1, \quad \forall \ell \in \{0,\dots,m_k\}, \quad B_1^*\phi_{k,\ell} = 0 \quad \text{and} \quad B_2^*\phi_{k,\ell} = 0.$$

The above relation combined with (2.17) and (2.20) implies that

$$B_1^* e^{-tA^*}\mathbb{P}_1^*\phi^0 = 0 \quad \text{and} \quad B_2^* e^{-tA^*}\mathbb{P}_1^*\phi^0 = 0 \quad (t \geqslant 0).$$

Thus, $\|\mathbb{P}_1^*\phi^0\|_{\mathcal{X}} = 0$ and we deduce that $\mathbb{P}_1^*\phi^0 = 0$. Therefore, $\phi^0 \in (\operatorname{Im}\mathbb{P}_1)^\perp$, where

$$\mathbb{P}_1 = \frac{1}{2\pi\imath} \oint_{\Gamma_1} (\lambda I - A)^{-1}\, d\lambda.$$

In particular,

$$\phi^0 \in \left(\bigoplus_{k\in K_1} \operatorname{Ker}(A - \lambda_k I)^{m_k}\right)^\perp.$$

By induction, we deduce that ϕ^0 is orthogonal to all the root vectors of A. Using (H3), we deduce that $\phi^0 = 0$. □

Let us finish this section by giving some ideas to adapt the above proof in order to show Theorem 1.4. The arguments are classical and thus we skip the details. We only consider the case $N = 2$ and the case of $B_j \in \mathcal{L}(\mathcal{U}_i, \mathcal{H})$, the other cases can be done similarly (note that we would need in particular to replace the space $\mathcal{H}$ by the space $[\mathcal{H}_{1-\gamma}, \mathcal{H}_{-\gamma}]_{1/2}$, see Remark 1.2 and Remark 1.5). Assume $z^0, z^1 \in \mathcal{H}$ and assume $\varepsilon > 0$. We want to obtain $u_1 \in L^2(0,T;\mathcal{U}_1)$, $u_2 \in L^2(0,T;\mathcal{U}_2)$ satisfying (1.5) such that the solution of (1.4) (with initial condition z^0) verifies

$$\|z(T) - z^1\|_{\mathcal{H}} \leqslant \varepsilon.$$

Then we replace the functional J of the proof of Theorem 1.1 by

$$J_\varepsilon(\varphi^0) \stackrel{\text{def}}{=} \frac{1}{2}\left\|\varphi^0\right\|_{\mathcal{X}}^2 + \left(z^0, e^{-TA^*}\varphi^0\right)_{\mathcal{H}} - \left(z^1, \varphi^0\right)_{\mathcal{H}} + \varepsilon\left\|\varphi^0\right\|_{\mathcal{H}} \quad (\varphi^0 \in \mathcal{H}).$$

Then we have that J_ε is convex, continuous in $\mathcal{H}$. Moreover, one can show that

$$\liminf_{\|\varphi^0\|_{\mathcal{H}}\to\infty} \frac{J_\varepsilon(\varphi^0)}{\|\varphi^0\|_{\mathcal{H}}} \geqslant \varepsilon$$

so that J_ε is coercive in $\mathcal{H}$ (see, for instance, [10]). Consequently, there exists a minimizer $\phi^0 \in \mathcal{H}$ of J_ε and the rest of the proof is the same as the proof of Theorem 1.1.

3 Examples

Here, we present some examples of application of Theorem 1.1. We focus in the case where A is not self-adjoint, and we refer the reader to [6] for several interesting examples in the self-adjoint case.

3.1 Coupled heat equations

Assume Ω is a bounded open set of $\mathbb{R}^d$, with regular boundary. Let us consider $m \geqslant 2$ and a nonempty open subset ω of Ω. We consider the following coupled system of heat equations:

$$\left\{\begin{array}{rcl} \partial_t y_j - d_j \Delta y_j + \sum_{k=1}^m (a_{j,k}\cdot\nabla y_k + b_{j,k} y_k) & = & 1_\omega u_j \quad \text{in } (0,T)\times\Omega, \\ y_j & = & 0 \quad \text{on } (0,T)\times\partial\Omega, \\ y_j(0,\cdot) & = & y_j^0 \quad \text{in } \Omega. \end{array}\right. \tag{3.1}$$

Here $d_j \in \mathbb{R}_+^*$, $a_{j,k} \in \mathbb{R}^d$, $b_{j,k} \in \mathbb{R}$ $(j,k \in \{1,\dots,m\})$. Using the Carleman estimates for the heat equation (see, [14] or [12]), it is well-known that the above system is null-controllable in $L^2(\Omega)^m$ with controls $u_j \in L^2(0,T;L^2(\omega))$ $(1 \leqslant j \leqslant m)$. In the literature, there are lot of works devoted to the possibility to decrease the number of controls while keeping the null-controllability property. One can refer for instance to [1] for a survey of results until 2011.

Applying Theorem 1.1, we deduce here the following result:

Corollary 3.1. *Assume $y^0 = \left(y_j^0\right)_{j=1,\dots,m} \in L^2(\Omega)^m$. Then for any $T > 0$, there exist $u_j \in L^2(0,T;L^2(\omega))$, $j = 1,\dots,m$ with*

$$\prod_{j=1}^m \left(\sum_{k\neq j} \|u_k(t,\cdot)\|_{L^2(\omega)}\right) = 0 \quad \textit{a.e. } t \in (0,T)$$

such that the solution y of (3.1) satisfies $y(T,\cdot) = 0$.

Proof. We set

$$\mathcal{H} \stackrel{\text{def}}{=} L^2(\Omega)^m, \quad \mathcal{D}(A) \stackrel{\text{def}}{=} \left[H^2(\Omega)\cap H_0^1(\Omega)\right]^m, \quad A = A_0 + A_1,$$

where

$$\mathcal{D}(A_0) \stackrel{\text{def}}{=} \mathcal{D}(A), \quad \mathcal{D}(A_1) \stackrel{\text{def}}{=} \left[H_0^1(\Omega)\right]^m = \mathcal{D}(A_0^{1/2}),$$

$$A_0 y \stackrel{\text{def}}{=} (-d_j\Delta y_j)_{j=1,\dots,m}, \quad A_1 y \stackrel{\text{def}}{=} \left(\sum_{k=1}^m [a_{j,k}\cdot\nabla y_k + b_{j,k} y_k]\right)_{j=1,\dots,m}.$$

We also set

$$\mathcal{U}_i \stackrel{\text{def}}{=} L^2(\omega), \quad B_j u_j \stackrel{\text{def}}{=} 1_\omega u_j e_j \quad (j = 1,\dots,m),$$

where $(e_1,\dots,e_m)$ is the canonical basis of $\mathbb{R}^m$. To check (H1), (H2) and (H3), we use Remark 1.3 since the eigenvalues μ_j of A_0 satisfy (1.9) from some $1 \leqslant p < \infty$ (by using Weyl formula, see, for instance, [2, Section 1.6]). □

3.2 The Oseen system

Assume Ω is a bounded open set of $\mathbb{R}^d$, with regular boundary. Assume Γ is a nonempty open subset of $\partial\Omega$. Let us consider the controllability of the Oseen system:

$$\left\{\begin{array}{rcl} \partial_t y - \nu\Delta y + \nabla p + \left(y^S\cdot\nabla\right)y + (y\cdot\nabla)\,y^S &=& 0 \quad \text{in } (0,T)\times\Omega,\\ \operatorname{div} y &=& 0 \quad \text{in } (0,T)\times\Omega,\\ y &=& u1_\Gamma \quad \text{on } (0,T)\times\partial\Omega,\\ y(0,\cdot) &=& y^0 \quad \text{in } \Omega. \end{array}\right. \tag{3.2}$$

Here $\nu > 0$ is the viscosity of the fluid and $y^S \in W^{1,\infty}(\Omega)$. Usually, for physical motivation, one has $d = 2, 3$ and y^S is a stationary state. Note moreover that for the controllability of (3.2), one can take $y^S \in L^\infty(\Omega)$ (see [11]), but to simplify the presentation we take y^S more regular. In the above system, the control u is located at the boundary ; in the case of distributed controls, there are several works devoted to the controllability of the Stokes or the Navier-Stokes system. Let us quote for instance [11, 7, 16]. Some of these studies are devoted to the case of controls with some vanishing components: see, for instance, [13], [8] and [4]. Let us also quote [9] where the authors obtain the local null controllability of the Navier-Stokes system in dimension 3 with a control having two vanishing components. Their method is quite different to the previous works and is based on results of Gromov.

Let us define

$$\mathcal{H} \overset{\text{def}}{=} \left\{y \in L^2(\Omega)^d\ ;\ \operatorname{div} y = 0, \quad y\cdot n = 0 \quad \text{on } \partial\Omega\right\}.$$

We denote by $P : L^2(\Omega)^d \to \mathcal{H}$ the orthogonal projection (the Leray projector). Due to the incompressibility condition, the controls have to satisfy the condition

$$\int_\Gamma u\cdot n\, d\gamma = 0 \quad \text{in } (0,T). \tag{3.3}$$

Applying Theorem 1.1, we deduce here the following result:

Corollary 3.2. *Assume $y^0 \in \mathcal{H}$. Then for any $T > 0$, there exists $u \in L^2(0,T;L^2(\Gamma))^d$ satisfying* (3.3) *and*

$$\prod_{j=1}^d \left(\sum_{k\neq j} \|u_k(t,\cdot)\|_{L^2(\Gamma)}\right) = 0 \quad a.e.\ t \in (0,T)$$

such that the solution y of (3.2) *satisfies $Py(T,\cdot) = 0$. In particular, taking $u \equiv 0$ in (T,∞), we deduce that the solution y of* (3.2) *satisfies $y \equiv 0$ in (T,∞).*

Remark 3.3. As in [6], instead of using the canonical basis of $\mathbb{R}^d$ to decompose u, we could use an orthonormal basis $(\tau_1(x),\ldots,\tau_{d-1}(x),n(x))$, $x\in\Gamma$, where n the normal to $\partial\Omega$ (so that $\tau_j(x)$ is a tangential vector of $\partial\Omega$ at the point x). The proof below would be exactly the same and we would obtain a controllability result for which at each instant of time, the control is either tangential or normal.

Proof of Corallry 3.2. We consider the following operators

$$\mathcal{D}(A) \overset{\text{def}}{=} \left\{y \in \left[H^2(\Omega)\cap H^1_0(\Omega)\right]^d\ ;\ \operatorname{div} y = 0\right\}, \quad A \overset{\text{def}}{=} A_0 + A_1,$$

where

$$\mathcal{D}(A_0) \overset{\text{def}}{=} \mathcal{D}(A), \quad \mathcal{D}(A_1) \overset{\text{def}}{=} \left\{y \in \left[H^1_0(\Omega)\right]^d\ ;\ \operatorname{div} y = 0\right\} = \mathcal{D}(A_0^{1/2}),$$

$$A_0 y \overset{\text{def}}{=} P\left(-\nu\Delta y\right), \quad A_1 y \overset{\text{def}}{=} P\left(\left(y^S\cdot\nabla\right)y + (y\cdot\nabla)\,y^S\right).$$

To introduce the control operator, we write

$$\mathcal{U}_i \overset{\text{def}}{=} \left\{u_i \in L^2(\Gamma)\ ;\ \int_\Gamma u_i n_i\, d\gamma = 0\right\},$$

and we consider the Dirichlet operators $D_i : \mathcal{U}_i \to \mathcal{H}$ defined by $w \stackrel{\text{def}}{=} D_i u_i$ is the solution of

$$\begin{cases} \lambda_0 w - \nu \Delta w + \nabla \pi + \left(y^S \cdot \nabla\right) w + (w \cdot \nabla)\, y^S & = & 0 \quad \text{in } \Omega, \\ \operatorname{div} w & = & 0 \quad \text{in } \Omega, \\ w & = & u_i e_i 1_\Gamma \quad \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

for some $\lambda_0 > 0$ large enough. Then we define $B_i : \mathcal{U}_i \to \mathcal{D}(A^*)'$ by $B_i \stackrel{\text{def}}{=} (\lambda_0 I - A) P D_i$. Following [20], one can write (3.2) under the form

$$\begin{cases} Py' + APy & = & \displaystyle\sum_{j=1}^{3} B_j u_j \quad \text{in } (0,T), \\ Py(0) & = & Py^0, \end{cases} \tag{3.5}$$

and

$$(I-P)y = \sum_{j=1}^{3} (I-P) D_j u_j \quad \text{in } (0,T). \tag{3.6}$$

To check (H1), (H2) and (H3), we use Remark 1.3 since the eigenvalues μ_j of A_0 satisfy (1.9) from some $1 \leqslant p < \infty$ (see [18]). □

3.3 The Boussinesq system

Assume Ω is a bounded open set of $\mathbb{R}^3$, with regular boundary. Assume ω is a nonempty open subset of Ω. Let us consider the controllability of the Boussinesq system:

$$\begin{cases} \partial_t y - \nu \Delta y + \nabla p & = & u_1 1_\omega e_1 + \theta e_3 \quad \text{in } (0,T) \times \Omega, \\ \partial_t \theta - \Delta\theta + y \cdot \nabla \theta^S & = & u_2 1_\omega \quad \text{in } (0,T) \times \Omega, \\ \operatorname{div} y & = & 0 \quad \text{in } (0,T) \times \Omega, \\ (y,\theta) & = & 0 \quad \text{on } (0,T) \times \partial\Omega, \\ (y,\theta)(0,\cdot) & = & (y^0, \theta^0) \quad \text{in } \Omega, \end{cases} \tag{3.7}$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. Here $\nu > 0$ is the viscosity of the fluid and $\theta^S \in W^{3,\infty}(\Omega)$. In [5], the author obtained that the above system is null-controllable for any $T > 0$. We keep the same notation for $\mathcal{H}$ as in the previous section. Applying Theorem 1.1, we deduce here the following result:

Corollary 3.4. *Assume $(y^0, \theta^0) \in \mathcal{H} \times L^2(\Omega)$. Then for any $T > 0$, there exists $(u_1, u_2) \in L^2(0,T; L^2(\omega))^2$ satisfying*

$$\|u_1(t,\cdot)\|_{L^2(\omega)} \|u_2(t,\cdot)\|_{L^2(\omega)} = 0 \quad \text{a.e. } t \in (0,T)$$

such that the solution (y,θ) of (3.7) satisfies $(y,\theta)(T,\cdot) = 0$.

The proof is similar to the proofs in the previous sections and we skip it.

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Received: November 5, 2021.
Accepted: March 22, 2022.