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CONFORMAL VECTOR FIELDS ON FINSLER SQUARE METRICS VIA NAVIGATION DATA

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Abstract In this paper, we study the Finsler manifold with square metric and navigation data (h, W). We proved the necessary and sufficiant condition for conformal vector field satisfies the system of PDEs. Further, we find the conformal vector fields on Finsler manifold with square metric of weakly isotropic flag curvature.

1 Introduction

There is a special and important class of Finsler metrics in Finsler geometry which can be expressed in the form $F = \alpha\phi(\frac{\beta}{\alpha})$, where α is a Riemannian metric and β is a 1-form and $\phi = \phi(s)$ is a C^∞ positive function on an open interval. We call this class of metrics as (α, β)-metrics. When $\phi = 1 + s$, the Finsler metric $F = \alpha + \beta$ is called Randers metric. When $\phi = 1 + s^2 + 2s$, the Finsler metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ is called Square metric. Square metrics were first introduced by L. Berwald in 1929 (see [8, 7]).

Zermelo's navigation problem is to determine the shortest time paths for an object with constant internal force in $\mathbb{R}^2$ under the influence of an external force. Later, Z. Shen discussed the navigation problem in a more general setting (see [2, 3]).

The conformal vector fields on a manifold M are vector fields induced by a local one-parameter group of conformal transformations of M. Z. Shen and Q. Xia were studied conformal vector fields on Randers manifolds of weakly isotropic flag curvature (see [6]). X. Cheng, Li Yin and T. Li are completely determined conformal vector fields on conic Kropina manifolds via navigation data of weakly isotropic flag curvature (see [4]). In this paper, we give several equivalent characterizations for conformal vector fields on a Finsler space and establish the relationships between the curvatures of a given Finsler metric F. Finally, we determine the conformal vector fields on a Finsler manifold with square metric via navigation data of weakly isotropic flag curvature. Above results are established in the section-3 and section-4.

2 Preliminaries

Definition 2.1. [1] A Minkowski norm on vector field V is a continuous function is denoted by $F : V \longrightarrow [0, \infty)$, such that it satisfies the following properties:

(i) F is smooth on $V \backslash \{0\}$,

(ii) F is 1-homogeneous function,

(iii) $\forall\ y = y^i e_i \in V \backslash \{0\}$, the symmetric bilinear form $g_y : V \times V \longrightarrow R$ is defined by

$$(u, v) \longmapsto \frac{1}{2}\frac{\partial^2}{\partial s \partial t}(F^2(y + su + tv))/_{t=s=0}$$

is positive definite.

Definition 2.2. [1] By a Finsler space, we mean a triple $F^n = (M, D, L)$, where M denotes n-dimensional differential manifolds, D is an open subset of tangent vector bundle TM endowed with the differentiable structure of the manifold TM and $L : D \longrightarrow R$ is a differentiable mapping having the following properties:

(i) $L(x, y) > 0$, for $(x, y) \in D$,

(ii) $L(x, \lambda y) = |\lambda| L(x, y)$, for any $(x, y) \in D$ and $\lambda \in R$, such that $(x, \lambda y) \in D$,

(iii) The d-tensor field $g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(x, y)$, $(x, y) \in D$, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$, is non-degenerate on D.
The metric tensor and angular metric tensor of Finsler space are given by
$g_{ij} = \dot{\partial}_i\dot{\partial}_j \frac{L^2}{2}$, $\quad g^{ij} = (g_{ij})^{-1}$ and $\quad h_{ij} = g_{ij} - l_i l_j$.
where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$, $\quad l_i = \dot{\partial}_i L$ and $\quad l^i = y^i/L$.

Let (M, F) be an n-dimensional smooth Finsler manifold. Let $\varphi : M \longrightarrow M$ be a diffeomorphism on M and $\varphi_* : T_xM \longrightarrow T_{\varphi(x)}$ be the tangent map at the point of x. φ is called a conformal transformation if there is a smooth scalar function $\rho = \rho(x)$ on M, such that

$$F(\varphi(x), \varphi_*(y)) = e^{2\rho(x)} F(x, y), \tag{2.1}$$

where $y \in T_xM$. $\rho = \rho(x)$ is called a conformal factor of φ.

A vector field V on a Finsler manifold (M, F) is called a conformal vector field with a conformal factor $\rho = \rho(x)$ if the one-parameter transformation φ_t generated by V is a conformal transformation, that is

$$F(\varphi_t(x), (\varphi_t)_*(y)) = e^{2\rho_t(x)} F(x, y), \quad \forall x \in M, y \in T_xM, \tag{2.2}$$

where $\rho_t(x) := \int_0^t \sigma(\varphi_s(x))ds$. In this case, it is easy to see that $\sigma(x) = \frac{d\rho_t(x)}{dt}|_{t=0}$ and $\rho_0(x) = 0$. Let Φ_t be a lifting of φ_t onto the tangent bundle TM, that is

$$\Phi_t(x, y) := (\varphi_t(x), (\varphi_t)_*(y)). \tag{2.3}$$

Obviously, $\{\Phi_t\}$ is also a one-parameter transformation group on TM. Then (2.2) is equivalent to the following,

$$\Phi_t^* F = e^{2\rho_t(x)} F. \tag{2.4}$$

Then the vector field V is called a homothetic vector field on M if ρ is constant and V is called a Killing vector field if $\rho = 0$. Recently, Z.Shen and Q.Xia have obtained the equivalent conditions that V is a conformal vector field on a Finsler manifold (M, F).

It is known that a Finsler metric is a Randers metric if and only if it is a solution of Zermelo's navigation problem on a manifold M with a Riemannian metric $h = \sqrt{h_{ij}(x)y^iy^j}$ under the influence of an external force field $W = W^i(x)\frac{\partial}{\partial x^i}$ with $\parallel W \parallel_h < 1$. Here, $\parallel W \parallel_h$ denotes the norm of W with respect to h. The condition $\parallel W \parallel_h < 1$ is essential for obtaining a positive definite Randers metric by Zermelo's navigation problem. Let $h = \sqrt{h_{ij}(x)y^iy^j}$ be a Riemannian metric and $W = W^i(x)\frac{\partial}{\partial x^i}$ be a vector field with $\parallel W \parallel_h < 1$. Then the solution of the following Zermelo's navigation problem

$$h\left(x, \frac{y}{F(x, y)} - V_x\right) = 1, \quad y \in T_xM, \tag{2.5}$$

that is,

$$\sqrt{h_{ij}(x)\left(\frac{y^i}{F} - W^i\right)\left(\frac{y^j}{F} - W^j\right)} = 1,$$

is a function, where $W_i = h_{ij}W^j$, $W_0 = W_iy^i$ and can be expressed as

$$F = \sqrt{h^2 + W_0^2} - W_0 \tag{2.6}$$

Then we can get (h, W) with $\parallel W \parallel_h = 1$ from (2.6) such that F is determined from (2.5) by h and W. Thus there is an one-to-one correspondence between a square metric F and a pair

(h, W) with $\| W \|_h = 1$ and it is easy to see that a square metric can be regarded as the limit of the navigation problem for Randers metrics when $\| W \|_h \to 1$ (see [2, 4]). We also call the pair (h,W) the navigation data of a square metric F.

The flag curvature in Finsler geometry is a natural analogue of sectional curvature in Riemannian geometry. For a Finsler manifold (M, F), the flag curvature $K = K(p, y)$ of F is a function of flag $P \in T_xM$ and flagpole $y \in T_xM$ at x with $y \in P$. We say Finsler matric F is of weakly isotropic flag curvature K if $K = \frac{3\theta}{F} + \sigma$, where $\sigma = \sigma(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M.

Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a square metric on a Finsler manifold M of dimention $n \geq 3$. Suppose that F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \sigma$. According to Schur lemma, h is of constant curvature $\mu(x)$(μ = constant, when $n \geq 3$) and W is conformal with respect to h,

$$W_{i;j} + W_{j;i} = 0, \tag{2.7}$$

where ";" denotes the covariant derivative with respect to Levi-Civita connection of h. In this case, $K_F = \sigma = \mu \geq 0$ and $\theta = 0$. Further, we can completely determine the local structure of square metrics of weakly isotropic flag curvature.

Lemma 2.3. *[6] Let (M, F) be an n-dimensional smooth Finsler manifold and V be a vector field on M. Then, in local coordinates, the following conditions are equivalent:*

(i) $V = V^i \frac{\partial}{\partial x^i}$ is a conformal vector field on (M, F) with a conformal factor $\rho = \rho(x)$;

(ii) $\frac{\partial g_{ij}}{\partial x^p} V^p = g_{pj} \frac{\partial V^p}{\partial x^i} + g_{ip} \frac{\partial V^p}{\partial x^j} + 2C_{ijp} \frac{\partial V^p}{\partial x^q} y^q = 4\rho g_{ij}$;

(iii) $V_{i|j} + V_{j|i} + 2C^p_{ij} V_{p|q} y^q = 4\rho g_{ij}$;

(iv) $\frac{\partial F}{\partial x^i} V^i + \frac{\partial F}{\partial y^i} \frac{\partial V^i}{\partial x^j} y^j = 2\rho F$,

where g_{ij} and C_{ijp} are the coefficients of the fundamental tensor g and Cartan torsion C respectively, $C^p_{ij} = g^{pq} C_{ijq}$. $V_i = g_{ij} V^j$ and "$|$" denotes the covariant derivative with respect to Chern connection of F.

Now, let $X_V = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i}$ be a vector field on TM. Actually, X_V is the induced tangent vector field of the one-parameter transformation group $\{\Phi_t\}$ on TM. From condition (4) of above Lemma 2.1, it is easy to see that a vector field $V = V^i \frac{\partial}{\partial x^k}$ on M is a conformal vector field on Finsler manifold (M, F) with conformal factor $\rho = \rho(x)$ if and only if $X_V(F^2) = 4\rho F^2$.

3 Conformal Vector Field on Square Metric

In this section we investigate the explicit expression of conformal vector field on square metric.

Theorem 3.1. *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a square metric on a Finsler manifold M with navigation data (h, W). Then a vecor field V on (M, F) is a conformal vector field with conformal factor $\lambda = \lambda(x)$ and $s = \frac{\beta}{\alpha}$ if and only if V satisfies the following system*

$$V_{i;j} + V_{j;i} = 4\lambda(1+s)^2 h_{ij}, \tag{3.1}$$

$$V^i b_{j;i} + b^i V_{i;j} = \frac{\lambda s(1+3s)}{b_j}. \tag{3.2}$$

In Theorem (3.1), the conditions (3.1) and (3.2) for a vector field on a square metric to be a conformal vector field are similar to the conditions in (3.3) of Proposition 3.1 in [6] for conformal vector fields on Randers manifolds. However, their proofs are quite different.
We need the following lemma to prove the main Theorem (3.1).

Lemma 3.2. *[5] Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be square metric on a Finsler manifold M. Then a vector field V on (M, F) is a conformal vector field with conformal factor $\rho = \rho(x)$ and $s = \frac{\beta}{\alpha}$ if and only if V satisfies the following*

$$V_{i;j} + V_{j;i} = 4\rho(1+s)^2 a_{ij}, \tag{3.3}$$

$$V^i b_{j;i} + b^i V_{i;j} = \frac{s(\rho(1+s) - (1-s)\lambda)}{b_j}. \tag{3.4}$$

where we use (a_{ij}) to raise and lower the indices of V and β, ";" denotes the covariant derivative with respect to the Levi-Civita connection of Riemannian metric α and $\lambda = \lambda(x)$ is a scalar function on M.

Now, we prove our main Theorem (3.1).

Proof. of Theorem (3.1): It is easy to check that (3.1) and (3.2) mean that V is a conformal vector field by Lemma (3.2). Hence, we just need to prove the necessity. By the assumption, V is a conformal vector field on square metric (M, F) with conformal factor $\lambda(x)$. By a direct computation, we have

$$X_V(\alpha^2) = 2V_{i;j}y^i y^j, \quad X_V(\beta) = (V^i b_{j;i} + b^i V_{i;j})y^j,$$

Then, by Lemma (3.2), we have

$$X_V(\alpha^2) = 4\rho(1+s)^2 a_{ij}, \qquad X_V(\beta) = \frac{s(\rho(1+s) - (1-s)\lambda)}{b_j}.$$

Since square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ with navigation data (h,W) and the above identities that

$$\lambda^2[X_V(\alpha^2) - 4\rho(1+s)^2 a_{ij}] = -h^2 X_V(\lambda^2), \tag{3.5}$$

$$\lambda^2 \left[X_V(W_0) - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_0}\right] = -W_0^2 X_V(\lambda^2). \tag{3.6}$$

By (3.5) and (3.6), we get

$$h^2 \left[X_V(W_0) - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_0}\right] = -W_0[X_V h^2 - 4\rho(1+s)^2 h^2]. \tag{3.7}$$

On the other hand, a direct computation shows

$$X_V(h^2) = 2V_{i;j}y^i y^j, \quad X_V(W_0) = (V^i W_{j;i} + W^i V_{i;j})y^j, \tag{3.8}$$

where ";" denotes the covariant derivative with respect to Levi-Civita connection of Riemannian metric h. Plugging (3.7) into (3.6) yields

$$h^2 \left[V^i W_{j;i} y^j + W^i V_{i;j} y^j - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_0}\right] = 2W_0[V_{i;j}y^i y^j - 2\rho(1+s)^2 h_{ij} y^i y^j]. \tag{3.9}$$

Taking the derivatives with respect to y^j, y^k, y^l on both sides of (3.7) respectively, we get the following

$$\begin{aligned} &h_{jk}\left[V^i W_{l;i} + W^i V_{i;l} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_l}\right] + h_{jl}\left[V^i W_{k;i} + W^i V_{i;k} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_k}\right] \\ &+ h_{kl}\left[V^i W_{j;i} + W^i V_{i;j} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_j}\right] = W_j(V_{k;l} + V_{l;k} - 4\rho(1+s)^2 h_{lk}) \\ &+ W_k(V_{j;l} + V_{l;j} - 4\rho(1+s)^2 h_{jl}) + W_l(V_{k;j} + V_{j;k} - 4\rho(1+s)^2 h_{kj}). \end{aligned} \tag{3.10}$$

At any point $x \in M$, take an orthonormal frame e_i with respect to h on T_xM such that $e_1 = W$. Then $h_{ij} = \delta_{ij}$, $W^i = W^1\delta_{1i}$ and $W_i = W_1\delta_{1i}$. Taking $k = l \neq j$ in (3.10), we can get

$$V^i W_{j;i} + W^i V_{i;j} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_j} = 2W_j(V_{l;l} - 2\rho(1+s)^2) + 2W_l(V_{l;j} + V_{j;l}), l \neq j. \tag{3.11}$$

Taking $k = j = l$ in (3.10), we get

$$V^i W_{j;i} + W^i V_{i;j} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_j} = 2W_j(V_{j;j} - 2\rho(1+s)^2), \quad 1 \leq j \leq n. \tag{3.12}$$

Case I: dim$M = n \leq 2$. Assuming $n = 2$ and letting $j = 1$, $l = 2$ in (3.11) and $j = 1$ in (3.12) respectively, we obtain

$$V^i W_{1;i} + W^i V_{i;1} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_1} = 2W_1(V_{2;2} - 4\rho(1+s)^2) = 2W_1(V_{1;1} - 2\rho(1+s)^2). \tag{3.13}$$

which implies that $V_{1;1} = V_{2;2}$. Put $\tau(x) = \frac{1}{2}(V_{1;1} - 2(1+s)^2\rho)$. The above identies become

$$V^i W_{1;i} + W^i V_{i;1} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_1} = 4W_1\tau(x). \tag{3.14}$$

Similarly, letting $j = 2$ and $l = 1$ in (3.11) and $j = 2$ in (3.12), we obtain

$$V^i W_{2;i} + W^i V_{i;2} = V_{1;2} + V_{2;1}. \tag{3.15}$$

Hence, by $V_{1;1} = V_{2;2} = 2(\tau + \lambda)$ and (3.14), we obtain

$$V_{i;j} + V_{j;i} = 4(\tau + \lambda)h_{ij}, \tag{3.16}$$

for $1 \leq i, j \leq 2$. By (3.14) and (3.15), we can get

$$V^i W_{j;i} + W^i V_{i;j} = 2(2\tau + 2\lambda - \rho)W_j = \frac{s((2\tau + 2\rho)(1+s) - (1-s)\lambda)}{W_j}, \tag{3.17}$$

for $1 \leq i, j \leq 2$. In the case of $n = 1$, we can get (3.16) and (3.14) from (3.10) obviously.
Case II : dim $M = n \geq 3$. Since the left-hand sides of (3.11) is independent of the index l, for each j, we can take $k \neq j$ and $l \neq j$, such that we have

$$W_j(V_{l;l} - 2\rho(1+s)^2) + W_l(V_{l;j} + V_{j;l}) = W_j(V_{k;k} - 2\rho(1+s)^2) + W_k(V_{k;j} + V_{j;k}). \tag{3.18}$$

Letting $j \neq 1$, $k \neq 1$ and $l = 1$ in (3.18), we obtain

$$V_{1;j} + V_{j;1} = 0, \quad j \neq 1. \tag{3.19}$$

Letting $j \neq 1$, $l = 1$ in (3.11) and by use of (3.19), we obtain

$$V^i W_{j;i} + W^i V_{i;j} = 0, \quad j \neq 1. \tag{3.20}$$

Taking $j = 1$ in (3.12) yields

$$V^i W_{1;i} + W^i V_{i;1} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_1} = 2W_1(V_{j;j} - 2\rho(1+s)^2).$$

Now, we put $\tau(x) = \frac{1}{2}((V_{1;1} - 2\rho(1+s)^2))$. Then the identity above can be written as

$$V^i W_{1;i} + W^i V_{i;1} - \frac{s(\rho(1+s) - (1-s)\lambda)}{W_1} = 4\tau W_1. \tag{3.21}$$

By (3.20) and (3.21), we get

$$V^i W_{j;i} + W^i V_{i;j} = \frac{s((2\tau + 2\rho)(1+s) - (1-s)\lambda)}{W_j}, \tag{3.22}$$

in the original coordinate system for $1 \leq i, j \leq n$. Plugging (3.22) into (3.10) and letting $l = 1$, $j \neq 1$, $k \neq 1$, we have

$$V_{k;j} + V_{j;k} = 4(\tau + \rho)h_{ij}, \tag{3.23}$$

in the original coordinate system for $1 \leq i, j \leq n$. From (3.16) and (3.17), we see that (3.22) and (3.23) hold for any n.
In the following, we would like to determine τ. By $\| W \|_h^2 = W^i W_i = 1$ we can get $W_{i;j} W^i = 0$. Contracting (3.22) with W^j yields

$$V_{i;j} W^i W^j = s((2\tau + 2\rho)(1+s) - (1-s)\lambda), \tag{3.24}$$

On the other hand, contracting (3.23) with $W^i W^j$, we have

$$V_{i;j} W^i W^j = 2(\tau + \rho). \tag{3.25}$$

Comparing (3.25) with (3.24), we get $\tau = \lambda - \rho$. Then, by (3.23) and (3.22), we get (3.1) and (3.2). □

4 The Conformal Vector Fields on Square Metrics of Weakly Isotropic Flag Curvature

In this section, we will give the explicit expressions of conformal vector fields on a square metric (M, F) of weakly isotropic flag curvature via solving the system of PDEs (3.1) and (3.2).

Theorem 4.1. *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a square metric on a Finsler manifold M of dimension $n \geq 3$ given by (2.6) with navigation data (h, W). Suppose V is a conformal vector field on (M, F) with conformal factor $\lambda(x)$, and F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \sigma$. Then in some local co-ordinate system for the local standard expression of F, V is given by one of the following,*

(i) $V = 2\epsilon x(1 + s)^2 + d$, where ϵ is constant and d is non-zero constant vector in R^n with $|d| = 1$.

(ii) $V = Qx + \mu\langle d, x\rangle x + d$, where μ is positive constant, d is non-zero constant vector in R^n with $|d| = 1$ and Q is a skew symmetric matrix with $Qd = 0$ and $Q^TQ + \mu dd^T = \mu E$, T denote the transpose of matrix and E is an identity matrix. In this case $\lambda = \rho = 0$ and $K = 0$.

For the above purpose, we first give the following lemma.

Lemma 4.2. *Let (M, h) be a Riemannian manifold and V be a conformal vector field on (M, h) with conformal factor $\lambda(x)$ defined by (3.1). Suppose $\bar{W}(x)$ is a general solution of the equation (2.7) and $T_0(x)$ is a special solution of the equation $T_{i;j} + T_{j;i} = 4\lambda(1+s)^2 h_{ij}$. Then the general solution of (3.1) is given by $V = \bar{W}(x) + T_0(x)$, where we raise and lower the indices of V, $\bar{W}$ and T by $\alpha = (h_{ij})$, "; " denotes the covariant derivative with respect to the Levi-Civita connection of Riemannian metric h.*

Proof. Suppose the general solution of equation (3.1) is given by

$$V = \bar{W}(x) + T_0(x). \tag{4.1}$$

Then $\bar{W}(x)$ is the general solution of the equation (2.7) and $T_0(x)$ is the special solution of equation

$$T_{i;j} + T_{j;i} = 4\lambda(1 + s)^2 h_{ij}. \tag{4.2}$$

By Theorem (3.1), we know that, if V is a conformal vector field on square metric (M, F), then V must be a conformal vector field of h, that is, V satisfies (3.1).
Now substituting (2.7) and (4.2) into (4.1) we get (3.1), that is $V_{i;j} + V_{j;i} = 4\lambda(1 + s)^2 h_{ij}$. □
When h is of constant sectional curvature, we have the Lemma 5.2.9 in [3]. Now, we prove the main Theorem (4.1) in this section.
Proof. Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a square metric on a manifold M of dimensional $n \geq 3$ given by (2.6) with navigation data (h,W). Assume that F is of weakly isotropic flag curvature. $K_F = \frac{3\theta}{F} + \sigma(x)$. Then, h is of nonnegative constant sectional curvature μ and W is a Killing vector field. Hence, there is a local coordinate system, in which h can be expressed by

$$h = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle y, x\rangle^2}}{1 + \mu|x|^2}, \tag{4.3}$$

and $W = (W^i)$ is determined by

$$W^i = q^i_k x^k + \mu\langle d, x\rangle x^i + d^i, \tag{4.4}$$

where $Q = (q^i_j)$ is a skew symmetric matrix with $Qd = 0$ and $Q^TQ + \mu d^Td = \mu E$ and $d = (d^i) \in R^n$ is a nonzero constant vector with $|d| = 1$. Here we have unified the conclusion (i) and (ii) in Lemma (2.3). Actually, we can prove that $Q = 0$ when $\mu = 0$(see case I below). It means that $W = d$ is a nonzero constant vector field when $\mu = 0$. Denote $\xi = \xi(x) := \sqrt{1 + \mu|x|^2}$. By (4.3) we have

$$h_{ij} = \xi^{-2}\delta_{ij} - \mu\xi^{-4}x^i x^j, \qquad h^{ij} = \xi^2(\delta^{ij} + \mu x^i x^j). \tag{4.5}$$

Then, we have

$$W_i := h_{ij}W^j = \frac{q_{ij}x^j + d^i}{1+\mu|x|^2}, \tag{4.6}$$

Further, the connection coefficents Γ^k_{ij} of h is given by

$$\Gamma^k_{ij} = -\frac{\mu(x_i\delta^k_j + x_j\delta^k_i)}{1+\mu|x|^2}. \tag{4.7}$$

Hence, by (4.5) and (4.6), we get

$$W_{i;j} = \xi^{-2}q_{ij} + \mu\xi^{-4}(x_iq_{jk}x^k - x_jq_{ik}x^k + d_jx_i - d_ix_j), \tag{4.8}$$

where we use δ_{ij} to raise and to lower the indices of the vectors x, d in R^n, and $q_{ij} = \delta_{ik}q^k_j$. It is easy to see that (W_i) determined by (4.8) is the general solution of $W_{i;j} + W_{j;i} = 0$.
Further, by (4.6) and (4.5), we have

$$\|W\|^2_h = h^{ij}W_iW_j = \frac{\langle Qx, Qx\rangle + 2\langle Qx, d\rangle + \mu\langle d, x\rangle^2 + \langle d, d\rangle}{1+\mu|x|^2}.$$

Since $\|W\|_h = 1$, the above is equivalent to

$$1 + \mu|x|^2 = \langle Qx, Qx\rangle + 2\langle Qx, d\rangle + \mu\langle x, d\rangle^2 + \langle d, d\rangle. \tag{4.9}$$

Replacing x with $-x$ in (4.9) and adding the obtained equation to (4.9) yield

$$\langle Qx, d\rangle = 0, \tag{4.10}$$

$$\mu|x|^2 = \langle Qx, Qx\rangle + \mu\langle x, d\rangle^2. \tag{4.11}$$

In the following, we focus to determine V and the related coefficients. By the assumption, V is conformal vector field with conformal factor λ on square metric (M, F) satisfying (3.1), i.e., $T_{i;j} + T_{j;i} = 4\lambda(1+s)^2h_{ij}$. Note that h is of nonnegative constant sectional curvature μ. By Lemma 5.2.9 in [3], the conformal factor $\lambda = \lambda(x)$ satisfies $\lambda_{;i;j} + \mu\lambda h_{ij} = 0$ and we can get a general formula for λ which is given by

$$\lambda = \frac{\epsilon + \langle a, x\rangle}{\sqrt{1+\mu|x|^2}}, \tag{4.12}$$

where ϵ is a constant and $a = (a^i)$ is a constant column vector in R^n.
Further, we construct a special solution of equation $T_{i;j} + T_{j;i} = 4\lambda(1+s)^2h_{ij}$.
Let

$$T_i = 2\xi^{-2}\lambda x^i(1+s)^2 - 2\xi^{-2}(1+\xi)^{-1}|x|^2a^i. \tag{4.13}$$

By (4.7) and (4.13), we get

$$T_{i;j} = 2\lambda(1+s)^2a_{ij} + 2\xi^{-4}(a_jx_i - a_ix_j). \tag{4.14}$$

Obviously, T is a special solution of the equation $T_{i;j} + T_{j;i} = 4\lambda(1+s)^2h_{ij}$.
From Lemma 5.2.9 in [3], $V_i = W_i + T_i$ is the general solution of equation (3.1). Moreover, we obtain

$$V^i = 2(\epsilon\xi + \langle a, x\rangle)x^i(1+s)^2 - 2(1+\xi)^{-1}|x|^2a^i + (q^i_kx^k + \mu\langle d, x\rangle x^i + d^i). \tag{4.15}$$

Now, we are going to determine λ, ϵ and vector a and matrix Q in (4.15). By (4.5) and (4.13), we obtain

$$T^i = 2[\epsilon\xi + \langle a, x\rangle]x^i(1+s)^2 - 2(1+\xi)^{-1}|x|^2a^i. \tag{4.16}$$

Substituting $V^i = W^i + T^i$ and $V_i = W_i + T_i$ into (3.2) yield

$$T^iW_{j;i} + W^iT_{i;j} = \frac{\lambda s(1+3s)}{W_j}. \tag{4.17}$$

Substituting (4.4), (4.6), (4.8), (4.14) and (4.16) into (4.17), we obtain

$$2[(1+\xi)\mu\epsilon\langle d,x\rangle x-(1+\xi)\langle a,Qx+d\rangle x+\xi(1+\xi)\langle d,x\rangle a-\xi|x|^2Qa$$
$$+(1+\xi)(\epsilon+\langle a,x\rangle)Qx-(1+\xi)\mu\epsilon|x|^2d-\mu\langle a,x\rangle|x|^2d](1+s)^2$$
$$-\frac{(\epsilon+\langle a,x\rangle)s(1+3s)\xi^5(1+\xi)}{(Qx+d)}=0. \tag{4.18}$$

Case I: μ = 0. In this case, $\xi=1$. By (4.11), we get $Q=0$. Thus, (4.17) is reduced to

$$2(\langle d,x\rangle a-\langle a,d\rangle x)(1+s)^2-2(\epsilon+\langle a,x\rangle)s(1+3s)=0. \tag{4.19}$$

Multiplying d^2 and d_T on both sides of (4.18), we have

$$2(\epsilon+\langle a,x\rangle)s(1+3s)=0. \tag{4.20}$$

Plugging (4.20) into (4.19) yields

$$(\langle d,x\rangle a-\langle a,d\rangle x)2(1+s)^2=0.$$

Since the above identity holds for any $x\in R^n$ and d is a nonzero constant vector, we assert that $a=0$. By (4.15), we get

$$V=d+2\epsilon x(1+s)^2.$$

In this case, by (4.12), $\lambda=\epsilon$ and V is a homothetic vector field. Furthermore, $h=|y|$, $W=d$ is a nonzero constant vector with $|d|=1$.

Case II: $\mu>0$. In this case, $\xi=\sqrt{1+\mu|x|^2}$ is an irrational function. Then (4.18) can be written as in the following form

$$A(x)\xi^6+B(x)\xi^5+C(x)\xi^2+D(x)\xi+E(x)=0, \tag{4.21}$$

wehre

$$\begin{aligned}
A(x) &= -\frac{(\epsilon+\langle a,x\rangle)s(1+3s)}{(Qx+d)}=0.\\
B(x) &= -\frac{(\epsilon+\langle a,x\rangle)s(1+3s)}{(Qx+d)}=0.\\
C(x) &= 2\langle d,x\rangle a(1+s)^2,\\
D(x) &= 2\mu\epsilon\langle d,x\rangle x(1+s)^2-2\langle a,Qx+d\rangle x(1+s)^2+2\langle d,x\rangle a(1+s)^2\\
&\quad -2|x|^2Qa+2(\epsilon+\langle a,x\rangle)Qx(1+s)^2-2\mu\epsilon|x|^2d(1+s)^2,\\
E(x) &= 2\mu\epsilon\langle d,x\rangle x(1+s)^2-2\langle a,Qx+d\rangle x(1+s)^2+2(\epsilon+\langle a,x\rangle)Qx(1+s)^2\\
&\quad -2\mu\epsilon|x|^2d(1+s)^2-2\mu\langle d,x\rangle|x|^2d(1+s)^2.
\end{aligned}$$

Further, (4.21) is equivalent to

$$A(x)\xi^6+C(x)\xi^2+E(x)=0,\quad B(x)\xi^4+D(x)\xi=0.$$

which are equivalent to the following equations, respectively,

$$-\frac{(\epsilon+\langle a,x\rangle)s(1+3s)\xi^6}{(Qx+d)}+2\xi^2\langle d,x\rangle a(1+s)^2+2\mu\epsilon\langle d,x\rangle x(1+s)^2-2\langle a,Qx+d\rangle x(1+s)^2$$
$$+2(\epsilon+\langle a,x\rangle)Qx(1+s)^2-2\mu\epsilon|x|^2d(1+s)^2-2\mu\langle a,x\rangle|x|^2d(1+s)^2=0, \tag{4.22}$$

$$-\frac{(\epsilon+\langle a,x\rangle)s(1+3s)\xi^4}{(Qx+d)}+2\mu\epsilon\langle d,x\rangle x(1+s)^2-2\langle a,Qx+d\rangle x(1+s)^2+2\langle d,x\rangle a(1+s)^2$$
$$-2|x|^2Qa+2(\epsilon+\langle a,x\rangle)Qx(1+s)^2-2\mu\epsilon|x|^2d(1+s)^2=0. \tag{4.23}$$

Plugging $\xi^2 - 1 = \mu|x|^2$ and (4.23) into (4.22), we have

$$\mu|x|^2\left[-\frac{(\epsilon+\langle a,x\rangle)s(1+3s)\xi^4}{(Qx+d)}+2(\langle d,x\rangle a-\langle a,x\rangle d)(1+s)^2\right]+2|x|^2Qa=0.$$

i.e,

$$\mu\left[-\frac{(\epsilon+\langle a,x\rangle)s(1+3s)\xi^4}{(Qx+d)}+2(\langle d,x\rangle a-\langle a,x\rangle d)(1+s)^2\right]+2Qa=0. \tag{4.24}$$

By replacing x with $-x$ in (4.23) and adding the obtained equation to (4.23), we obtained the following

$$\begin{aligned}&-\frac{(\epsilon+\langle a,x\rangle)s(1+3s)\xi^4}{(Qx+d)}+2\mu\epsilon\langle d,x\rangle(1+s)^2-2\langle a,Qx\rangle x(1+s)^2\\&+2\langle a,x\rangle Qx(1+s)^2-2\mu\epsilon|x|^2d(1+s)^2-2|x|^2Qa=0,\end{aligned} \tag{4.25}$$

$$2\langle d,x\rangle a(1+s)^2-2\langle a,d\rangle x+\epsilon Qx(1+s)^2=0. \tag{4.26}$$

Similarly, by replacing x with $-x$ in (4.24) and adding the obtained equation to (4.24), we can get

$$2|x|^2Qa=0, \tag{4.27}$$

$$(2\langle d,x\rangle a-2\langle a,x\rangle d)(1+s)^2=0. \tag{4.28}$$

Since the above identities hold for all $x\in R^n$, we have $Qa=0$ by (4.27). Hence, (4.25) becomes

$$-\frac{(\epsilon+\langle a,x\rangle)s(1+3s)\xi^4}{(Qx+d)}+[\mu\epsilon\langle d,x\rangle(1+s)^2-\langle a,Qx\rangle(1+s)^2]x+\langle a,x\rangle Qx(1+s)^2-\mu\epsilon|x|^2d(1+s)^2=0.$$

Multiplying x^T on both sides of the above identity yields $\langle a,Qx\rangle=0$ for all $x\in M$.
We assert that $a=0$. Or else, if $a\neq 0$, by (4.28), we have

$$a(1+s)^2=\frac{\langle a,x\rangle}{\langle d,x\rangle}d(1+s)^2. \tag{4.29}$$

By (4.26) and by $\langle a,Qx\rangle=0$ we have

$$\langle a,x\rangle(1+s)^2=\frac{\langle d,x\rangle}{\langle a,d\rangle}|a|^2(1+s)^2. \tag{4.30}$$

From (4.29) and (4.30), we get

$$a=\bar\lambda d,\ \ \bar\lambda(1+s)^2=\frac{|a|^2}{\langle a,d\rangle}(1+s)^2. \tag{4.31}$$

Multiplying x^T on both sides of (4.24) yields

$$\mu\left[-\xi^4\frac{(\epsilon+\langle a,x\rangle)s(1+3s)\xi^4}{(Qx+d)}\right]x^T-2x^TQa=0. \tag{4.32}$$

Letting $x=d$ in (4.24) and by $Qd=0$, we have

$$\mu\left[-\xi^4\frac{(\epsilon+\langle a,d\rangle)s(1+3s)\xi^4}{d}\right]d^T=0.$$

Letting $x=d$ in (4.24) yields

$$2\mu[a|d|^2-\langle a,d\rangle d](1+s)^2+2Qa=0, \tag{4.33}$$

From (4.33), we can assert that $a = 0$. In fact, if $a \neq 0$, by multiplying a^T on both sides of (4.33), we have the following

$$2\mu[\langle a, d\rangle^2 - |a|^2|d|^2](1+s)^2 = 0.$$

which means that $a = \bar{\rho}d$ by $\mu > 0$ and Cauchy-Schwarz inequality. Substituting it into (4.24) and by $Qd = 0$, we get (see [5])

$$\mu\left[-\xi^4\frac{(\epsilon + \langle a, x\rangle)s(1+3s)\xi^4}{(Qx+d)}\right] = 0. \tag{4.34}$$

By (4.23), $Qa = 0$ and $\langle a, Qx\rangle = 0$, we can obtain

$$\begin{aligned} &-\mu\left[-\xi^4\frac{(\epsilon + \langle a, x\rangle)s(1+3s)\xi^4}{(Qx+d)}\right] + [2\mu\epsilon\langle d, x\rangle - \langle a, d\rangle]\langle a, x\rangle(1+s)^2 \\ &-2\mu\epsilon|x|^2\langle a, d\rangle(1+s)^2 + \langle d, x\rangle|a|^2(1+s)^2 = 0, \end{aligned}$$

from we get

$$(\mu\epsilon\langle a, x\rangle + |a|^2)\langle d, x\rangle(1+s)^2 = \langle a, d\rangle(\langle a, x\rangle + \mu\epsilon|x|^2)(1+s)^2. \tag{4.35}$$

By $a = \bar{\lambda}d$, we can obtain the following

$$\bar{\lambda}\mu\epsilon\langle d, x\rangle^2(1+s)^2 = \bar{\lambda}\mu\epsilon|d|^2|x|^2(1+s)^2, \tag{4.36}$$

for all $x \in M$. (4.36) implies that $\bar{\lambda} = 0$ or $\epsilon = 0$.
(i) If $\bar{\lambda} = 0$, then $a = 0$. It is a contradiction.
(ii) If $\epsilon = 0$, by (4.26), $\langle d, x\rangle a - \langle a, d\rangle x = 0$. Then $\bar{\lambda}\langle d, x\rangle d = \bar{\lambda}|d|^2 x$, which implies that $\bar{\lambda} = 0$, and then $a = 0$. It is also a contradiction.
Hence, we have $a = 0$. From (4.25), we obtain $(\mu\epsilon\langle d, x\rangle x - \mu\epsilon|x|^2 d)(1+s)^2 = 0$, which implies that $\epsilon = 0$. By (4.12), we have $\lambda = 0$. Further, by (4.15), we have

$$V = Qx + \mu\langle d, x\rangle x + d.$$

In this case, $\rho = \lambda = 0$, $K = 0$. This completes the proof. □

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