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CR-submanifolds of a golden Riemannian manifold

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Abstract In this paper, we define and study CR-submanifolds of a golden Riemannian manifold. We investigate some properties for CR-submanifolds. Moreover, we obtain many interesting results of totally umbilical CR-submanifolds on a golden Riemannian manifold.

1 Introduction

The theory of submanifolds of a manifold is one of the most interesting topics in differential geometry. According to the behaviour of the tangent bundle of a submanifold, we have three classes of submanifolds: holomorphic submanifolds, totally real submanifolds [17] and CR-submanifolds has been introduced by the author in [15] as follows:
Let $\overline{M}$ be an almost Hermitian manifold and let J be the almost complex structure of N. A real submanifold M of $\overline{M}$ is called a CR submanifold if there exists a differential distribution D on M satisfying

$$(i) J(D_X) = D_X$$

and

$$(ii) J(D_X^{\perp}) \subset T_X M^{\perp}$$

for each $x \in M$, where $D^{\perp}$ is the complementary orthogonal distribution to D and $T_X M^{\perp}$ is normal space to M at X. Holomorphic submanifolds and totally real submanifolds are particular cases of CR-submanifolds.
The CR-submanifolds have been extensively studied by several geometers [4], [6], [12], [13], [16]. Also, some properties of CR-submanifolds have been investigated in [1], [2], [3], [5], [6], [7], [11], [19], [27], [28].
Crasmareanu and Hretcanu [18] constructed the golden structure on a differentiable manifold $(\overline{M}, g)$ as a particular case of polynomial structure [24].
Gezer et. al investigated the integrability conditions of golden Riemannian structure. M. Ahmad and M. A. Qayyoom [8], C.E. Hretcanu [25] studied submanifolds in Riemannian manifolds with golden structure. The golden structure was also studied in [8], [9], [10], [20], [21], [22], [23], [26].
Motivated by above studies in this paper, we study CR-submanifolds of a golden Riemannian manifolds. The paper is organized as follows.
In section 2, we define golden structure manifold and CR-submanifolds of golden Riemannian manifold. In section 3, we established several properties of CR-submanifolds on golden Riemannian manifolds. In section 4, we investigate some properties of totally umbilical CR-submanifolds.

2 Definition and preliminaries

In this section, we give a brief information of golden Riemannian manifolds.

Definition 2.1. [18] Let $(\overline{M}, g)$ be a Riemannian manifold. A golden structure on $(\overline{M}, g)$ is a non-null tensor J of type (1,1) which satisfies the equation

$$J^2 = J + I, \tag{2.1}$$

where I is the identity transformation. We say that the metric g is J-compatible if

$$g(JX, Y) = g(X, JY) \tag{2.2}$$

for all X,Y vector fields on $\overline{M}$. If we substitute JX into X in (2.2), then we have

$$g(JX, JY) = g(JX, Y) + g(X, Y).$$

The Riemannian metric (2.2) is called J-compatible and $(\overline{M}, J, g)$ is called a golden Riemannian manifold.

Proposition 2.2. *[18] A golden structure on the manifold $\overline{M}$ has the power*

$$J^n = F_n J + F_{n-1} I \tag{2.3}$$

for any integer n, where (F_n) is the Fibonacci sequence.
Using an explicit expression for the Fibonacci sequence namely the Binet's formula

$$F_n = \frac{J^n - (1-J)^n}{\sqrt{5}},$$

we obtain a new form for the equality (2.3) *as*

$$J^n = (\frac{\phi^n - (1-\phi)^n}{\sqrt{5}})J + (\frac{\phi^{n-1} - (1-\phi)^{n-1}}{\sqrt{5}})I.$$

The straight forward computations yield:

Proposition 2.3. *[18] (i) The eigenvalues of a golden structure J are the golden ratio ϕ and $1 - \phi$.*
(ii) A golden structure J is an isomorphism on the tangent space $T_x\overline{M}$ of the manifold $\overline{M}$ for every $x \in \overline{M}$.
(iii) It follows that J is invertible and its inverse $\widehat{J} = J^{-1}$ satisfies

$$\widehat{\phi}^2 = -\widehat{\phi} + 1.$$

Let $\overline{M}$ be an $m-$ dimensional Riemannian manifold with a golden structure and M is an $n-$ dimensional Riemannian manifold isometrically immersed in $\overline{M}$. We denote g the Riemannian metric of $\overline{M}$ as well as M. Let ∇ and $\overline{\nabla}$ be the covariant differentiation on M and $\overline{M}$ respectively. Then the Gauss and Weingarten formulas for M are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.4}$$

$$\overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{2.5}$$

for any vector fields $X, Y \in TM$ and $\xi \in TM^\perp$, where h denotes the second fundamental form and $\nabla^\perp$ the linear connection induced in the normal bundle $T^\perp M$. The second fundamental tensor A_ξ is related to h by

$$g(A_\xi X, Y) = g(h(X, Y), \xi) \tag{2.6}$$

for any vector field X tangent to M, we have

$$JX = PX + QX, \tag{2.7}$$

where PX and QX are tangential and normal components of JX, respectively.
For any vector field ξ normal to M, we put

$$J\xi = t\xi + f\xi, \tag{2.8}$$

where $t\xi$ and $f\xi$ are tangential and normal components of $J\xi$ respectively.

Definition 2.4. A submanifold M of a golden Riemannian manifold $\overline{M}$ is called a CR-submanifold if there is a differentiable distribution $D : x \to D_x \subseteq T_xM$ on M satisfying the following conditions:
(i) D is holomorphic, i.e $JD_x = D_x$ for each $x \in M$, and
(ii) The complementary orthogonal distribution $D^{\perp} : x \to D_x^{\perp} \subseteq T_xM$ is totally real, i.e $JD^{\perp} \subset T_x^{\perp}M$ for each $x \in M$.

If $dimD_x^{\perp} = 0$ (respectively, $dimD_x = 0$), then CR-submanifold M is a holomorphic submanifold (respectively, totally real submnaifold). If $dimD_x^{\perp} = dimT_x^{\perp}M$, then the CR-submanifold is an anti-holomorphic submanifold (or generic submanifold). A submnaifold is called a proper CR - submnaifold if it is neither holomorphic nor totally real.
We shall denote by p the complex dimension of D_x and by q the real dimension of $D_x^{\perp}$, i.e. $p = dimD_x$ and $q = dimD_x^{\perp}$. We denote the ν the complementary orthogonal subbundle of $JD^{\perp}$ in $TM^{\perp}$. Hence we have

$$TM^{\perp} = JD^{\perp} \oplus \nu.$$

3 CR-submanifolds of golden Riemannian manifolds

Let $\overline{M}$ be a locally golden Riemannian manifold and M is a CR-submanifold of $\overline{M}$.
From (2.4) and (2.5), we have

$$J(\nabla_X Y) + J(h(X,Y)) = J(\overline{\nabla}_X Y - h(X,Y)) + Jh(X,Y)$$

$$J(\nabla_X Y) + J(h(X,Y)) = (\overline{\nabla}_X JY - (\overline{\nabla}_X J)Z.$$

Since $\overline{\nabla}_X J = 0$,

$$J(\nabla_X Y) + J(h(X,Y)) = (\overline{\nabla}_X JY)Z.$$

Since $Y \in D^{\perp}$ and M is CR-submanifold, then $JY \in TM^{\perp}$, we have

$$J(\nabla_X Y) + Jh(X,Y) = -A_{JY}X + \nabla^{\perp} JY \tag{3.1}$$

for $X \in TM$ and $Y \in D^{\perp}$.

Proposition 3.1. *Let M be a CR-submanifold of a locally golden Riemannian manifold $\overline{M}$. Then*

$$g(JA_{JY}X,Z) + g(\nabla_X Y, JZ) + g(\nabla_X Y, Z) = 0, \tag{3.2}$$

$$A_{J\xi}Z = -A_{\xi}JZ, \tag{3.3}$$

$$A_{JY}W = A_{JW}Y \tag{3.4}$$

for $X \in TM, Z \in D, Y, W \in D^{\perp}$ and $\xi \in \nu$.

Proof.

$$g(JA_{JY}X,Z) = g(J(\nabla_X^{\perp} JY) - J(\overline{\nabla}_X JY), Z)$$

$$g(JA_{JY}X,Z) = -g(J(\overline{\nabla}_X Y), JZ)$$

$$g(JA_{JY}X,Z) = -g(\overline{\nabla}_X Y, J^2 Z)$$

$$g(JA_{JY}X,Z) = -g((\nabla_X Y + h(X,Y)), JZ) - g(\nabla_X Y + h(X,Y), Z)$$

$$g(JA_{JY}X,Z) + g(\nabla_X Y, JZ) + g(\nabla_X Y, Z) = 0,$$

which is (3.2).
From equation (2.6), we have

$$g(A_{\xi}Z,U) = g(h(Z,U),\xi)$$

$$g(A_{J\xi}Z,U) = g(h(Z,U),J\xi).$$

Using (2.4), we have

$$g(A_{J\xi}Z, U) = g(\overline{\nabla}_Z U - \nabla_Z U, J\xi)$$
$$g(A_{J\xi}Z, U) = g(J(\overline{\nabla}_Z U), \xi)$$
$$g(A_{J\xi}Z, U) = g((\overline{\nabla}_Z(JU) - (\overline{\nabla}_Z J)Y), \xi)$$
$$g(A_{J\xi}Z, U) = g(\overline{\nabla}_Z(JU), \xi).$$

Using (3.4), we have

$$g(A_{J\xi}Z, U) = -g((\overline{\nabla}_U(JZ)), \xi)$$
$$g(A_{J\xi}Z, U) = -g(\nabla_U(JZ), \xi) - g(h(JZ, U), \xi)$$
$$g(A_{J\xi}Z, U) = -g(h(JZ, U), \xi).$$

Using (2.6), we have

$$g(A_{J\xi}Z, U) = -g(A_\xi JZ, U)$$
$$A_{J\xi}Z = -A_\xi JZ,$$

which is (3.3).
Now,

$$g(A_{JY}W, Z) = g(h(W, Z), JY).$$

Using (2.4), we have

$$g(A_{JY}W, Z) = g(\overline{\nabla}_W Z - \nabla_W X, JY).$$

Since $Y \in D^\perp, JY \in TM^\perp$,

$$g(A_{JY}W, Z) = g(\overline{\nabla}_W Z, JY)$$
$$g(A_{JY}W, Z) = g(J(\overline{\nabla}_W Z), Y)$$
$$g(A_{JY}W, Z) = g((\overline{\nabla}_W JZ - (\overline{\nabla}_W J)Z, Y)$$
$$g(A_{JY}W, Z) = g((\overline{\nabla}_W JZ, Y).$$

Using (3.4), we get

$$g(A_{JY}W, Z) = -g((\overline{\nabla}_Z JW, Y)$$
$$g(A_{JY}W, Z) = g(JW, \overline{\nabla}_Z Y).$$

Using (2.4), we have

$$g(A_{JY}W, Z) = g(JW, \nabla_Z Y) + g(JW, h(Z, Y))$$
$$g(A_{JY}W, Z) = g(JW, h(Z, Y)).$$

Using (2.6), we get

$$g(A_{JY}W, Z) = g(A_{JW}Y, Z)$$
$$A_{JY}W = A_{JW}Y.$$

□

Lemma 3.2. *Let M be a CR-submanifold of a locally golden Riemannian manifold $\overline{M}$. Then for any $Y, W \in D^\perp$,*

$$(\nabla^\perp_W JY - \nabla^\perp_Y JW) \in JD^\perp.$$

Proof. For any ξ in ν and $Y, W \in D^\perp$ and using (2.5), we have

$$g(A_{J\xi}Y, W) = g(-\overline{\nabla}_Y J\xi, W) + g(\nabla^\perp_Y J\xi, w)$$
$$g(A_{J\xi}Y, W) = -g((\overline{\nabla}_Y J)\xi, W) - g(J(\overline{\nabla}_Y \xi), W) + g(\nabla^\perp_Y J\xi, W)$$
$$g(A_{J\xi}Y, W) = -g(\overline{\nabla}_Y \xi, JW).$$

Using (2.5), we have

$$g(A_{J\xi}Y, W) = -g(-A_\xi Y + \nabla^\perp_Y \xi, JW)$$

$$g(A_{J\xi}Y,W) = -g(\nabla^{\perp}_{Y}\xi, JW)$$
$$g(A_{J\xi}Y,W) = g(\xi, \nabla^{\perp}_{Y}JW),$$

then

$$g(A_{J\xi}W,Y) = g(\xi, \nabla^{\perp}_{W}JY)$$
$$g(\xi, \nabla^{\perp}_{W}JY - \nabla^{\perp}_{Y}JW) = g(A_{J\xi}W,Y) - g(A_{J\xi}Y,W) = 0.$$

Thus,

$$\nabla^{\perp}_{W}JY - \nabla^{\perp}_{Y}JW \in JD^{\perp}.$$

□

Lemma 3.3. *Let M be a CR-submanifold of a locally golden Riemannian manifold. Then the distribution D is integrable and its leaves are totally geodesic in M if*

$$g(h(X,Y), JZ) = 0$$

for all $X, Y \in D$ and $Z \in D^{\perp}$.

Proof. Suppose distribution D is integrable and each leaf of D is totally geodesic in M and $\nabla_X Y \in D$ for any $X, Y \in D$, $Z \in D^{\perp}$ and using (2.4), we have

$$g(h(X,Y), JZ) = g(\overline{\nabla}_X Y - \nabla_X Y, JZ)$$
$$g(h(X,Y), JZ) = g(J(\overline{\nabla}_X Y), Z)$$
$$g(h(X,Y), JZ) = g((\overline{\nabla}_X JY - (\overline{\nabla}_X J)Y), Z)$$
$$g(h(X,Y), JZ) = g(\nabla_X JY + h(X, JY), Z)$$
$$g(h(X,Y), JZ) = g(\nabla_X JY, Z) = 0$$

for any $X, Y \in D$ and $Z \in D^{\perp}$. □

Proposition 3.4. *Let M be a CR-submanifold of a locally golden Riemannian manifold $\overline{M}$. Then D is integrable if*

$$g(h(X, JY), JZ) - g(h(Y, JX), JZ) = g(\nabla_Y Z, JX) - g(\nabla_X Z, JY)$$

for any vector $X, Y \in D$ and $Z \in D^{\perp}$.

Proof. From (2.6) and (2.5), we have

$$g(h(X, JY), JZ) = g(\overline{\nabla}_X JZ, JY) - g(\nabla^{\perp}_X JZ, JY)$$
$$g(h(X, JY), JZ) = g((\overline{\nabla}_X J)Z, JY) + g(J(\overline{\nabla}_X Z), JY) - g(\nabla^{\perp}_X JZ, JY)$$
$$g(h(X, JY), JZ) = g((\overline{\nabla}_X Z), J^2 Y).$$

Using (2.1), we have

$$g(h(X, JY), JZ) = g((\overline{\nabla}_X Z), (J + I)Y).$$

Using (2.4), we have

$$g(h(X, JY), JZ) = g(\nabla_X Z, JY) + g(\nabla_X Z, Y)$$
$$g(Y, \nabla_X Z) = -g(\nabla_X Z, JY) + g(h(X, JY), JZ)$$
$$g(Z, \nabla_X Y) = g(\nabla_X Z, JY) - g(h(X, JY), JZ)$$

and

$$g(Z, \nabla_Y X) = g(\nabla_Y Z, JX) - g(h(Y, JX), JZ)$$
$$g(Z, \nabla_X Y) - g(Z, \nabla_Y X) = g(\nabla_X Z, JY) - g(h(X, JY), JZ) - g(\nabla_Y Z, JX) + g(h(Y, JX), JZ)$$
$$g(Z, [X, Y]) = g(\nabla_X Z, JY) - g(h(X, JY), JZ) - g(\nabla_Y Z, JX) + g(h(Y, JX), JZ).$$

Since D is integrable, we have

$$g(\nabla_Y Z, JX) - g(\nabla_X Z, JY) = -g(h(X, JY), JZ) + g(h(Y, JX), JZ).$$

□

Lemma 3.5. *If D is integrable, then for any X in D and ξ in $J(D^{\perp})$, we have*

$$g(\nabla_Y Z, JX) - g(\nabla_X Z, JY) = g(A_\xi X, JY) - g(A_\xi JX, Y).$$

Proof. From Proposition 3.4 and (2.6), we have

$$g(\nabla_Y Z, JX) - g(\nabla_X Z, JY) = -g(A_{JZ} X, JY) + g(A_{JZ} Y, JX)$$

$$g(\nabla_Y Z, JX) - g(\nabla_X Z, JY) = -g(JA_{JZ} X, Y) + g(JA_{JZ} Y, X).$$

Replace JZ by ξ, we have

$$g(\nabla_Y Z, JX) - g(\nabla_X Z, JY) = -g(JA_\xi X, Y) + g(JA_\xi Y, X).$$

□

4 Totally Umbilical CR-Submanifolds

If M is a totally umbilical CR-submanifold in a Riemannian manifold $\overline{M}$, then we have

$$h(X, Y) = g(X, Y)H \tag{4.1}$$

for $X, Y \in TM$, where H is mean curvature.

Lemma 4.1. *If M is a totally umbilical CR-submanifold of a locally golden Riemannian manifold $\overline{M}$. Then either the totally real distribution $D^{\perp}$ is $1-$ dimensional or the mean curvature vector H is perpendicular to $JD^{\perp}$.*

Proof. From equation (4.1), we get

$$g(h(X, X), JW) = g(X, X)g(H, JW) \tag{4.2}$$

for $X \in TM$ and $W \in D^{\perp}$.
Using equality (3.1) and (3.4) in equation (4.2),

$$g(H, JW) = g(h(X, X), JW)$$

$$g(H, JW) = g(A_{JW} X, X)$$

$$g(H, JW) = g(A_{JX} W, X)$$

$$g(H, JW) = g(h(X, W), JX).$$

For unit vector $Z \in D^{\perp}$ perpendicular to W,

$$g(H, JW) = g(h(X, W), JX) = 0.$$

Hence proved.

□

Theorem 4.2. *Let M be a totally umbilical CR-submanifold of a locally golden Riemannian manifold $\overline{M}$. Then the CR-sectional curvature of $\overline{M}$ vanish, i.e. $\overline{K}(\pi) = 0$ for all CR-section π.*

Proof. Since M is a totally umbilical submanifold, we have

$$(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

$$(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X (g(Y, Z)H) - g(\nabla_X Y, Z)H - g(Y, \nabla_X Z)H$$

$$(\overline{\nabla}_X h)(Y, Z) = g(Y, Z)\nabla^{\perp}_X H - g(\nabla_X Y, Z)H + g(\nabla_X Y, Z)H$$

$$(\overline{\nabla}_X h)(Y, Z) = g(Y, Z)\nabla^{\perp}_X H. \tag{4.3}$$

Let $\xi \in T^{\perp}N$, then from Codazzi equation, we have

$$(\widetilde{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z).$$

From equation (4.3), we get

$$(\widetilde{R}(X,Y)Z)^{\perp} = g(Y,Z)\nabla^{\perp}_X H - g(X,Z)\nabla^{\perp}_X H$$

$$g(\widetilde{R}(X,Y)Z,\xi) = g(Y,Z)g(\nabla^{\perp}_X H,\xi) - g(X,Z)g(\nabla^{\perp}_X H,\xi)$$

$$\widetilde{R}(X,Y;Z,\xi) = g(Y,Z)g(\nabla^{\perp}_X H,\xi) - g(X,Z)g(\nabla^{\perp}_X H,\xi).$$

For any unit vectors $X \in D$ and $Z \in D^{\perp}$,

$$\widetilde{R}(X,Z;JX,JZ) = g(Z,JX)g(\nabla^{\perp}_X H,JZ) - g(X,JX)g(\nabla^{\perp}_Z H,JZ) = 0.$$

Similarly,

$$\widetilde{R}(X,Z;X,Z) = g(Z,X)g(\nabla^{\perp}_X H,Z) - g(X,X)g(\nabla^{\perp}_Z H,Z) = 0.$$

Since,

$$K(\gamma) = K_N(X \wedge Y)$$

$$K(\gamma) = g(R(X,Y)Y,X)$$

$$\widetilde{R}(X,Z;,JX,JZ) = g(\widetilde{R}(X,Z)JX,JZ) = 0,$$

i.e.

$$\widetilde{K}(\pi) = 0$$

for all CR-section.

□

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