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Extension of Phillips type q-Bernstein Operators on Triangle with one Curved side

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Abstract In this paper, approximation properties by Phillips type q-Bernstein operators on a triangle with one curved side are studied. Their products and their Boolean sums are defined. Interpolation properties of all these operators are discussed. Error bounds and the remainder terms are computed using modulus of continuity and Peano's theorem for the corresponding operators. Graphs are added to demonstrate consistency in theoretical findings.

1 Introduction

A constructive proof of the Weierstrass approximation theorem [10] by S.N. Bernstein in 1912 is based on uniform continuity and law of large numbers. These polynomials are now known as Bernstein polynomials in Approximation theory. In Computer-aided geometric design (CAGD), the basis of these Bernstein type polynomials plays a significant role in preserving the shape of the curves and surfaces [16, 31, 32].

It is well known that the space of all continuous functions is not strictly convex concerning uniform norm. Therefore best approximation may not be unique. Thus several authors constructed various operators to approximate continuous functions.

In finite element method for differential equations with given boundary conditions, approximation operators on polygonal domains are required. Thus many researchers generalized Bernstein type operators on different domains and constructed some other operators for improved approximation. After the papers [7, 8, 9] of R.E. Barnhill et al., Lagrange, Birkhoff and Hermite type operators have been studied, which interpolate a given function and certain of its derivative on the boundary of triangle (as in Dirichlet, Neumann or Robin boundary conditions for differential equation problems). They considered interpolation operators on triangles with curved sides (one, two or all curved sides), many of them in connection with finite element method and Computer aided geometric design.

D. D. Stancu studied polynomial interpolation on boundary data on triangles and error bound for smooth interpolation [33, 34]. Catinas extended some interpolation operators to triangle with one curved side [15]. T. Acar et al. studied approximation properties of Bivariate Bernstein-Stancu-Chlodowsky, Bernstein-Kantorovich type operators etc. in [1, 2, 3, 19]. Q. B. Cai constructed λ-Bernstein operators and studied its approximation properties in [13, 14]. N. Braha et al. studied λ-Bernstein operators via power series summability methods in [12]. Mursaleen et al. studied approximation properties by q-Bernstein shifted operators and q-Bernstein Schurer operators in [25, 26]. Recently Khalid et al. generalised Bernstein type operators and studied applications of its basis in Computer Aided Geometric Design (CAGD) [21, 20, 27, 30, 36, 37]. For other applications of Bernstein type operators related to construction of Bezier curves and surfaces, one can see [29] and [5, 17, 21, 20, 23, 24].

Further, After the development of Quantum calculus (q-analogue). In 1997 Phillips [28] introduced polynomial by introducing extra parameter q, which is a generalization of Bernstein polynomial. A lot of attention is given to q-Bernstein polynomials and by many researchers and it were studied broadly.

2 Essential preliminaries of quantum calculus

Let $q > 0$. For any $\mu \in \mathbb{N} \cup \{0\}$, the q-integer $[\mu]_q$ is defined by

$$[\mu]_q := 1 + q + \cdots + q^{\mu-1}, \quad \text{when } \mu \in \mathbb{N}, \quad [0]_q := 0;$$

and the q-factorial $[\mu]_q!$ by

$$[\mu]_q! := [1]_q[2]_q \cdots [\mu]_q, \quad \text{when } \mu \in \mathbb{N}, \quad [0]_q! = 1,$$

where $\mathbb{N}$ is the set of natural numbers [35].

For integers $0 \leq i \leq \mu$, we define the q-binomial coefficient as

$$\begin{bmatrix} \mu \\ i \end{bmatrix}_q := \frac{[\mu]_q!}{[i]_q![\mu - i]_q!},$$

for $q = 1$,

$$[\mu]_1 = \mu, \quad [\mu]_1! = \mu!, \quad \begin{bmatrix} \mu \\ i \end{bmatrix}_1 = \binom{\mu}{i}.$$

In Cauchy's q-binomial theorem, the q-binomial coefficients are used. In the following equation the first equation is a q-analogue of Newton's binomial formula:

$$(aw + bz)_q^{\mu} := \sum_{i=0}^{\mu} q^{\frac{i(i-1)}{2}} \begin{bmatrix} \mu \\ i \end{bmatrix}_q a^{\mu-i} b^i w^{\mu-i} z^i, \tag{2.1}$$

$$(1 + w)(1 + qw) \cdot \cdots \cdot (1 + q^{\mu-1}w) = \sum_{i=0}^{\mu} \begin{bmatrix} \mu \\ i \end{bmatrix}_q q^{i(i-1)/2} w^i. \tag{2.2}$$

Following Phillips we denote

$$b_{m,i}(w, z) = \begin{bmatrix} m \\ i \end{bmatrix}_q w^i \prod_{s=0}^{m-i-1} (1 - q^s w), \tag{2.3}$$

it follows from (2.2) that

$$\sum_{i=0}^{m} b_{m,i}(q, w) = 1, \qquad w \in [0, 1], \tag{2.4}$$

for integers $\mu \geq i \geq 0$, the q-binomial coefficients satisfy the following recurrence relations

$$\begin{bmatrix} \mu + 1 \\ i \end{bmatrix}_q = q^{\mu-i+1} \begin{bmatrix} \mu \\ i - 1 \end{bmatrix}_q + \begin{bmatrix} \mu \\ i \end{bmatrix}_q \tag{2.5}$$

and

$$\begin{bmatrix} \mu + 1 \\ i \end{bmatrix}_q = \begin{bmatrix} \mu \\ i - 1 \end{bmatrix}_q + q^i \begin{bmatrix} \mu \\ i \end{bmatrix}_q. \tag{2.6}$$

In the paper [15], T. Cătinaş defined classical Bernstein-type operators on triangle with one curve side. In [22] Asif et al. constructed Phillips q-Bernstein operator on triangle via quantum calculus. Motivated by the work in [15] and [22], we extend Phillips type q-Bernstein operator on triangle with one curve side in the next section.

3 Extension of new univariate operators on triangles with one curved side

Consider $\mathcal{R}_h$ be triangle with one curve side and F be a real valued function, which is defined on $\mathcal{R}_h$, as done in [15]. Through the point $(w,z) \in \mathcal{R}_h$, one considers the parallel lines to the coordinate axes which intersect the edges $\Gamma_i, i = 1,2,3,$ of the triangle at the points $(0,z)$ and $(g(z),z)$, respectively $(w,0)$ and $(w,f(w))$ ([15, Figure 1]).

By using quantum calculus, we define the new Phillips type q-Bernstein operators $\mathcal{B}^w_{m,q}$ and $\mathcal{B}^z_{n,q}$ on triangle with one curve side as follows:

$$(\mathcal{B}^w_{m,q}F)(w,z) = \begin{cases} \sum_{i=0}^{m} \tilde{p}_{m,i}(w,z) F\left(\frac{[i]_q}{[m]_q} g(z), z\right), & (w,z) \in \mathcal{R}_h \setminus (0,h), \\ F(0,h), & (0,h) \in \mathcal{R}_h, \end{cases} \tag{3.1}$$

and

$$(\mathcal{B}^z_{n,q}F)(w,z) = \begin{cases} \sum_{j=0}^{n} \tilde{q}_{n,j}(w,z) F\left(w, \frac{[j]_q}{[n]_q} f(w)\right), & (w,z) \in \mathcal{R}_h \setminus (h,0), \\ F(h,0), & (h,0) \in \mathcal{R}_h, \end{cases} \tag{3.2}$$

where

$$\tilde{p}_{m,i}(w,z) = \frac{\begin{bmatrix} m \\ i \end{bmatrix}_q w^i \prod_{s=0}^{m-i-1}(g(z) - q^s w)}{[g(z)]^m}, \quad 0 \le w + z \le g(z), \tag{3.3}$$

and

$$\tilde{q}_{n,j}(w,z) = \frac{\begin{bmatrix} n \\ j \end{bmatrix}_q z^j \prod_{t=0}^{n-j-1}(f(w) - q^t z)}{[f(w)]^n}, \quad 0 \le w + z \le f(w), \tag{3.4}$$

respectively.

For calculating the moments of above operators, the following notation are used:

$$[g(z)]^m := \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q w^i \prod_{s=0}^{m-i-1} (g(z) - q^s w), \tag{3.5}$$

and

$$[f(w)]^n := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q z^j \prod_{s=0}^{n-j-1} (f(w) - q^s z). \tag{3.6}$$

Here we have present the proof of equation

$$[g(z)]^m := \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q w^i \prod_{s=0}^{m-i-1} (g(z) - q^s w).$$

We will prove it by the method of induction. For $m = 1$ the right hand side of equation is

$$\sum_{i=0}^{1} \begin{bmatrix} 1 \\ i \end{bmatrix}_q w^i \prod_{s=0}^{-i}(g(z) - q^s w) = (g(z) - w) + w = g(z).$$

For $m=2$,

$$\sum_{i=0}^{2}\left[\begin{array}{c}2\\ i\end{array}\right]_q w^i\prod_{s=0}^{1-i}(g(z)-q^sw)$$
$$=\left[\begin{array}{c}2\\ 0\end{array}\right]_q w^0(g(z)-w)((z)-qw)+\left[\begin{array}{c}2\\ 1\end{array}\right]_q w^1(g(z)-w)+\left[\begin{array}{c}2\\ 2\end{array}\right]_q w^2$$
$$=(g(z)-w)(g(z)-qw)+(1+q)w(g(z)-w)+w^2$$
$$=[g(z)]^2,$$

let us assume that the equation (3.5) is true for $m=k$, i.e.

$$[g(z)]^k:=\sum_{i=0}^{k}\left[\begin{array}{c}k\\ i\end{array}\right]_q w^i\prod_{s=0}^{k-i-1}(g(z)-q^sw). \tag{3.7}$$

Now, only we have to show that

$$[g(z)]^{k+1}:=\sum_{i=0}^{k+1}\left[\begin{array}{c}k+1\\ i\end{array}\right]_q w^i\prod_{s=0}^{k-i}(g(z)-q^sw),$$

$$R.H.S=\sum_{i=0}^{k+1}\left[\begin{array}{c}k+1\\ i\end{array}\right]_q w^i\prod_{s=0}^{k-i}(g(z)-q^sw)$$
$$=\sum_{i=0}^{k+1}\left(q^{k-i+1}\left[\begin{array}{c}k\\ i-1\end{array}\right]_q+\left[\begin{array}{c}k\\ i\end{array}\right]_q\right)w^i\prod_{s=0}^{k-i}(g(z)-q^sw)\qquad \textit{by using equation (2.5)}$$
$$=\sum_{i=0}^{k}q^{k-i}\left[\begin{array}{c}k\\ i\end{array}\right]_q w^{i+1}\prod_{s=0}^{k-i-1}(g(z)-q^sw)+\sum_{i=0}^{k}\left[\begin{array}{c}k\\ i\end{array}\right]_q w^i\prod_{s=0}^{k-i-1}(g(z)-q^sw)(g(z)-q^{k-i}w)$$
$$=g(z)\sum_{i=0}^{k}\left[\begin{array}{c}k\\ i\end{array}\right]_q w^i\prod_{s=0}^{k-i-1}(g(z)-q^sw)$$
$$=g(z)[g(z)]^k=[g(z)]^{k+1}\qquad \textit{by using equation (3.7)}.$$

Hence, the proof is completed. □

Similarly we can prove equation (3.4).

Theorem 3.1. *For a real-valued function F, which is defined on $\mathcal{R}_h$, we have*

(i) $\mathcal{B}^w_{m,q}F=F$ *on* $\Gamma_2\cup\Gamma_3$,

$$(ii)\ \left(\mathcal{B}^w_{m,q}e_{i0}\right)(w,z)=w^i,\ \ i=0,1\ (dex(\mathcal{B}^w_{m,q})=1), \tag{3.8}$$

$$(iii)\ \left(\mathcal{B}^w_{m,q}e_{20}\right)(w,z)=w^2+\frac{w(g(z)-w)}{[m]_q}, \tag{3.9}$$

$$\left(\mathcal{B}^w_{m,q}e_{ij}\right)(w,z)=\begin{cases} z^jw^i, & i=0,1,\ \ j\in\mathbb{N};\\ z^j\left(w^2+\frac{w(g(z)-w)}{[m]_q}\right), & i=2,\ \ j\in\mathbb{N};\end{cases} \tag{3.10}$$

where $e_{ij}(w,z)=w^iz^j$ *and* $dex(\mathcal{B}^w_{m,q})$ *stand for the degree of exactness of* $\mathcal{B}^w_{m,q}$.

Proof. The property (i) obtains from the relations

$$\tilde{p}_{m,i}\,(0,z)=\begin{cases}1, & \text{if } i=0,\\ 0, & i\neq 0,\end{cases}$$

and

$$\tilde{p}_{m,i}(g(z),z)=\begin{cases}1, & \text{if } i=m,\\ 0, & i\neq m.\end{cases}$$

Now we prove property (ii), we have

$$\begin{aligned}\left(\mathcal{B}^{w}_{m,q}e_{00}\right)(w,z)&=\sum_{i=0}^{m}\frac{\begin{bmatrix} m\\ i\end{bmatrix}_q w^i\prod_{s=0}^{m-i-1}(g(z)-q^s w)}{[g(z)]^m}\\ &=\frac{[g(z)]^m}{[g(z)]^m}=1,\\ (\mathcal{B}^{w}_{m,q}e_{10})(w,z)&=\sum_{i=0}^{m}\frac{\begin{bmatrix} m\\ i\end{bmatrix}_q w^i\prod_{s=0}^{m-i-1}(g(z)-q^s w)}{[g(z)]^m}\frac{[i]_q}{[m]_q}g(z)\\ &=\sum_{i=0}^{m-1}\frac{\begin{bmatrix} m-1\\ i\end{bmatrix}_q w^{i+1}\prod_{s=0}^{m-i-2}(g(z)-q^s w)}{[g(z)]^{m-1}}\\ &=w\sum_{i=0}^{m-1}\frac{\begin{bmatrix} m-1\\ i\end{bmatrix}_q w^{i}\prod_{s=0}^{(m-1)-i-1}(g(z)-q^s w)}{[g(z)]^{m-1}}\\ &=w,\end{aligned}$$

$$\left(\mathcal{B}^{w}_{m,q}e_{20}\right)(w,z)=\sum_{i=0}^{m}\frac{\left[\begin{array}{c}m\\ i\end{array}\right]_q w^i\prod_{s=0}^{m-i-1}(g(z)-q^s w)}{[g(z)]^m}\frac{[i]_q^2}{[m]_q^2}[g(z)]^2$$

$$=[g(z)]^2\sum_{i=0}^{m-1}\frac{\frac{[i+1]_q}{[m]_q}\left[\begin{array}{c}m-1\\ i\end{array}\right]_q w^{i+1}\prod_{s=0}^{m-i-2}(g(z)-q^s w)}{[g(z)]^m}$$

$$=[g(z)]^2 w\sum_{i=0}^{m-1}\frac{\frac{(1+q[i]_q)}{[m]_q}\left[\begin{array}{c}m-1\\ i\end{array}\right]_q w^{i}\prod_{s=0}^{m-i-2}(g(z)-q^s w)}{[g(z)]^m}$$

$$=g(z)\frac{w}{[m]_q}+\frac{q[m-1]_q\,w^2}{[m]_q}\sum_{i=0}^{m-2}\frac{\left[\begin{array}{c}m-2\\ i\end{array}\right]_q w^{i}\prod_{s=0}^{(m-2)-i-1}(g(z)-q^s w)}{[g(z)]^{m-2}}$$

$$\left(\mathcal{B}^{w}_{m,q}e_{20}\right)(w,z)=g(z)\frac{w}{[m]_q}+\frac{q[m-1]_q\,w^2}{[m]_q},$$

$$=g(z)\frac{w}{[m]_q}+w^2\left(1-\frac{1}{[m]_q}\right)$$

$$=w^2+\frac{w(g(z)-w)}{[m]_q}.$$

□

Remark 3.2. In the same manner one can prove that:
For a real-valued function F, which is defined on $\mathcal{R}_h$, we have

(i) $\mathcal{B}^{z}_{n,q}F=F$ *on* $\Gamma_1\cup\Gamma_3$, ‘

(ii) $\left(\mathcal{B}^{z}_{n,q}e_{0j}\right)(w,z)=z^j,\ \ j=0,1\ (dex(\mathcal{B}^{z}_{n,q})=1),$

(iii) $\left(\mathcal{B}^{z}_{n,q}e_{02}\right)(w,z)=z^2+\dfrac{z(f(w)-z)}{[n]_q},$

$$\left(\mathcal{B}^{z}_{n,q}e_{ij}\right)(w,z)=\begin{cases}w^i z^j, & j=0,1,\ \ i\in\mathbb{N};\\ w^i\left(z^2+\frac{z(f(w)-z)}{[n]_q}\right), & j=2,\ \ i\in\mathbb{N}.\end{cases}\tag{3.11}$$

For calulating error, we have approximation formula

$$F=\mathcal{B}^{w}_{m,q}F+\mathcal{R}^{w}_{m,q}F.$$

Theorem 3.3. *If* $F(.,z)\in C[0,g(z)]$, *then*

$$\left|(\mathcal{R}^{w}_{m,q}F)(w,z)\right|\le\left(1+\frac{h}{2\delta\sqrt{[m]_q}}\right)\omega\big(F(.,z);\delta\big),\quad z\in[0,h].$$

Also, if $\delta=\frac{1}{\sqrt{[m]_q}}$, *then*

$$\left|(\mathcal{R}^{w}_{m,q}F)(w,z)\right|\le\left(1+\frac{h}{2}\right)\omega\left(F(.,z);\frac{1}{\sqrt{[m]_q}}\right),\quad z\in[0,h],\tag{3.12}$$

where $\omega(F(.,z);\delta)$ *denotes the modulus of continuity of* F *with respect to* w.

Proof. We have

$$\left|(\mathcal{R}^w_{m,q}F)(w,z)\right| \leq \sum_{i=0}^{m} \tilde{p}_{m,i}(w,z)\left|F(w,z) - F\left(\frac{[i]_q g(z)}{[m]_q}, z\right)\right|.$$

As

$$\left|F(w,z) - F\left(\frac{[i]_q g(z)}{[m]_q}, z\right)\right| \leq \left(\frac{1}{\delta}\left|w - \frac{[i]_q g(z)}{[m]_q}\right| + 1\right)\omega(F(\centerdot,z);\delta),$$

we get

$$\begin{aligned}\left|(\mathcal{R}^w_{m,q}F)(w,z)\right| &\leq \sum_{i=0}^{m} \tilde{p}_{m,i}(w,z)\left(\frac{1}{\delta}\left|w - \frac{[i]_q g(z)}{[m]_q}\right| + 1\right)\omega(F(\centerdot,z);\delta) \\ &\leq \left[1 + \frac{1}{\delta}\left(\sum_{i=0}^{m} \tilde{p}_{m,i}(w,z)\left(w - \frac{[i]_q g(z)}{[m]_q}\right)^2\right)^{1/2}\right]\omega(F(\centerdot,z);\delta) \\ &= \left[1 + \frac{1}{\delta}\sqrt{\frac{w(g(z)-w)}{[m]_q}}\ \right]\omega(F(\centerdot,z);\delta).\end{aligned}$$

Since

$$\max_{0\leq w\leq g(z)}[w(g(z)-w)] = \frac{[g(z)]^2}{4} \quad and \quad \max_{0\leq z\leq h}[g(z)]^2 = h^2$$

it follows that

$$\left|(\mathcal{R}^w_{m,q}F)(w,z)\right| \leq \left(1 + \frac{h}{2\delta\sqrt{[m]_q}}\right)\omega(F(\centerdot,z);\delta).$$

For taking $\delta = \frac{1}{\sqrt{[m]_q}}$, we get

$$\left|(\mathcal{R}^w_{m,q}F)(w,z)\right| \leq \left(1 + \frac{h}{2}\right)\omega\left(F(\centerdot,z); \frac{1}{\sqrt{[m]_q}}\right).$$

□

Theorem 3.4. *If $F(\centerdot,z) \in C^2[0,h]$, then*

$$(\mathcal{R}^w_{m,q}F)(w,z) = \frac{w(w-g(z))}{2[m]_q}F^{(2,0)}(\xi,z), \qquad \xi \in [0,g(z)], \tag{3.13}$$

and

$$\left|(\mathcal{R}^w_{m,q}F)(w,z)\right| \leq \frac{h^2}{8[m]_q}\mathcal{M}_{20}F, \quad (w,z) \in \mathcal{R}_h,$$

where

$$\mathcal{M}_{ij}F = \max_{\mathcal{R}_h}\left|F^{(i,j)}(w,z)\right|.$$

Proof. Since $dex(\mathcal{B}^w_{m,q}) = 1$, by Peano's theorem, we get

$$\left(\mathcal{R}^w_{m,q}F\right)(w,z) = \int_0^{g(z)} \mathcal{K}_{20}(w,z;t)F^{(2,0)}(t,z)dt,$$

where the kernel

$$\mathcal{K}_{20}(w,z;t) := \mathcal{R}^w_{m,q}\left[(w-t)_+\right] = (w-t)_+ - \sum_{i=0}^{m}\tilde{p}_{m,i}(w,z)\left([i]_q\frac{g(z)}{[m]_q} - t\right)_+$$

does not change the sign $(\mathcal{K}_{20}(w,z;t) \leq 0, \quad w \in [0,g(z)])$. Using Mean Value Theorem, we get

$$\left(\mathcal{R}^w_{m,q}F\right)(w,z) = F^{(2,0)}(\xi,z)\int_0^{g(z)} \mathcal{K}_{20}(w,z;t)dt, \qquad \xi \in [0,g(z)].$$

After some calculation, we obtain

$$(\mathcal{R}^w_{m,q}F)(w,z) = \frac{w(w-g(z))}{2[m]_q}\, F^{(2,0)}(\xi,z),$$

where $\xi \in [0,g(z)]$.
As

$$\frac{w(w-g(z))}{2[m]_q} \leq \frac{h^2}{8[m]_q}$$

we get

$$\left|(\mathcal{R}^w_{m,q}F)(w,z)\right| \leq \frac{h^2}{8[m]_q}\mathcal{M}_{20}F, \quad (w,z)\in\mathcal{R}_h.$$

□

Remark 3.5. For obtaining the remainder $\mathcal{R}^z_{n,q}F$, we consider the formula

$$F = \mathcal{B}^z_{n,q}F + \mathcal{R}^z_{n,q}F.$$

We have $\boldsymbol{A}.$ if $F(w,.)\in C[0,f(w)]$, then

$$\left|(\mathcal{R}^z_{n,q}F)(w,z)\right| \leq \left(1+\frac{h}{2\delta\sqrt{[n]_q}}\right)\omega\big(F(w,.);\delta\big), \quad w\in[0,h],$$

and

$$\left|(\mathcal{R}^z_{n,q}F)(w,z)\right| \leq \left(1+\frac{h}{2}\right)\omega\left(F(w,.);\frac{1}{\sqrt{[n]_q}}\right), \quad w\in[0,h]. \tag{3.14}$$

$\boldsymbol{B}.$ If $F(w,.)\in C^2[0,h]$, then

$$(\mathcal{R}^z_{n,q}F)(w,z) = \frac{z(z-f(w))}{2[n]_q}\, F^{(0,2)}(w,\eta), \qquad \eta\in[0,f(w)],$$

and

$$\left|(\mathcal{R}^z_{n,q}F)(w,z)\right| \leq \frac{h^2}{8[n]_q}\mathcal{M}_{02}F, \quad (w,z)\in\mathcal{R}_h,$$

where

$$\mathcal{M}_{ij}F = \max_{\mathcal{R}_h}\left|F^{(i,j)}(w,z)\right|.$$

4 Product operators

Let $\mathcal{P}_{mn,q} = \mathcal{B}^w_{m,q}\mathcal{B}^z_{n,q}$ and $\mathcal{Q}_{mn,q} = \mathcal{B}^z_{n,q}\mathcal{B}^w_{m,q}$ be the products of operators $\mathcal{B}^w_{m,q}$ and $\mathcal{B}^z_{n,q}$.
We have

$$(\mathcal{P}_{mn,q}F)(w,z) = \sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(w,z)\tilde{q}_{n,j}\left([i]_q\frac{g(z)}{[m]_q},z\right)F\left([i]_q\frac{g(z)}{[m]_q},\frac{[j]_q}{[n]_q}f\left(\frac{[i]_q}{[m]_q}g(z)\right)\right).$$

Remark 4.1. The nodes for operator $P_{mn,q}$ are q-analogue of the nodes for P_{mn} and the nodes for P_{mn} are given in [15, Figure 2].

Theorem 4.2. *The following relations are satisfied by the product operator $\mathcal{P}_{mn,q}$:*
(i) $(\mathcal{P}_{mn,q}F)(w,0) = (\mathcal{B}^w_{m,q}F)(w,0)$,
(ii) $(\mathcal{P}_{mn,q}F)(0,z) = (\mathcal{B}^z_{n,q}F)(0,z)$,
(iii) $(\mathcal{P}_{mn,q}F)(w,f(w)) = F(w,f(w)), \qquad w,z\in[0,h]$.

One can easly prove above relations by following easy computation.
The property (i) or (ii) imply that $(\mathcal{P}_{mn,q}F)(0,0) = F(0,0)$. □

Remark 4.3. . The operator $\mathcal{P}_{mn,q}$ interpolates the function F on the curve $g(z)$ and at the vertex $(0,0)$ of the one curve side triangle $\mathcal{R}_h$. □

The product operator $\mathcal{Q}_{mn,q}$ is defined as

$$(\mathcal{Q}_{nm,q}F)(w,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} \tilde{p}_{m,i}\left(w, [j]_q \frac{f(w)}{[n]_q}\right)\tilde{q}_{n,j}(w,z)\; F\left(\frac{[i]_q}{[m]_q} g\left(\frac{[j]_q}{[n]_q} f(w)\right), [j]_q \frac{f(w)}{[n]_q}\right)$$

and the nodes for this operator are q-analogue of the nodes for operator Q_{mn} and the nodes for Q_{mn} are given in [15, Figure 2].
Also the operator $\mathcal{Q}_{nm,q}$ satisfies the Properties;
(i) $(\mathcal{Q}_{nm,q}F)(w,0) = (\mathcal{B}^w_{m,q}F)(w,0)$,
(ii) $(\mathcal{Q}_{nm,q}F)(0,z) = (\mathcal{B}^z_{n,q}F)(0,z)$,
(iii) $(\mathcal{Q}_{nm,q}F)(g(z),z) = F(g(z),z), \quad w,z \in [0,h]$.

Consider the formula for the Product operator

$$F = \mathcal{P}_{mn,q}F + \mathcal{R}^{\mathcal{P}}_{mn,q}F.$$

Theorem 4.4. *If $F \in C(\mathcal{R}_h)$ and $0 < q \le 1$ then*

$$\left|(\mathcal{R}^{\mathcal{P}}_{mn,q}F)(w,z)\right| \le (1+h)\omega\left(F; \frac{1}{\sqrt{[m]_q}}, \frac{1}{\sqrt{[n]_q}}\right) \quad (w,z) \in \mathcal{R}_h. \tag{4.1}$$

Proof. We have

$$\begin{aligned}\left|(\mathcal{R}^{\mathcal{P}}_{mn,q}F)(w,z)\right| \le & \left[\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(w,z)\tilde{q}_{n,j}\left([i]_q\frac{g(z)}{[m]_q}, z\right)\left|w - [i]_q\frac{g(z)}{[m]_q}\right|\right.\\ & + \frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(w,z)\tilde{q}_{n,j}\left([i]_q\frac{g(z)}{[m]_q}, z\right)\left|z - [j]_q\frac{([m]_q - [i]_q)h + [i]_q z}{[m]_q[n]_q}\right|\\ & \left. + \sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(w,z)\tilde{q}_{n,j}\left([i]_q\frac{g(z)}{[m]_q}, z\right)\right]\omega(F;\delta_1,\delta_2).\end{aligned}$$

After an easy calculation, we get

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(w,z)\tilde{q}_{n,j}\left([i]_q\frac{g(z)}{[m]_q}, z\right)\left|w - [i]_q\frac{g(z)}{[m]_q}\right| \le \sqrt{\frac{w(g(z)-w)}{[m]_q}},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(w,z)\tilde{q}_{n,j}\left([i]_q\frac{g(z)}{[m]_q}, z\right)\left|z - [j]_q\frac{([m]_q - [i]_q)h + [i]_q z}{[m]_q[n]_q}\right| \le \sqrt{\frac{z(f(w)-z)}{[n]_q}},$$

while

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(w,z)\tilde{q}_{n,j}\left([i]_q\frac{g(z)}{[m]_q}, z\right) = 1.$$

It follows

$$\left|(\mathcal{R}^{\mathcal{P}}_{mn,q}F)(w,z)\right| \le \left(\frac{1}{\delta_1}\sqrt{\frac{w(g(z)-w)}{[m]_q}} + \frac{1}{\delta_2}\sqrt{\frac{z(f(w)-z)}{[n]_q}} + 1\right)\omega(F;\delta_1,\delta_2).$$

As

$$\frac{w(g(z)-w)}{[m]_q} \le \frac{h^2}{4[m]_q}, \qquad \frac{z(f(w)-z)}{[n]_q} \le \frac{h^2}{4[n]_q}, \qquad \textit{for all } (w,z) \in \mathcal{R}_h,$$

we have

$$\left|\left(\mathcal{R}^{\mathcal{P}}_{mn,q}F\right)(w,z)\right| \leq \left(\frac{h}{2\delta_1\sqrt{[m]_q}} + \frac{h}{2\delta_2\sqrt{[n]_q}} \; + \; 1\right)\omega(F;\delta_1,\delta_2)$$

$$\left|\left(\mathcal{R}^{\mathcal{P}}_{mn,q}F\right)(w,z)\right| \leq (1+h)\,\omega\left(F;\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right).$$

□

5 Boolean sum operators

let the Boolean sums operators of the Phillips type q-Bernstein operators $\mathcal{B}^{w}_{m,q}$ and $\mathcal{B}^{z}_{n,q}$ are defined as

$$\mathcal{S}_{mn,q} := \mathcal{B}^{w}_{m,q} \oplus \mathcal{B}^{z}_{n,q} = \mathcal{B}^{w}_{m,q} + \mathcal{B}^{z}_{n,q} - \mathcal{B}^{w}_{m,q}\mathcal{B}^{z}_{n,q},$$
$$\mathcal{T}_{nm,q} := \mathcal{B}^{z}_{n,q} \oplus \mathcal{B}^{w}_{m,q} = \mathcal{B}^{z}_{n,q} + \mathcal{B}^{w}_{m,q} - \mathcal{B}^{z}_{n,q}\mathcal{B}^{w}_{m,q},$$

Theorem 5.1. *For the real-valued function F, which is defined on $\mathcal{R}_h$, we have*

$$\mathcal{S}_{mn,q}F\Big|_{\partial\mathcal{R}_h} = F\Big|_{\partial\mathcal{R}_h}.$$

Proof. We have

$$\mathcal{S}_{mn,q}F = (\mathcal{B}^{w}_{m,q} + \mathcal{B}^{z}_{n,q} - \mathcal{B}^{w}_{m,q}\mathcal{B}^{z}_{n,q})F.$$

All three properties of the operator $\mathcal{P}_{mn,q}$ in theorem 4.2 together with the interpolation properties of $\mathcal{B}^{w}_{m,q}$ and $\mathcal{B}^{z}_{n,q}$, imply that

$$(\mathcal{S}_{mn,q}F)(w,0) = (\mathcal{B}^{w}_{m,q}F)(w,0) + F(w,0) - (\mathcal{B}^{w}_{m,q}F)(w,0) = F(w,0),$$
$$(\mathcal{S}_{mn,q}F)(0,z) = F(0,z) - (\mathcal{B}^{z}_{n,q}F)(0,z) + (\mathcal{B}^{z}_{n,q}F)(0,z) = F(0,z),$$
$$(\mathcal{S}_{mn,q}F)(w,f(w)) = F(w,f(w)) + F(w,f(w)) - F(w,f(w)) = F(w,f(w)),$$

for all $w,z \in [0,h]$. □

For the remainder of Boolean sum operator, we have

$$F = \mathcal{S}_{mn,q}F + \mathcal{R}^{\mathcal{S}}_{mn,q}F.$$

Theorem 5.2. *If $F \in C(\mathcal{R}_h)$, then*

$$\left|\left(\mathcal{R}^{\mathcal{S}}_{mn,q}F\right)(w,z)\right| \leq \left(1+\frac{h}{2}\right)\omega\left(F(\cdot,z);\frac{1}{\sqrt{[m]_q}}\right) + \left(1+\frac{h}{2}\right)\omega\left(F(w,\cdot);\frac{1}{\sqrt{[n]_q}}\right)$$
$$+(1+h)\,\omega\left(F;\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right), \tag{5.1}$$

For all $(w,z) \in \mathcal{R}_h$.

Proof. We have

$$F - \mathcal{S}_{mn,q}F = F - \mathcal{B}^{w}_{m,q}F + F - \mathcal{B}^{z}_{n,q}F - (F - \mathcal{P}_{mn,q}F),$$

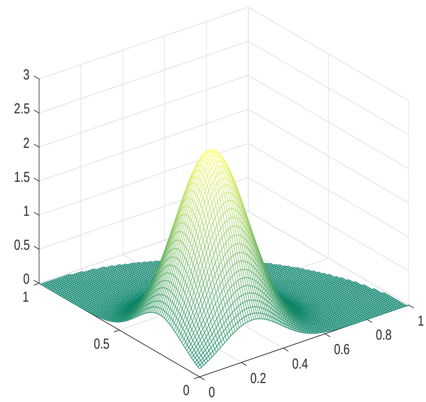
we obtain

$$\left|\left(\mathcal{R}^{\mathcal{S}}_{mn,q}F\right)(w,z)\right| \leq \left|\left(\mathcal{R}^{w}_{m,q}F\right)(w,z)\right| + \left|\left(\mathcal{R}^{z}_{n,q}F\right)(w,z)\right| + \left|\left(\mathcal{R}^{\mathcal{P}}_{mn,q}F\right)(w,z)\right|.$$

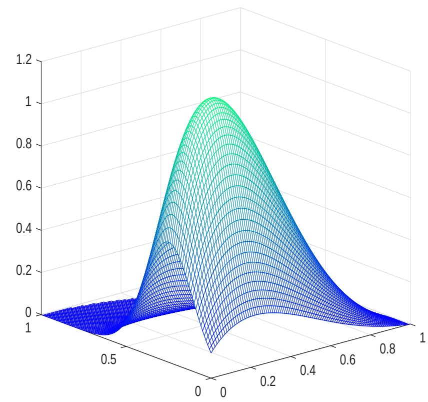
Now, from (3.12, 3.14, 4.1), we get the proof (5.1). □

6 Graphical analysis

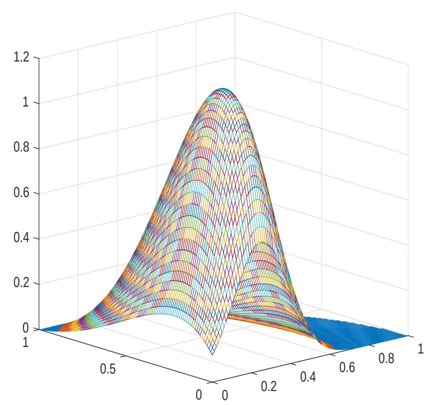
In Figures 1a and 2a, we have shown the graph of function $f(w,z) = exp[1 - 25(w - 0.25)^2 - 25(v - 0.25)^2]$ on triangle with one curve side. The curve side of triangle is given by the function $f(w) = \sqrt{1 - w^2}$. The graphs of operator $\mathcal{B}^u_{m,q}F$ are presented in figures 1b and 2b. Also we have presented the graphs of the operators $\mathcal{B}^v_{n,q}F$, $\mathcal{P}_{mn,q}F$ and $\mathcal{S}_{mn,q}F$. Interpolation properties of all above operators can be seen through these graphs. One can observe from the figures 2b, 2c, 2d and 2e that the graph of operators are approximating the graph of function better as q approaches to 1 for fixed value of m and n.
Thus, one can observe that by introducing a parameter q in Phillips type q-Bernstein operator on triangle with one curve side, we get more modeling flexivility.



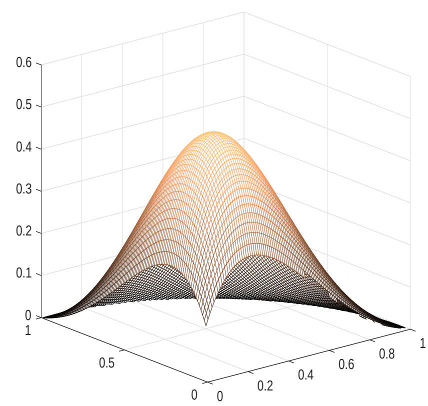
(a) $f(w,z) = e^{1-25(w-0.25)^2-25(v-0.25)^2}$



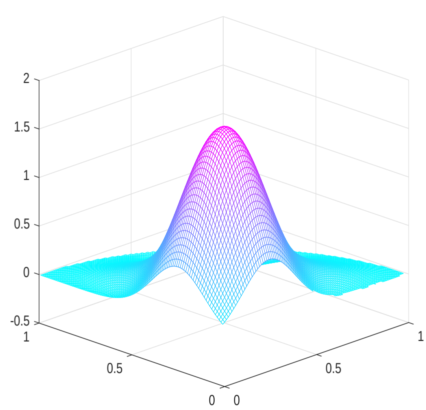
(b) Operator $B^{w}_{m,q}f$ for $m = n = 3, q = 0.70$



(c) Operator $B^{z}_{n,q}f$ for $m = n = 3, q = 0.70$

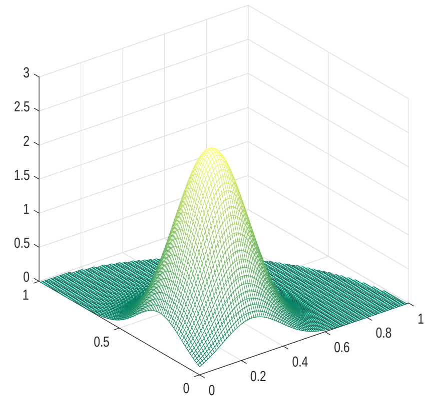


(d) Operator $P_{mn,q}f$ for $m = n = 3, q = 0.70$

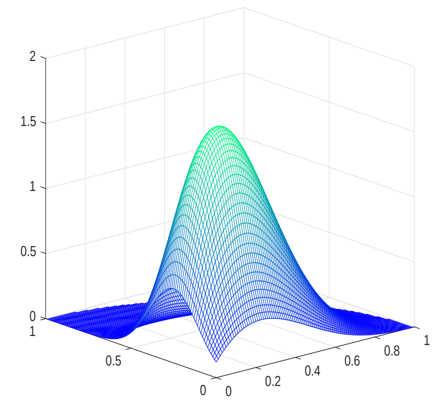


(e) Operator $S_{mn,q}f$ for $m = n = 3, q = 0.70$

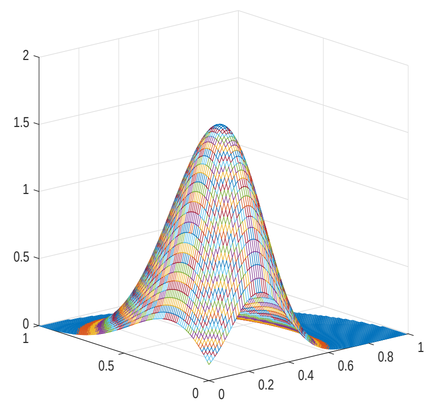
Figure 1: Graphs of the operators, which approximate the funtion and interpolate the function on some edges of one side curved triangle for $q = 0.70$.



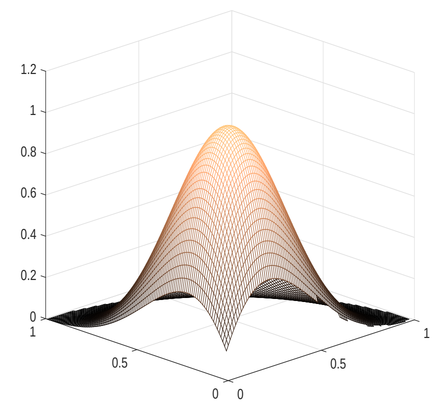
(a) $f(w,z) = e^{1-25(w-0.25)^2-25(v-0.25)^2}$



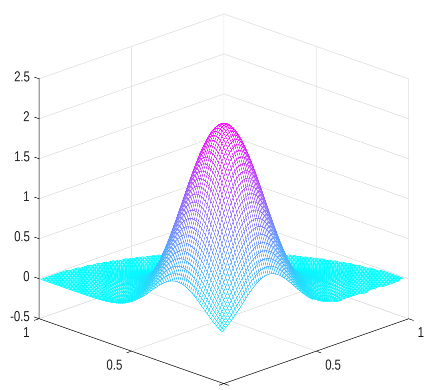
(b) Operator $B^w_{m,q}f$ for $m = n = 3, q = 0.90$



(c) Operator $B^z_{n,q}f$ for $m = n = 3, q = 0.90$



(d) Operator $P_{mn,q}f$ for $m = n = 3, q = 0.90$



(e) Operator $S_{mn,q}f$ for $m = n = 3, q = 0.90$

Figure 2: Graphs of the operators, which approximate the function and interpolate the function on some edges of one side curved triangle for $q = 0.90$.

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