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A NOVEL APPROACH TO EVALUATE THE VOLUME OF A PENTACHORON

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Abstract A simplex is an object defined in n-dimensional space with n+1 points. A 0-simplex is a point. A 1-simplex is defined as a line connecting two points. A 2-simplex is a triangle connecting three points. Similarly a 3-simplex is a tetrahedron having four points in a three dimensional space. In this paper we proposed a volume formula for a 4-simplex. Geometrically, a 4-simplex is a pentachoron having five points , ten edges and ten triangular faces. The main objective of this article is to propose a formula for volume of pentachoron. The methodology used to calculate the volume for such simplex is based on the basic trigonometric formulas of triangles. It is assumed that the pentachoron is circumscribed by a sphere of radius R in 4-dimensional space. The proposed volume formula is described in terms of inradii, circumradii, face angles of the triaglular faces of simplex and the radius of the sphere circumscribing the pentachoron.

1 Introduction

The area of a polygon can be calculated by dividing it into a number of triangles. Similarly a polyhedron can be decomposed into a number of tetrahedrons to calculate its volume. It is very difficult to calculate the volume of a polyhedron of dimension more than three. Caley Menger determinant is a very useful tool to calculate the volume of higher dimensional simplex but if it is not a simplex then one has to decompose it into the nearest considerable simplex which is nothing but a tetrahedron. In this article we have calculated the volume of a four dimensional simplex which is called pentachoron. It is assumed that the pentachoron is circumscribed by a hypersphere of radius R. The volume is represented in terms of inradii, circumradii and the face angles of the faces of the pentachoron. The inradii of a face is the distance of that face from the center of the pentachoron and circumradii can be defined as the distance of the vertices of the pentachoron from its center. In his article Cho[2] derived a formula for volume of tetrahedron in terms of circumradius and dihedral angles of a tetrahedron. He used the internal dihedral angles of the tetrahedron to calculate the value of a determinant along with the circumradius to derive a relation which gives the volume of a tertrahedron. Bhattacharyya and Pal[1] proposed a volume formula for tetrahedron in terms of inradii, circumradii and face angles of the tetrahedron. They used the basic trigonometrical formulas to define the edges of the tetrahedron in terms of face angles of the faces of tetrahedron, inradii of the faces, circumradii corresponding to vertices and the radius of the sphere circumscribing the tetrahedron. They also defined relations between inradii and between circumradii respectively. These edges are used as the terms of Cayley Menger determinant which is basically used to calculate the volume of the tetrahedron. Choudhury, Dey and Bhattacharyya[3] extended these investigation to calculate the volume of pentahedron in terms dihedral angles, face angles and circumradius. Pentahedron is a polyhedron which is not a simplex. Therefore Caley Menger determinant cannot be used directly. Hence, they decomposed the pentahedron into number of component tetrahedrons and derived the volume formula for the proposed polyhedron. They also tried to give a tensorial formula for the volume of the pentahedron.

2 Volume of a pentachoron in terms of r_{ij}, θ_k^{ij} and R

Caley Menger determinant is a tool which can be used to determine the volume of higher dimensional simplices. If L be a simplex with k+1 points $v_0, v_1, v_2, ..., v_k$, then a matrix $M = (\alpha_{ij})$ of order (k+1) can be defined with the property

$$\beta_{ij} = |v_i - v_j|^2$$

The volume of the simplex L can be written as

$$V_k^2(L) = \frac{(-1)^{k+1}}{2^k(k!)^2} det(M')$$

where $det(M')$ is termed as Caley Menger determinant of order (k+2). As it is considered that the pentachoron is a 4-simplex , therefore taking $k = 4$, we can write the volume of pentachoron as

$$V_4^2(L) = \frac{(-1)^{4+1}}{2^4(4!)^2} det(M') \tag{2.1}$$

The term $det(M')$ which is the Caley Menger determinant can be written for 4-simplex as

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a_{01}^2 & a_{02}^2 & a_{03}^2 & a_{04}^2 \\ 1 & a_{10}^2 & 0 & a_{12}^2 & a_{13}^2 & a_{14}^2 \\ 1 & a_{20}^2 & a_{21}^2 & 0 & a_{23}^2 & a_{24}^2 \\ 1 & a_{30}^2 & a_{31}^2 & a_{32}^2 & 0 & a_{34}^2 \\ 1 & a_{40}^2 & a_{41}^2 & a_{42}^2 & a_{43}^2 & 0 \end{vmatrix}$$

It is known that [4, section 9.7] circumradius R can be given by the formula

$$R^2 = -\frac{det(M'')}{2det(M')} \tag{2.2}$$

where $det(M'')$ is the determinant obtained by removing first row and first column from the Caley Menger determinant i.e. $det(M')$ and hence we can write $det(M'')$ as

$$\begin{vmatrix} 0 & a_{01}^2 & a_{02}^2 & a_{03}^2 & a_{04}^2 \\ a_{10}^2 & 0 & a_{12}^2 & a_{13}^2 & a_{14}^2 \\ a_{20}^2 & a_{21}^2 & 0 & a_{23}^2 & a_{24}^2 \\ a_{30}^2 & a_{31}^2 & a_{32}^2 & 0 & a_{34}^2 \\ a_{40}^2 & a_{41}^2 & a_{42}^2 & a_{43}^2 & 0 \end{vmatrix}$$

Now, substituting the value of $det(M')$ from equation (2) in equation(1), the volume formula can be written as

$$V_4 = \frac{\sqrt{det(M'')}}{96\sqrt{2}R} \tag{2.3}$$

Considering the pentachoron P with five vertices x_0, x_1, x_2, x_3, x_4 having ten inradii $r_{ij}, i, j\epsilon\{0, 1, 2, 3, 4\}$ corresponding to ten triangular faces $F_{ij}, i, j\epsilon\{0, 1, 2, 3, 4\}$ ten edges $a_{ij}, i, j\epsilon\{0, 1, 2, 3, 4\}$ and five circumradii $R_i, i\epsilon\{0, 1, 2, 3, 4\}$ corresponding to five vertices. It is obvious that $a_{ij} = a_{ji}, \forall i, j$

Theorem 2.1. *Let r_{ij} be the inradii of the faces $F_{ij}, i, j\epsilon\{0, 1, 2, 3, 4\}, i \neq j$ and $\theta_k^{ij}, i, j\epsilon\{0, 1, 2, 3, 4\}, i \neq j$, and $k\epsilon\{0, 1, 2, 3, 4\} - \{i, j\}$ be the face angles of the pentachoron then the volume of the pentachoron is given by*

$$V_4 = \frac{\sqrt{M_1 + M_2 + M_3 - M_4}}{96\sqrt{2}R}$$

where,

$$M_1 = (r_{12}^2 r_{13}^2 r_{23}^2 B_1^2 + r_{13}^2 r_{14}^2 r_{34}^2 B_2^2)(r_{03}^2 r_{13}^2 A_3^2 + r_{03}^2 r_{23}^2 A_2^2 - r_{03}^2 r_{23}^2 A_1^2)$$
$$M_2 = (r_{12}^2 r_{13}^2 r_{23}^2 B_3^2 + r_{12}^2 r_{14}^2 r_{24}^2 B_4^2)(r_{03}^2 r_{23}^2 A_1^2 + r_{03}^2 r_{13}^2 A_3^2 - r_{03}^2 r_{23}^2 A_2^2)$$

$$M_3 = (r_{12}^2 r_{14}^2 r_{24}^2 B_5^2 + r_{12}^2 r_{13}^2 r_{23}^2 B_6^2)(r_{03}^2 r_{23}^2 A_1^2 + r_{03}^2 r_{23}^2 A_2^2 - r_{03}^2 r_{13}^2 A_3^2)$$

$$M_4 = r_{01}^2 r_{12}^2 r_{13}^2 r_{23}^4 C_1^2 + r_{01}^2 r_{23}^2 r_{24}^2 r_{13}^4 C_2^2 + r_{13}^2 r_{23}^2 r_{34}^2 r_{12}^4 C_3^2 + r_{14}^2 r_{24}^2 r_{34}^2 r_{12}^4 C_4^2$$

$$A_1 = \frac{\sin\theta_4^{03}\sin\theta_1^{02}\sin\theta_0^{23}}{4\sin\frac{\theta_1^{03}}{2}\sin\frac{\theta_2^{03}}{2}\sin\frac{\theta_4^{03}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}\sin\theta_3^{02}}$$

$$A_2 = \frac{\sin\theta_4^{02}\sin\theta_1^{03}\sin\theta_0^{23}}{4\sin\frac{\theta_1^{03}}{2}\sin\frac{\theta_2^{03}}{2}\sin\frac{\theta_4^{03}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}\sin\theta_3^{02}}$$

$$A_3 = \frac{\sin\theta_4^{01}\sin\theta_2^{03}\sin\theta_0^{13}}{4\sin\frac{\theta_1^{03}}{2}\sin\frac{\theta_2^{03}}{2}\sin\frac{\theta_4^{03}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\theta_3^{01}}$$

$$B_1 = \frac{\sin\theta_0^{12}\sin\theta_4^{13}\sin\theta_4^{23}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}}$$

$$B_2 = \frac{\sin\theta_2^{13}\sin\theta_2^{14}\sin\theta_0^{34}}{8\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin\frac{\theta_0^{34}}{2}\sin\frac{\theta_1^{34}}{2}\sin\frac{\theta_2^{34}}{2}}$$

$$B_3 = \frac{\sin\theta_4^{12}\sin\theta_0^{13}\sin\theta_4^{23}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}}$$

$$B_4 = \frac{\sin\theta_3^{12}\sin\theta_3^{14}\sin\theta_0^{24}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin\frac{\theta_0^{24}}{2}\sin\frac{\theta_1^{24}}{2}\sin\frac{\theta_3^{24}}{2}}$$

$$B_5 = \frac{\sin\theta_3^{12}\sin\theta_0^{14}\sin\theta_3^{24}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin\frac{\theta_0^{24}}{2}\sin\frac{\theta_1^{24}}{2}\sin\frac{\theta_3^{24}}{4}}$$

$$B_6 = \frac{\sin\theta_4^{12}\sin\theta_4^{13}\sin\theta_0^{23}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}}$$

$$C_1 = \frac{\sin\theta_0^{14}\sin\theta_0^{12}\sin\theta_0^{13}\sin^2\theta_4^{23}}{32\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin^2\frac{\theta_0^{23}}{2}\sin^2\frac{\theta_1^{23}}{2}\sin^2\frac{\theta_4^{23}}{2}}$$

$$C_2 = \frac{\sin\theta_2^{01}\sin\theta_0^{23}\sin\theta_0^{24}\sin^2\theta_4^{13}}{32\sin\frac{\theta_2^{01}}{2}\sin\frac{\theta_3^{01}}{2}\sin\frac{\theta_4^{01}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}\sin\frac{\theta_0^{24}}{2}\sin\frac{\theta_1^{24}}{2}\sin\frac{\theta_3^{24}}{2}\sin^2\frac{\theta_0^{13}}{2}\sin^2\frac{\theta_2^{13}}{2}\sin^2\frac{\theta_4^{13}}{2}}$$

$$C_3 = \frac{\sin\theta_0^{13}\sin\theta_0^{23}\sin\theta_0^{34}\sin^2\theta_4^{12}}{32\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}\sin\frac{\theta_0^{34}}{2}\sin\frac{\theta_1^{34}}{2}\sin\frac{\theta_2^{34}}{2}\sin^2\frac{\theta_0^{12}}{2}\sin^2\frac{\theta_3^{12}}{2}\sin^2\frac{\theta_4^{12}}{2}}$$

$$C_4 = \frac{\sin\theta_0^{14}\sin\theta_0^{24}\sin\theta_0^{34}\sin^2\theta_3^{12}}{32\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin\frac{\theta_0^{24}}{2}\sin\frac{\theta_1^{24}}{2}\sin\frac{\theta_3^{24}}{2}\sin\frac{\theta_0^{34}}{2}\sin\frac{\theta_1^{34}}{2}\sin\frac{\theta_2^{34}}{2}\sin^2\frac{\theta_0^{12}}{2}\sin^2\frac{\theta_3^{12}}{2}\sin^2\frac{\theta_4^{12}}{2}}$$

Proof. Considering a circle of radius r inscribed in a triangle with the edges a, b, c, we have the following relation

$$a = r(\cot\tfrac{B}{2} + \cot\tfrac{C}{2}),$$

$$b = r(\cot\tfrac{A}{2} + \cot\tfrac{C}{2}),$$

$$c = r(\cot\tfrac{A}{2} + \cot\tfrac{B}{2})$$

Using the above relations, we can write the general relation for the edges of the pentachoron as

$$a_{ij} = \frac{r_{kl} \cos \frac{\theta_p^{kl}}{2}}{\sin \frac{\theta_i^{kl}}{2} \sin \frac{\theta_j^{kl}}{2}}. \tag{2.4}$$

where, $i, j \epsilon \{0, 1, 2, 3, 4\}, i \neq j$ and $k, l \epsilon \{0, 1, 2, 3, 4\} - \{i, j\}, k \neq l$ and $p \epsilon T, T = \{0, 1, 2, 3, 4\} - \{k, l\}$
It is obvious that for any edge of the pentachoron, $a_{ij} = a_{ji}, \forall i, j$
Therefore, we have ten relations as

$$\frac{r_{xy}}{r_{yz}} = \frac{\cos \frac{\theta_x^{yz}}{2} \sin \frac{\theta_d^{xy}}{2} \sin \frac{\theta_e^{xy}}{2}}{\cos \frac{\theta_z^{xy}}{2} \sin \frac{\theta_d^{yz}}{2} \sin \frac{\theta_e^{yz}}{2}} \tag{2.5}$$

where, $x \epsilon \{0, 1, 2\}, y \epsilon \{1, 2, 3\}, z \epsilon \{2, 3, 4\}, x \neq y \neq z$ and
$d \epsilon \{0, 1, 2, 3\} - \{x, y, z\}, e \epsilon \{1, 2, 3, 4\} - \{x, y, z\}, d \neq e$
Now expanding the determinant in equation (3), we can write

$$V_4 = \frac{\sqrt{K_1 + K_2 + K_3 - K_4}}{96\sqrt{2}R} \tag{2.6}$$

where,
$K_1 = 2(a_{01}^2 a_{02}^2 a_{34}^2 + a_{03}^2 a_{04}^2 a_{12}^2)(a_{14}^2 a_{23}^2 + a_{13}^2 a_{24}^2 - a_{12}^2 a_{34}^2),$

$K_2 = 2(a_{01}^2 a_{03}^2 a_{24}^2 + a_{02}^2 a_{04}^2 a_{13}^2)(a_{12}^2 a_{34}^2 + a_{14}^2 a_{23}^2 - a_{13}^2 a_{24}^2),$

$K_3 = 2(a_{01}^2 a_{04}^2 a_{23}^2 + a_{02}^2 a_{03}^2 a_{14}^2)(a_{12}^2 a_{34}^2 + a_{13}^2 a_{24}^2 - a_{14}^2 a_{23}^2),$

$K_4 = 2a_{01}^4 a_{23}^2 a_{24}^2 a_{34}^2 + 2a_{02}^4 a_{13}^2 a_{14}^2 a_{34}^2 + 2a_{03}^4 a_{12}^2 a_{14}^2 a_{24}^2 + 2a_{04}^4 a_{12}^2 a_{13}^2 a_{23}^2$

Using equation (2.4) and equation (2.5) in equation (2.6), we have

$$V_4 = \frac{\sqrt{M_1 + M_2 + M_3 - M_4}}{96\sqrt{2}R}$$

which proves the theorem. □

[See the Appendix A for the calculations of this part]

3 Volume of a pentachoron in terms of R_{ij}, θ_k^{ij} and R

In this section, we calculate the volume of the pentachoron in terms of ten circumradii, thirty face angles and the radius of the sphere circumscribing the pentachoron.

Theorem 3.1. *Let $R_{ij}, i, j \epsilon \{0, 1, 2, 3, 4\}, i \neq j$ be the circumradii corresponding to faces of the pentachoron and $\theta_k^{ij}, i, j \epsilon \{0, 1, 2, 3, 4\}, i \neq j$ and $k \epsilon \{0, 1, 2, 3, 4\} - \{i, j\}$ be the thirty face angles of the pentachoron. Then the volume of the pentachoron is given by*

$$V_4 = \frac{\sqrt{N_1 + N_2 + N_3 - N_4}}{96\sqrt{2}R}$$

where,
$N_1 = (R_{12}^2 R_{13}^2 R_{23}^2 B_1^2 + R_{13}^2 R_{14}^2 R_{34}^2 B_2^2)(R_{03}^2 R_{13}^2 A_3^2 + R_{03}^2 R_{23}^2 A_2^2 - R_{03}^2 R_{23}^2 A_1^2)$

$N_2 = (R_{12}^2 R_{13}^2 R_{23}^2 B_3^2 + R_{12}^2 R_{14}^2 R_{24}^2 B_4^2)(R_{03}^2 R_{23}^2 A_1^2 + R_{03}^2 R_{13}^2 A_3^2 - R_{03}^2 R_{23}^2 A_2^2)$

$N_3 = (R_{12}^2 R_{14}^2 R_{24}^2 B_5^2 + R_{12}^2 R_{13}^2 R_{23}^2 B_6^2)(R_{03}^2 R_{23}^2 A_1^2 + R_{03}^2 R_{23}^2 A_2^2 - R_{03}^2 R_{13}^2 A_3^2)$

$N_4 = R_{01}^2 R_{12}^2 R_{13}^2 R_{23}^4 C_1^2 + R_{01}^2 R_{23}^2 R_{24}^2 R_{13}^4 C_2^2 + R_{13}^2 R_{23}^2 R_{34}^2 R_{12}^4 C_3^2 + R_{14}^2 R_{24}^2 R_{34}^2 R_{12}^4 C_4^2$

$A_1 = \frac{4\sin\theta_4^{03}\sin\theta_1^{02}\sin\theta_0^{23}}{\sin\theta_3^{02}},$

$A_2 = \frac{4\sin\theta_4^{02}\sin\theta_1^{03}\sin\theta_0^{23}}{\sin\theta_3^{02}},$

$A_3 = \frac{4\sin\theta_4^{01}\sin\theta_2^{03}\sin\theta_0^{13}}{\sin\theta_3^{01}},$

$B_1 = 8\sin\theta_0^{12}\sin\theta_4^{13}\sin\theta_4^{23},$

$B_2 = 8\sin\theta_4^{03}\sin\theta_1^{23}\sin\theta_4^{12},$

$B_3 = 8\sin\theta_2^{13}\sin\theta_2^{14}\sin\theta_0^{34},$

$B_4 = 8\sin\theta_3^{12}\sin\theta_3^{14}\sin\theta_0^{24},$

$B_5 = 8\sin\theta_4^{01}\sin\theta_4^{23}\sin\theta_3^{12},$

$B_6 = 8\sin\theta_4^{12}\sin\theta_4^{13}\sin\theta_0^{23},$

$C_1 = 32\sin\theta_0^{14}\sin\theta_0^{12}\sin\theta_0^{13}\sin^2\theta_4^{23},$

$C_2 = 32\sin\theta_2^{01}\sin\theta_0^{23}\sin\theta_0^{24}\sin^2\theta_4^{13},$

$C_3 = 32\sin\theta_0^{13}\sin\theta_0^{23}\sin\theta_0^{34}\sin^2\theta_4^{12},$

$C_4 = 32\sin\theta_0^{14}\sin\theta_0^{24}\sin\theta_0^{34}\sin^2\theta_3^{12}$

Proof. It is observed that for a triangle with the edges a, b and c circumscribed by a sphere of radius R, a relation between the angles and the corresponding opposite edges can be given as

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

Using the above relation, we can write the general equation for the edge of the pentachoron as

$$a_{ij} = 2R_{kl}\sin\theta_p^{kl} \tag{3.1}$$

where, $i, j\epsilon\{0,1,2,3,4\}, i \neq j$ and $k, l\epsilon\{0,1,2,3,4\} - \{i,j\}, k \neq l$ and $p\epsilon S - \{k,l\}, S = \{0,1,2,3,4\} - \{i,j\}$
As it is clear that for any edge of the pentachoron, $a_{ij} = a_{ji}, \forall i, j$. Therefore we have the ten relations as

$$\frac{R_{mn}}{R_{nt}} = \frac{\sin\theta_m^{nt}}{\sin\theta_t^{mn}} \tag{3.2}$$

where, $m\epsilon\{0,1,2\}, n\epsilon\{1,2,3\}, t\epsilon\{2,3,4\}, m \neq n \neq t$.
Now, using the relations (3.1) and (3.2) in equation(2.6), we have,

$$V_4 = \frac{\sqrt{N_1 + N_2 + N_3 - N_4}}{96\sqrt{2}R}$$

which proves the theorem. □

[See the Appendix B for the calculations of this part]

4 Conclusion

In this article, we proposed a volume formula for pentachoron which is basically a 4-simplex. In the first section, we calculated the volume in terms of inradii, face angles of the pentachoron and the radius of the sphere cicumscribing it. Next we derived the same volume formula in terms of circumradii, face angles of the pentachoron and the radius of the sphere circumscribing the simplex. This method can also be used to calculate the volume for next higher dimensional simplices and can be generalized to n-dimensions also. This method is also very useful to analyze the volume formula in non Euclidean spaces.

5 Appendix A

From equation(2.4), we have the terms,

$$a_{12} = \frac{r_{03}\cos\frac{\theta_4^{03}}{2}}{\sin\frac{\theta_1^{03}}{2}\sin\frac{\theta_2^{03}}{2}}$$

$$a_{34} = \frac{r_{02}\cos\frac{\theta_1^{02}}{2}}{\sin\frac{\theta_3^{02}}{2}\sin\frac{\theta_4^{02}}{2}}$$

After multiplying the two terms, we get the following relation

$$a_{12}a_{34} = \frac{r_{02}r_{03}\cos\frac{\theta_4^{03}}{2}\cos\frac{\theta_1^{02}}{2}}{\sin\frac{\theta_1^{03}}{2}\sin\frac{\theta_2^{03}}{2}\sin\frac{\theta_3^{02}}{2}\sin\frac{\theta_4^{02}}{2}} \tag{5.1}$$

Using equation(2.5), we have

$$\frac{r_{02}}{r_{23}} = \frac{\cos\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{02}}{2}\sin\frac{\theta_4^{02}}{2}}{\cos\frac{\theta_3^{02}}{2}\cos\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}} \tag{5.2}$$

Substituting the value of r_{02} from eqution(5.2) in equation(5.1) and simplifying, we have

$$a_{12}a_{34} = \frac{r_{03}r_{23}\sin\theta_4^{03}\sin\theta_1^{02}\sin\theta_0^{23}}{4\sin\frac{\theta_1^{03}}{2}\sin\frac{\theta_2^{03}}{2}\sin\frac{\theta_4^{03}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}\sin\theta_3^{02}} = r_{03}r_{23}A_1 \tag{5.3}$$

Similarly, we have

$$a_{13}a_{24} = \frac{r_{03}r_{23}\sin\theta_4^{02}\sin\theta_1^{03}\sin\theta_0^{23}}{4\sin\frac{\theta_1^{03}}{2}\sin\frac{\theta_2^{03}}{2}\sin\frac{\theta_4^{03}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}\sin\theta_3^{02}} = r_{03}r_{23}A_2 \tag{5.4}$$

$$a_{14}a_{23} = \frac{r_{03}r_{13}\sin\theta_4^{01}\sin\theta_2^{03}\sin\theta_0^{13}}{4\sin\frac{\theta_1^{03}}{2}\sin\frac{\theta_2^{03}}{2}\sin\frac{\theta_4^{03}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\theta_3^{01}} = r_{03}r_{13}A_3 \tag{5.5}$$

$$a_{01}a_{02}a_{34} = \frac{r_{12}r_{13}r_{23}\sin\theta_0^{12}\sin\theta_4^{13}\sin\theta_4^{23}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}} = r_{12}r_{13}r_{23}B_1 \tag{5.6}$$

$$a_{03}a_{04}a_{12} = \frac{r_{13}r_{14}r_{34}\sin\theta_2^{13}\sin\theta_2^{14}\sin\theta_0^{34}}{8\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin\frac{\theta_0^{34}}{2}\sin\frac{\theta_1^{34}}{2}\sin\frac{\theta_2^{34}}{2}} = r_{13}r_{14}r_{34}B_2 \tag{5.7}$$

$$a_{01}a_{03}a_{24} = \frac{r_{12}r_{13}r_{23}\sin\theta_4^{12}\sin\theta_0^{13}\sin\theta_4^{23}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}} = r_{12}r_{13}r_{23}B_3 \tag{5.8}$$

$$a_{02}a_{04}a_{13} = \frac{r_{12}r_{14}r_{24}\sin\theta_3^{12}\sin\theta_3^{14}\sin\theta_0^{24}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin\frac{\theta_0^{24}}{2}\sin\frac{\theta_1^{24}}{2}\sin\frac{\theta_3^{24}}{2}} = r_{12}r_{14}r_{24}B_4 \tag{5.9}$$

$$a_{01}a_{04}a_{23} = \frac{r_{12}r_{14}r_{24}\sin\theta_3^{12}\sin\theta_0^{14}\sin\theta_3^{24}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin\frac{\theta_0^{24}}{2}\sin\frac{\theta_1^{24}}{2}\sin\frac{\theta_3^{24}}{4}} = r_{12}r_{14}r_{24}B_5 \tag{5.10}$$

$$a_{02}a_{03}a_{14} = \frac{r_{12}r_{13}r_{23}\sin\theta_4^{12}\sin\theta_4^{13}\sin\theta_0^{23}}{8\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}} = r_{12}r_{13}r_{23}B_6 \tag{5.11}$$

$$a_{01}^2a_{23}a_{24}a_{34} = \frac{r_{12}r_{13}r_{14}r_{23}^2\sin\theta_4^{01}\sin\theta_0^{12}\sin\theta_0^{13}\sin^2\theta_4^{23}}{32\sin\frac{\theta_0^{12}}{2}\sin\frac{\theta_3^{12}}{2}\sin\frac{\theta_4^{12}}{2}\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin^2\frac{\theta_0^{23}}{2}\sin^2\frac{\theta_1^{23}}{2}\sin^2\frac{\theta_4^{23}}{2}} = r_{12}r_{13}r_{14}r_{23}^2C_1 \tag{5.12}$$

$$a_{02}^2a_{13}a_{14}a_{34} = \frac{r_{01}r_{23}r_{24}r_{13}^2\sin\theta_2^{01}\sin\theta_0^{23}\sin\theta_0^{24}\sin^2\theta_4^{13}}{32\sin\frac{\theta_2^{01}}{2}\sin\frac{\theta_3^{01}}{2}\sin\frac{\theta_4^{01}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}\sin\frac{\theta_0^{24}}{2}\sin\frac{\theta_1^{24}}{2}\sin\frac{\theta_3^{24}}{2}\sin^2\frac{\theta_0^{13}}{2}\sin^2\frac{\theta_2^{13}}{2}\sin^2\frac{\theta_4^{13}}{2}} = r_{01}r_{23}r_{24}r_{13}^2C_2 \tag{5.13}$$

$$a_{03}^2a_{12}a_{14}a_{24} = \frac{r_{13}r_{23}r_{34}r_{12}^2\sin\theta_0^{13}\sin\theta_0^{23}\sin\theta_0^{34}\sin^2\theta_4^{12}}{32\sin\frac{\theta_0^{13}}{2}\sin\frac{\theta_2^{13}}{2}\sin\frac{\theta_4^{13}}{2}\sin\frac{\theta_0^{23}}{2}\sin\frac{\theta_1^{23}}{2}\sin\frac{\theta_4^{23}}{2}\sin\frac{\theta_0^{34}}{2}\sin\frac{\theta_1^{34}}{2}\sin\frac{\theta_2^{34}}{2}\sin^2\frac{\theta_0^{12}}{2}\sin^2\frac{\theta_3^{12}}{2}\sin^2\frac{\theta_4^{12}}{2}} = r_{13}r_{23}r_{34}r_{12}^2C_3 \tag{5.14}$$

$$a_{04}^2a_{12}a_{13}a_{23} = \frac{r_{14}r_{24}r_{34}r_{12}^2\sin\theta_0^{14}\sin\theta_0^{24}\sin\theta_0^{34}\sin^2\theta_3^{12}}{32\sin\frac{\theta_0^{14}}{2}\sin\frac{\theta_2^{14}}{2}\sin\frac{\theta_3^{14}}{2}\sin\frac{\theta_0^{24}}{2}\sin\frac{\theta_1^{24}}{2}\sin\frac{\theta_3^{24}}{2}\sin\frac{\theta_0^{34}}{2}\sin\frac{\theta_1^{34}}{2}\sin\frac{\theta_2^{34}}{2}\sin^2\frac{\theta_0^{12}}{2}\sin^2\frac{\theta_3^{12}}{2}\sin^2\frac{\theta_4^{12}}{2}} = r_{14}r_{24}r_{34}r_{12}^2C_4 \tag{5.15}$$

6 Appendix B

From equation(3.1), we can write

$$a_{12} = 2R_{03}\sin\theta_4^{03} = 2R_{04}\sin\theta_3^{04} = 2R_{34}\sin\theta_0^{34}$$
$$a_{34} = 2R_{01}\sin\theta_2^{01} = 2R_{02}\sin\theta_1^{02} = 2R_{12}\sin\theta_0^{12}$$

Now multiplying these two terms, we have

$$a_{12}a_{34} = 4R_{02}R_{03}\sin\theta_1^{02}\sin\theta_4^{03} \tag{6.1}$$

From equation(3.2), we have

$$\frac{R_{02}}{R_{23}} = \frac{\sin\theta_0^{23}}{\sin\theta_3^{02}} \tag{6.2}$$

Substituting the value of R_{02} from equation(6.2) in equation(6.1) and simplifying, we get the relation

$$a_{12}a_{34} = \frac{4R_{03}R_{23}\sin\theta_4^{03}\sin\theta_1^{02}\sin\theta_0^{23}}{\sin\theta_3^{02}} = R_{03}R_{23}A_1 \tag{6.3}$$

Similarly we have the other relations as follows

$$a_{13}a_{24} = \frac{4R_{03}R_{23}\sin\theta_4^{02}\sin\theta_1^{03}\sin\theta_0^{23}}{\sin\theta_3^{02}} = R_{03}R_{23}A_2 \tag{6.4}$$

$$a_{14}a_{23} = \frac{4R_{03}R_{13}\sin\theta_4^{01}\sin\theta_2^{03}\sin\theta_0^{13}}{\sin\theta_3^{01}} = R_{03}R_{13}A_3 \tag{6.5}$$

$$a_{01}a_{02}a_{34} = 8R_{12}R_{13}R_{23}\sin\theta_0^{12}\sin\theta_4^{13}\sin\theta_4^{23} = R_{12}R_{13}R_{23}B_1 \tag{6.6}$$

$$a_{03}a_{04}a_{12} = 8R_{13}R_{14}R_{34}\sin\theta_2^{13}\sin\theta_2^{14}\sin\theta_0^{34} = R_{13}R_{14}R_{34}B_2 \tag{6.7}$$

$$a_{01}a_{03}a_{24} = 8R_{12}R_{13}R_{23}\sin\theta_4^{12}\sin\theta_0^{13}\sin\theta_4^{23} = R_{12}R_{13}R_{23}B_3 \tag{6.8}$$

$$a_{02}a_{04}a_{13} = 8R_{12}R_{14}R_{24}\sin\theta_3^{12}\sin\theta_3^{14}\sin\theta_0^{24} = R_{12}R_{14}R_{24}B_4 \tag{6.9}$$

$$a_{01}a_{04}a_{23} = 8R_{12}R_{14}R_{24}\sin\theta_3^{12}\sin\theta_0^{14}\sin\theta_3^{24} = R_{12}R_{14}R_{24}B_5 \tag{6.10}$$

$$a_{02}a_{03}a_{14} = 8R_{12}R_{13}R_{23}\sin\theta_4^{12}\sin\theta_4^{13}\sin\theta_0^{23} = R_{12}R_{13}R_{23}B_6 \tag{6.11}$$

$$a_{01}^2a_{23}a_{24}a_{34} = 32R_{12}R_{13}R_{14}R_{23}^2\sin\theta_0^{14}\sin\theta_0^{12}\sin\theta_0^{13}\sin^2\theta_4^{23} = R_{12}R_{13}R_{14}R_{23}^2C_1 \tag{6.12}$$

$$a_{02}^2a_{13}a_{14}a_{34} = 32R_{01}R_{23}R_{24}R_{13}^2\sin\theta_2^{01}\sin\theta_0^{23}\sin\theta_0^{24}\sin^2\theta_4^{13} = R_{01}R_{23}R_{24}R_{13}^2C_2 \tag{6.13}$$

$$a_{03}^2a_{12}a_{14}a_{24} = 32R_{13}R_{23}R_{34}R_{12}^2\sin\theta_0^{13}\sin\theta_0^{23}\sin\theta_0^{34}\sin^2\theta_4^{12} = R_{13}R_{23}R_{34}R_{12}^2C_3 \tag{6.14}$$

$$a_{04}^2a_{12}a_{13}a_{23} = 32R_{14}R_{24}R_{34}R_{12}^2\sin\theta_0^{14}\sin\theta_0^{24}\sin\theta_0^{34}\sin^2\theta_3^{12} = R_{14}R_{24}R_{34}R_{12}^2C_4 \tag{6.15}$$

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