Palestine Journal of Mathematics

Vol. 12(1)(2023) , 722–731

Oscillatory behavior of even-order functional differential equations with a superlinear neutral term

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Communicated by Cemil Tunç

MSC 2010 Classifications: Primary 34C10, 34K11, 34K40.

Keywords and phrases: Oscillation, even-order differential equation, superlinear neutral term.

Abstract The authors establish some new oscillation criteria for a class of even-order functional differential equations with a superlinear neutral term using comparison methods and integral conditions. The results obtained are complement and improve a number of existing ones in the literature. An example is provided to illustrate the main results.

1 Introduction

This paper is concerned with the oscillatory behavior of solutions of the even-order nonlinear differential equation with a superlinear neutral term

$$(b(t)z^{(n-1)}(t))' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x^{\alpha}(\tau(t))$, $n \geq 4$ is an even natural number, α and β are the ratios of odd positive integers with $\alpha > 1$. Throughout this paper, we always assume that the following conditions are satisfied:

(H_1) $b : [t_0, \infty) \to (0, \infty)$ is a continuous function with $b'(t) \geq 0$, and

$$\int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty;$$

(H_2) $p, q : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $p(t) > 0$, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t;

(H_3) $\tau, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that $\tau'(t) > 0$, $\tau(t) \leq t$, $\sigma(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(H_4) there exists a constant $\kappa \in (0, 1)$ such that

$$\lim_{t\to\infty} \left(\frac{t}{\tau(t)}\right)^{(n-1)/\kappa} \frac{1}{p^{1/\alpha}(t)} = 0. \tag{1.2}$$

By a *solution* of (1.1), we mean a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \geq t_0$ such that $z \in C^{n-1}([t_x, \infty), \mathbb{R})$, $bz^{(n-1)} \in C^1([t_x, \infty), \mathbb{R})$ and x satisfies (1.1) on $[t_x, \infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_x, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \text{ for any } T_1 \geq t_x;$$

we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of equation (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$; otherwise it is called *nonoscillatory*. Equation (1.1) itself is termed oscillatory if all its solutions are oscillatory.

In recent years, the problem of investigating the oscillatory behavior of solutions of various classes of differential equations has received great attention; we refer the reader to the monographs [1, 14], the papers [2, 3, 4, 5, 6, 7, 10, 11, 16, 18, 21, 22, 23, 24] and the reference cited therein. However, there are few results dealing with the oscillation of differential equations with a superlinear neutral term; for example, see [9, 12, 17, 25, 26], where first, second and third-order differential equations of the type (1.1) are studied.

In this paper, we first reduce the nonlinear equation (1.1) into a linear one and then we obtain some new criteria for the oscillation of all solutions of (1.1) via couple of first-order delay differential equations whose oscillatory characters are known. Also the results presented in this paper will provide an answer to the interesting problem raised in the paper [13].

For the reader's convenience, and to simplify notation, we define:

$$R_0(t) := \left(\frac{1}{b(t)} \int_t^\infty q(s) p^{-\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s)))ds \right),$$

$$R_m(t) := \int_t^\infty R_{m-1}(s)ds \quad \text{for } m = 1, 2, ..., n-3;$$

$$F_0(t) := \left(\frac{1}{b(t)} \int_t^\infty q(s) \left(h^{1/\kappa}(s) \right)^{\frac{\beta}{\alpha}-1} p^{-\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s)))ds \right),$$

$$F_m(t) := \int_t^\infty F_{m-1}(s)ds \quad \text{for } m = 1, 2, ..., n-3;$$

and

$$h(t) := \tau^{-1}(\eta(t)),$$

where $\eta \in C^1([t_0, \infty), \mathbb{R})$ and τ^{-1} is the inverse function of τ.

In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of (1.1); since if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution.

2 Main Results

We begin with the following auxiliary lemmas that are essential in the proofs of our main results.

Lemma 2.1 ([19, Lemma 1]). *Let $f(t) \in C^n([T, \infty), (0, \infty))$ such that the derivative $f^{(n)}(t)$ is nonpositive on $[T, \infty)$ and not identically zero on any interval of the form $[T', \infty)$, $T' \geq T$. Then there exist a $T^* \geq T'$ and an integer ℓ, $0 \leq \ell \leq n-1$, with $n + \ell$ odd so that*

$$(-1)^{\ell+j} f^{(j)}(t) > 0 \quad \text{on } [T^*, \infty) \quad \text{for } j = \ell, \ldots, n-1,$$
$$f^{(i)}(t) > 0 \quad \text{on } [T^*, \infty) \quad \text{for } i = 1, \ldots, \ell - 1 \quad \text{when } \ell > 1.$$

Lemma 2.2 ([19, Lemma 2]). *Let $f(t)$ be as in Lemma 2.1 and $T^* \geq T'$ be assigned to $f(t)$ by Lemma 2.1. Moreover, let λ be a number with $0 < \lambda < 1$. If $\lim_{t\to\infty} f(t) \neq 0$, then there exists a $T^{**} \geq T^*/\lambda$ such that*

$$f(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} f^{(n-1)}(t) \quad \text{for } t \geq T^{**}. \tag{2.1}$$

Lemma 2.3 ([8, Lemma 1]). *Let $f(t)$ be as in Lemma 2.1 for $T' \geq T$, and $T^* \geq T'$ be assigned to $f(t)$ by Lemma 2.1. Then for every $\kappa \in (0, 1)$ there exists a $T^{**} \geq T^*$ such that*

$$\frac{f(t)}{f'(t)} \geq \kappa \frac{t}{\ell} \quad \text{for } t \geq T^{**}. \tag{2.2}$$

Lemma 2.4 ([15]). *If $a > 0$ and $0 < \gamma \leq 1$, then*

$$a^\gamma \leq \gamma a + (1 - \gamma),$$

where equality holds when $\gamma = 1$.

Now, we present our first oscillation result for (1.1) in the case where $\beta > \alpha$.

Theorem 2.5. *Let conditions* (H_1)–(H_4) *and* $\beta > \alpha$ *hold. Assume further that there exists a positive nondecreasing function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\eta(t) \le \sigma(t),\ \eta(t) < \tau(t),\ \text{and}\ \lim_{t\to\infty}\eta(t) = \infty. \tag{2.3}$$

If, for every $d_1 > 0$ *and* $d_2 > 0$*, the even-order linear delay differential inequalities*

$$(b(t)z^{(n-1)}(t))' + d_1 q(t)p^{-\beta/\alpha}(\tau^{-1}(\sigma(t)))z(h(t)) \le 0 \tag{2.4}$$

and

$$(b(t)z^{(n-1)}(t))' + d_2 q(t)p^{-\beta/\alpha}(\tau^{-1}(\sigma(t)))z(h(t)) \le 0 \tag{2.5}$$

have no positive solutions, then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. From the definition of $z(t)$, we see that

$$\begin{aligned} x^\alpha(t) &= \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right) \\ &\ge \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - \frac{z^{\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))}{p^{\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))}\right). \end{aligned} \tag{2.6}$$

Applying Lemma 2.4 in (2.6) yields

$$x^\alpha(t) \ge \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - \frac{1}{p^{\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))}\left(\frac{1}{\alpha}z(\tau^{-1}(\tau^{-1}(t))) + \left(1-\frac{1}{\alpha}\right)\right)\right]. \tag{2.7}$$

It follows from (H_1) and (1.1) that

$$z^{(n-1)}(t) > 0 \ \text{and}\ z^{(n)}(t) \le 0 \quad \text{for}\ t \ge t_1.$$

Thus, by Lemma 2.1, there exists a $t_2 \ge t_1$ and an odd integer $\ell \in \{1,3,5,\dots,n-1\}$ such that

$$\begin{aligned} (-1)^{\ell+j}z^{(j)}(t) > 0 \quad &\text{for}\ j = \ell,\dots,n-1, \\ z^{(i)}(t) > 0 \quad &\text{for}\ i = 1,\dots,\ell-1 \quad \text{when}\ \ell > 1, \end{aligned}$$

for $t \ge t_2$, and so we shall distinguish the following two cases:

$$\text{(I)}\ \ \ell \ge 3 \quad \text{for}\ t \ge t_2, \ \ \text{or}\ \ \text{(II)}\ \ \ell = 1 \quad \text{for}\ t \ge t_2.$$

Suppose (I) holds. Then,

$$z(t) > 0,\quad z'(t) > 0,\quad z''(t) > 0,\quad z'''(t) > 0,\quad \cdots\quad,\quad z^{(n-1)}(t) > 0,\quad z^{(n)}(t) \le 0 \tag{2.8}$$

for $t \ge t_2$. Since $(n-1) \ge \ell \ge 3$, in view of (2.2), there exists a $t_\kappa \ge t_2$ for every $\kappa \in (0,1)$ such that

$$\frac{z(t)}{z'(t)} \ge \kappa\frac{t}{\ell} \ge \kappa\frac{t}{n-1} \quad \text{for}\ t \ge t_\kappa, \tag{2.9}$$

which implies

$$\left(\frac{z(t)}{t^{(n-1)/\kappa}}\right)' = \frac{\kappa t z'(t) - (n-1)z(t)}{\kappa t^{(n-1)/\kappa+1}} \le 0 \ \ \text{for}\ t \ge t_\kappa. \tag{2.10}$$

Since $z(t) \to \infty$ as $t \to \infty$, there exists $t_3 \ge t_\kappa$ such that $z(\tau^{-1}(\tau^{-1}(t))) \ge 1$ and using this in (2.6), we get

$$x^\alpha(t) \ge \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - \frac{z(\tau^{-1}(\tau^{-1}(t)))}{p^{\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))}\right], \quad t \ge t_3. \tag{2.11}$$

Since $\tau(t) \leq t$ and $\tau'(t) > 0$, τ^{-1} is increasing and moreover $t \leq \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t)). \tag{2.12}$$

By virtue of (2.10) and (2.12), it follows that

$$(\tau^{-1}(t))^{(n-1)/\kappa} z(\tau^{-1}(\tau^{-1}(t))) \leq (\tau^{-1}(\tau^{-1}(t)))^{(n-1)/\kappa} z(\tau^{-1}(t)). \tag{2.13}$$

Combining (2.11) and (2.13), we conclude that

$$x^{\alpha}(t) \quad \geq \quad \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} \left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{(n-1)/\kappa} \frac{1}{p^{\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))}\right] \tag{2.14}$$

for $t \geq t_3$ for some $t_3 \geq t_\kappa$. From (H_4), there exist a $\epsilon_1 \in (0,1)$ and a $t_4 \geq t_3$ such that

$$\left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{(n-1)/\kappa} p^{-\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t))) \leq (1-\epsilon_1) \quad \text{for } t \geq t_4.$$

Using this in (2.14) gives

$$x^{\alpha}(t) \geq \frac{\epsilon_1 z(\tau^{-1}(t))}{p(\tau^{-1}(t))} \quad \text{for } t \geq t_4. \tag{2.15}$$

From (1.1) and (2.15), we obtain

$$(b(t)z^{(n-1)}(t))' + \epsilon_1^{\beta/\alpha} q(t) p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \leq 0. \tag{2.16}$$

In view of the fact that $\eta(t) \leq \sigma(t)$ and $z'(t) > 0$, inequality (2.16) takes the form

$$(b(t)z^{(n-1)}(t))' + \epsilon_1^{\beta/\alpha} q(t) p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z^{\beta/\alpha}(\tau^{-1}(\eta(t))) \leq 0, \ t \geq t_4. \tag{2.17}$$

Since $z(t) > 0$ and $z'(t) > 0$ on $[t_4, \infty)$, there exists a $t_5 \geq t_4$ and a constant $c > 0$ such that

$$z(t) \geq c \quad \text{for } t \geq t_5. \tag{2.18}$$

From (2.17) and (2.18), we see that z is a positive solution of the differential inequality

$$(b(t)z^{(n-1)}(t))' + d_1 q(t) p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z(h(t)) \leq 0, \ t \geq t_5, \tag{2.19}$$

where $d_1 = \epsilon_1^{\beta/\alpha} c^{\frac{\beta}{\alpha}-1} > 0$, i.e., (2.4) has a positive solution, which is a contradiction.

Next, we consider (II). Then,

$$z(t) > 0, \ (-1)^{j+1} z^{(j)}(t) > 0, \ j = 1, 2, \ldots, n-1, \quad \text{and} \ z^{(n)}(t) \leq 0 \tag{2.20}$$

for $t \geq t_2$. Since $\ell = 1$, in view of (2.2), there exists a $t_\kappa \geq t_2$ for every $\kappa \in (0,1)$ such that

$$\frac{z(t)}{z'(t)} \geq \kappa \frac{t}{1}, \quad t \geq t_\kappa, \tag{2.21}$$

from which we see that

$$\left(\frac{z(t)}{t^{1/\kappa}}\right)' \leq 0 \ \text{ for } t \geq t_\kappa. \tag{2.22}$$

By (2.12) and (2.22),

$$(\tau^{-1}(t))^{1/\kappa} z(\tau^{-1}(\tau^{-1}(t))) \leq (\tau^{-1}(\tau^{-1}(t)))^{1/\kappa} z(\tau^{-1}(t)) \tag{2.23}$$

for $t \geq t_3$ for some $t_3 \geq t_\kappa$. Combining (2.7) and (2.23), we obtain

$$x^{\alpha}(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} \left[1 - \frac{1}{p^{\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))} \left(\frac{1}{\alpha}\left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{1/\kappa} + \frac{(1-\frac{1}{\alpha})}{z(\tau^{-1}(t))}\right)\right], t \geq t_3.$$

Since $z(t) > 0$ and $z'(t) > 0$, we again see that (2.18) holds, and so the latter inequality takes the form

$$x^\alpha(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left[1 - p^{-\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))\left(\frac{1}{\alpha}\left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{1/\kappa} + \frac{(1-\frac{1}{\alpha})}{c}\right)\right] \tag{2.24}$$

for $t \geq t_4$ for some $t_4 \geq t_3$. From (H_5), for any $\epsilon_2 \in (0,1)$ there exists $t_5 \geq t_4$ such that

$$p^{-\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))\left(\frac{1}{\alpha}\left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{1/\kappa} + \frac{(1-\frac{1}{\alpha})}{c}\right) \leq 1 - \epsilon_2,\ t \geq t_5,$$

and using this in (2.24) implies

$$x^\alpha(t) \geq \frac{\epsilon_2 z(\tau^{-1}(t))}{p(\tau^{-1}(t))}, \quad \text{for } t \geq t_5. \tag{2.25}$$

Using (2.25) in (1.1) yields

$$(b(t)z^{(n-1)}(t))' + \epsilon_2^{\beta/\alpha} q(t) p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \leq 0.$$

Since $\eta(t) \leq \sigma(t)$ and $z'(t) > 0$, the latter inequality takes the form

$$(b(t)z^{(n-1)}(t))' + \epsilon_2^{\beta/\alpha} q(t) p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z^{\beta/\alpha}(\tau^{-1}(\eta(t))) \leq 0. \tag{2.26}$$

In view of (2.18) and $\beta > \alpha$, we see that z is a positive solution of the differential inequality

$$(b(t)z^{(n-1)}(t))' + d_2 q(t) p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z(h(t)) \leq 0,\ t \geq t_5, \tag{2.27}$$

where $d_2 = \epsilon_2^{\beta/\alpha} c^{\frac{\beta}{\alpha}-1} > 0$. That is, (2.5) has a positive solution, which is again a contradiction. The proof is now completed. □

Theorem 2.6. *Let conditions (H_1)–(H_4) and $\beta = \alpha$ hold. Assume further that there exists a positive nondecreasing function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that (2.3) holds. If, for any $\epsilon_1, \epsilon_2 \in (0,1)$, the even-order linear delay differential inequalities*

$$(b(t)z^{(n-1)}(t))' + \epsilon_1 q(t) p^{-1}(\tau^{-1}(\sigma(t))) z(h(t)) \leq 0 \tag{2.28}$$

and

$$(b(t)z^{(n-1)}(t))' + \epsilon_2 q(t) p^{-1}(\tau^{-1}(\sigma(t))) z(h(t)) \leq 0 \tag{2.29}$$

have no positive solutions, then (1.1) is oscillatory.

Proof. The proof follows from Theorem 2.5 with $\beta = \alpha$, and hence details are omitted. □

Theorem 2.7. *Let conditions (H_1)–(H_4) and $\beta < \alpha$ hold. Assume further that there exists a positive nondecreasing function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that (2.3) holds. If, for every $d_3 > 0$ and $d_4 > 0$, the even-order linear delay differential inequalities*

$$(b(t)z^{(n-1)}(t))' + d_3 q(t)\left(h^{(n-1)/\kappa}(t)\right)^{\frac{\beta}{\alpha}-1} p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z(h(t)) \leq 0 \tag{2.30}$$

and

$$(b(t)z^{(n-1)}(t))' + d_4 q(t)\left(h^{1/\kappa}(t)\right)^{\frac{\beta}{\alpha}-1} p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z(h(t)) \leq 0 \tag{2.31}$$

have no positive solutions, then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. As in the proof of Theorem 2.5, we again have two cases to consider: (I) $\ell \geq 3$ or (II) $\ell = 1$ for $t \geq t_2$. If case (I) holds, proceeding as in the proof of Theorem 2.5, we see that (2.10) holds for $t \geq t_\kappa \geq t_2$ and we again arrive at (2.17) for $t \geq t_4$. By (2.10), there exist a $t_3 \geq t_\kappa$ and a constant $d_5 > 0$ such that

$$z(t) \leq d_5 t^{(n-1)/\kappa} \quad \text{for } t \geq t_3. \tag{2.32}$$

Using (2.32) in (2.17) and applying the fact that $\beta/\alpha < 1$ yields

$$(b(t)z^{(n-1)}(t))' + d_3 q(t) \left(h^{(n-1)/\kappa}(t)\right)^{\frac{\beta}{\alpha}-1} p^{-\beta/\alpha}(\tau^{-1}(\sigma(t)))z(h(t)) \leq 0 \tag{2.33}$$

for $t \geq t_4$ for some $t_4 \geq t_3$, where $d_3 = \epsilon_1^{\beta/\alpha} d_5^{\frac{\beta}{\alpha}-1} > 0$. That is, (2.30) has a positive solution, a contradiction.

Next, assume that case (II) holds. Proceeding as in the proof of Theorem 2.5, we see that (2.22) holds for $t \geq t_\kappa \geq t_2$ and we again arrive at (2.26) for $t \geq t_5$. By (2.22), there exist a $t_3 \geq t_\kappa$ and a constant $d_6 > 0$ such that

$$z(t) \leq d_6 t^{1/\kappa} \quad \text{for } t \geq t_3. \tag{2.34}$$

Using (2.34) in (2.26) and applying the fact that $\beta/\alpha < 1$ yields

$$(b(t)z^{(n-1)}(t))' + d_4 q(t) \left(h^{1/\kappa}(t)\right)^{\frac{\beta}{\alpha}-1} p^{-\beta/\alpha}(\tau^{-1}(\sigma(t)))z(h(t)) \leq 0, \tag{2.35}$$

where $d_4 = \epsilon_2^{\beta/\alpha} d_6^{\frac{\beta}{\alpha}-1} > 0$ and $t \geq t_5$. That is, (2.31) has a positive solution, which is again a contradiction. This completes the proof of the Theorem. □

Next, we derive results concerning with the oscillatory behavior of (1.1) via comparison with first-order delay differential equations whose oscillatory characters are known.

Theorem 2.8. *Let conditions (H_1)–(H_4) and $\beta > \alpha$ hold. Assume further that there exists a positive nondecreasing function $\eta \in C^1([t_0,\infty),\mathbb{R})$ such that (2.3) holds. If for some constants $\lambda_1, \kappa_1 \in (0,1)$, the first-order linear delay differential equations*

$$y'(t) + \frac{d_1\lambda_1}{(n-1)!}\frac{h^{n-1}(t)}{b(h(t))} q(t)p^{-\beta/\alpha}(\tau^{-1}(\sigma(t)))y(h(t)) = 0 \tag{2.36}$$

and

$$w'(t) + \kappa_1 d_2 h(t) R_{n-3}(t) w(h(t)) = 0 \tag{2.37}$$

are oscillatory for every constants $d_1 > 0$ and $d_2 > 0$, then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. As in the proof of Theorem 2.5, we again have two cases to consider: (I) $\ell \geq 3$ or (II) $\ell = 1$ for $t \geq t_2$. If case (I) holds, proceeding as in the proof of Theorem 2.5, we again arrive at (2.19) for $t \geq t_5$. Since $z(t) > 0$ and $z'(t) > 0$ for $t \geq t_5$, we have $\lim_{t\to\infty} z(t) \neq 0$. Thus, by Lemma 2.2, for every $\lambda \in (0,1)$, there exists $t_\lambda \geq t_5$ such that

$$z(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t) \quad \text{for } t \geq t_\lambda, \tag{2.38}$$

from which we see that

$$z(h(t)) \geq \frac{\lambda}{(n-1)!} h^{n-1}(t) z^{(n-1)}(h(t)) \quad \text{for } t \geq t_6, \tag{2.39}$$

where $h(t) \geq t_\lambda$ for $t \geq t_6$ for some $t_6 \geq t_\lambda$. Using (2.39) in (2.19) yields

$$(b(t)z^{(n-1)}(t))' + \frac{d_1\lambda}{(n-1)!} h^{n-1}(t) q(t) p^{-\beta/\alpha}(\tau^{-1}(\sigma(t))) z^{(n-1)}(h(t)) \leq 0,$$

for every λ with $0 < \lambda < 1$. With $y(t) = b(t)z^{(n-1)}(t)$, we see that $y(t)$ is a positive solution of the first-order linear delay differential inequality

$$y'(t) + \frac{d_1\lambda}{(n-1)!}\frac{h^{n-1}(t)}{b(h(t))} q(t)p^{-\beta/\alpha}(\tau^{-1}(\sigma(t)))y(h(t)) \leq 0, \ t \geq t_6. \tag{2.40}$$

It follows from [20, Theorem 1] that the delay differential equation (2.36) corresponding to (2.40) also has a positive solution for all $\lambda_1 \in (0,1)$, but this contradicts our assumption on Eq. (2.36).

Next, assume that case (II) holds. As in the proof of Theorem 2.5, we again see that (2.21) and (2.27) hold for $t \geq t_5$. Integrating (2.27) from $t \geq t_5$ to ∞ gives

$$z^{(n-1)}(t) \geq d_2 R_0(t) z(h(t)).$$

Integrating the latter inequality from t to ∞ a total of $n-3$ times, we obtain

$$z''(t) + d_2 R_{n-3}(t) z(h(t)) \leq 0. \tag{2.41}$$

Using (2.21) in (2.41) yields

$$z''(t) + \kappa d_2 R_{n-3}(t) h(t) z'(h(t)) \leq 0. \tag{2.42}$$

With $w(t) = z'(t)$, we see that $w(t)$ is a positive solution of the first-order linear delay differential inequality

$$w'(t) + \kappa d_2 h(t) R_{n-3}(t) w(h(t)) \leq 0 \tag{2.43}$$

for every $\kappa \in (0,1)$. The remainder of the proof is similar to case (I) and hence it is omitted. This completes the proof of the theorem. □

Theorem 2.9. *Let conditions* (H_1)–(H_4) *and* $\beta = \alpha$ *hold. Assume further that there exists a positive nondecreasing function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that* (2.3) *holds. If for some constants* $\lambda_1, \kappa_1 \in (0,1)$*, the first-order linear delay differential equations*

$$y'(t) + \frac{\lambda_1 \epsilon_1}{(n-1)!} \frac{h^{n-1}(t)}{b(h(t))} q(t) p^{-1}(\tau^{-1}(\sigma(t))) y(h(t)) = 0 \tag{2.44}$$

and

$$w'(t) + \kappa_1 \epsilon_2 h(t) R_{n-3}(t) w(h(t)) = 0 \tag{2.45}$$

are oscillatory for any $\epsilon_1, \epsilon_2 \in (0,1)$*, then* (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 2.8 with $\beta = \alpha$, and hence the details are omitted. □

Corollary 2.10. *Let conditions* (H_1)–(H_4) *and* $\beta \geq \alpha$ *hold. Assume further that there exists a positive nondecreasing function* $\eta \in C^1([t_0,\infty),\mathbb{R})$ *such that* (2.3) *holds. If*

$$\lim_{t\to\infty} \int_{h(t)}^{t} \frac{h^{n-1}(s)}{b(h(s))} q(s) p^{-\beta/\alpha}(\tau^{-1}(\sigma(s))) ds = \infty \tag{2.46}$$

and

$$\lim_{t\to\infty} \int_{h(t)}^{t} h(s) R_{n-3}(s) ds = \infty, \tag{2.47}$$

then equation (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.8, we again arrive at (2.40) for $t \geq t_6$ and (2.43) for $t \geq t_5$. Integrating (2.40) from $h(t)$ to t and then using the fact that y is a decreasing function, we see that

$$\int_{h(t)}^{t} \frac{h^{n-1}(s)}{b(h(s))} q(s) p^{-\beta/\alpha}(\tau^{-1}(\sigma(s))) ds \leq \frac{(n-1)!}{d_1 \lambda},$$

which contradicts (2.46).

Next, integrating (2.43) from $h(t)$ to t and then using the fact that w is a decreasing function, we see that

$$\int_{h(t)}^{t} h(s) R_{n-3}(s) ds \leq \frac{1}{\kappa d_2}$$

which contradicts (2.47) and completes the proof. □

Theorem 2.11. *Let conditions (H_1)–(H_4) and $\beta < \alpha$ hold. Assume further that there exists a positive nondecreasing function $\eta \in C^1([t_0, \infty), \mathbb{R})$ such that (2.3) holds. If for some constants $\lambda_1, \kappa_1 \in (0, 1)$, the first-order linear delay differential equations*

$$y'(t) + \frac{d_3\lambda_1}{(n-1)!}\frac{q(t)\left(h^{(n-1)/\kappa}(t)\right)^{\frac{\beta}{\alpha}-1}}{b(h(t))p^{\beta/\alpha}(\tau^{-1}(\sigma(t)))}h^{n-1}(t)y(h(t)) = 0 \tag{2.48}$$

and

$$w'(t) + \kappa_1 d_4 h(t)F_{n-3}(t)w(h(t)) = 0 \tag{2.49}$$

are oscillatory for every constants $d_3 > 0$ and $d_4 > 0$, then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.7, we again have two cases to consider: (I) $\ell \geq 3$ or (II) $\ell = 1$ for $t \geq t_2$. If case (I) holds, we again arrive at (2.33) for $t \geq t_4$. Since $z(t) > 0$ and $z'(t) > 0$ for $t \geq t_2$, we have $\lim_{t\to\infty} z(t) \neq 0$ and so by Lemma 2.2, for every $\lambda \in (0, 1)$, there exists $t_\lambda \geq t_2$ such that (2.38) holds for $t \geq t_\lambda$. Using (2.38) in (2.33) gives

$$(b(t)z^{(n-1)}(t))' + \frac{d_3\lambda}{(n-1)!}\frac{q(t)\left(h^{(n-1)/\kappa}(t)\right)^{\frac{\beta}{\alpha}-1}}{p^{\beta/\alpha}(\tau^{-1}(\sigma(t)))}h^{n-1}(t)z^{(n-1)}(h(t)) \leq 0$$

for $t \geq t_4$. With $y(t) = b(t)z^{(n-1)}(t)$, we see that $y(t)$ is a positive solution of the first-order linear delay differential inequality

$$y'(t) + \frac{d_3\lambda}{(n-1)!}\frac{q(t)\left(h^{(n-1)/\kappa}(t)\right)^{\frac{\beta}{\alpha}-1}}{b(h(t))p^{\beta/\alpha}(\tau^{-1}(\sigma(t)))}h^{n-1}(t)y(h(t)) \leq 0. \tag{2.50}$$

It follows from [20, Theorem 1] that the delay differential equation (2.48) corresponding to (2.50) also has a positive solution for all $\lambda_1 \in (0, 1)$, but this contradicts our assumption on Eq. (2.48).

Next, assume that case (II) holds. Then again (2.21) holds for every $\kappa \in (0, 1)$ and for $t \geq t_\kappa \geq t_2$. Proceeding as in the proof of Theorem 2.7, we again arrive at (2.35) for $t \geq t_5$. Integrating (2.35) from $t \geq t_5$ to ∞, we obtain

$$z^{(n-1)}(t) \geq d_4 F_0(t)z(h(t)).$$

Integrating the latter inequality from t to ∞ a total of $n - 3$ times, we obtain

$$z''(t) + d_4 F_{n-3}(t)z(h(t)) \leq 0, \quad t \geq t_5. \tag{2.51}$$

Thus, if we set $w(t) = z'(t)$ and using (2.21) in (2.51), then we conclude that w is a positive solution of

$$w'(t) + \kappa d_4 h(t)F_{n-3}(t)w(h(t)) \leq 0.$$

The rest of the proof is similar to case (I) and hence the details are omitted. This completes the proof. □

Similar to what we did above, we obtain the following Corollary.

Corollary 2.12. *Let conditions (H_1)–(H_4) and $\beta < \alpha$ hold. Assume further that there exists a positive nondecreasing function $\eta \in C^1([t_0, \infty), \mathbb{R})$ such that (2.3) holds. If*

$$\lim_{t\to\infty}\int_{h(t)}^{t}\frac{\left(h^{(n-1)/\kappa}(s)\right)^{\frac{\beta}{\alpha}-1}}{b(h(s))p^{\beta/\alpha}(\tau^{-1}(\sigma(s)))}q(s)h^{n-1}(s)ds = \infty \tag{2.52}$$

and

$$\lim_{t\to\infty}\int_{h(t)}^{t} h(s)F_{n-3}(s)ds = \infty, \tag{2.53}$$

then equation (1.1) is oscillatory.

We conclude this paper with the following example.

Example 2.13. Consider the nonlinear delay differential equation with a superlinear neutral term

$$\left(x(t)+tx^{\alpha}\left(\frac{t}{3}\right)\right)^{(n)}+\frac{a}{t^{n-2}}x^{\alpha}\left(\frac{t}{2}\right)=0,\quad t\geq 1, \tag{2.54}$$

where $\alpha > 1$, $a > 0$ and $n \geq 4$.

Here $b(t) = 1$, $p(t) = t$, $\tau(t) = t/3$, $\sigma(t) = t/2$, and $q(t) = a/t^{n-2}$. Choosing $\eta(t) = \frac{t}{4}$, we see that (2.3) holds, and a simple calculation shows that

$$h(t)=\tau^{-1}(\eta(t))=3t/4,\ \tau^{-1}(\sigma(t))=3t/2,\ \tau^{-1}(t)=3t,\ \text{ and }\ \tau^{-1}(\tau^{-1}(t))=9t.$$

Choosing $\kappa = 1/3$, we see that

$$\lim_{t\to\infty}\left(\frac{t}{\tau(t)}\right)^{(n-1)/\kappa}\frac{1}{p^{1/\alpha}(t)}=\lim_{t\to\infty}3^{3(n-1)}\frac{1}{t^{1/\alpha}}=0,$$

i.e., condition (H_4) holds. Further

$$R_0(t)=\frac{2at^{2-n}}{3(n-2)}\quad\text{and}\quad R_{n-3}(t)=\frac{2a}{3(n-2)!}\frac{1}{t}.$$

Now conditions (2.46) and (2.47) become

$$\lim_{t\to\infty}\int_{\frac{3t}{4}}^{t}\left(\frac{3s}{4}\right)^{n-1}\frac{a}{s^{n-2}}\left(\frac{2}{3s}\right)ds=\lim_{t\to\infty}\frac{2a}{3}\left(\frac{3}{4}\right)^{n-1}\frac{t}{4}=\infty$$

and

$$\lim_{t\to\infty}\int_{\frac{3t}{4}}^{t}\frac{3s}{4}\frac{2a}{3(n-2)!}\frac{1}{s}ds=\lim_{t\to\infty}\frac{a}{2(n-2)!}\frac{t}{4}=\infty,$$

that is, conditions (2.46) and (2.47) are satisfied. Hence, by Corollary 2.10, equation (2.54) is oscillatory.

3 Conclusion

In this paper, we present new comparison theorems that compare the higher-order equation (1.1) with a couple of first-order delay differential equations. There are many results available in the literature on the oscillation of first order delay differential equations, and so it would be possible to formulate many criteria for the oscillation of (1.1) based on the results in this paper. Further, the results obtained in this paper provide an answer to the interesting problem mentioned in the paper [13] for $\alpha > 1$, that is, equation (1.1) with a superlinear neutral term.

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Received: September 24th, 2021
Accepted: December 26th, 2021