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SOME INEQUALITIES FOR GOLDEN RIEMANNIAN SPACE FORMS

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Abstract In the present paper, we prove sharp inequalities involving generalized normalized δ-Casorati curvature and normalized δ-Casorati curvature for slant submanifolds of Golden Riemannian space forms. Moreover, we also characterize those submanifolds for which the equality cases hold. Some special cases of these inequalities are given.

1 Introduction

The theory of Chen's invariants has been very interesting topic in the field of differential geometry of submanifolds after introducing Chen's δ- invariants by B. Y. Chen [3]. Since then, many geometers considered such invariants and Chen like inequalities in many classes of submanifolds in different ambient spaces (for instance, see [16, 17], [18], [21], [23, 24]). One can also observe that Casorati curvature of submanifolds in a Riemannian Geometry is an extrinsic invariant defined as the normalized square of the length of the second fundamental form and it was preferred by Casorati over the Gaussian curvature because corresponds better with the common intuition of curvature. We see that some optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces were derived in [8], [9], [14], [15],[22].

On the other hand, the golden ratio has attracted attention of many researchers of diverse interests for more than 2000 years. In fact, it will be fair to say that this number has inspired thinkers of all disciplines like no other number in the history of number theory.

C. Hretcanu and M. Crasmareanu ([11], [12]) studied induced structure on an invariant submanifold in a golden Riemannian manifold and showed that the golden structure induces on every invariant submanifold a golden structure. In 2014, M. Ozkan [19] investigated golden semi-Riemannian manifolds and defined the horizontal lift of golden structures in a tangent bundle.

In 1990, B. Y. Chen introduced some fundamental results concerning slant immersions [5]. O. Bahadir and S. Uddin characterized slant submanifolds of a Riemannian manifold with Golden structure and provided some non-trivial examples of slant submanifolds of Golden Riemannian manifolds [1].

In this paper, we study slant submanifolds in golden Riemannian manifolds. In Section 2, we provide some basic formulas and definition to make this paper self contained. In Section 3, we prove sharp inequalities that involve the generalized normalized δ-Casorati curvature and normalized δ-Casorati curvature for slant submanifolds in golden Riemannian space forms. Moreover, we give some special cases of these inequalities as a consequence for different classes of submanifolds.

2 Preliminaries

2.1 Riemannian Invariants

[4] Let $\mathcal{N}^n$ be n-dimensional Riemannian submanifold of m-dimensional Riemannian manifold $(\overline{\mathcal{N}}, \overline{g})$ and g be the metric tensor induced on $\mathcal{N}$. If $\overline{\nabla}$ is the Levi-Civita connection on $\overline{\mathcal{N}}$ and ∇ is the covariant differentiation induced on $\mathcal{N}$, then the Gauss and Weingarten formulas are given

by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \forall X, Y \in \Gamma(T\mathcal{N})$$

and

$$\overline{\nabla}_X N = -S_N X + \nabla^{\perp}_X N, \qquad \forall X \in \Gamma(T\mathcal{N}), \forall N \in \Gamma(T\mathcal{N}^{\perp})$$

where h is the second fundamental form of $\mathcal{N}$, $\nabla^{\perp}$ is the connection on the normal bundle and S_N is the shape operator of $\mathcal{N}$ with respect to N. The shape operator S_N and the second fundamental form h are related by

$$g(S_N X, Y) = \overline{g}(h(X,Y), N) \qquad \forall X, Y \in \Gamma(T\mathcal{N}), \forall N \in \Gamma(T\mathcal{N}^{\perp}).$$

We write the Gauss equation as follows [25]

$$\begin{aligned} \overline{R}(X,Y,Z,W) &= R(X,Y,Z,W) - g(h(X,W), h(Y,Z)) \\ &\quad + g(h(X,Z), h(Y,W)) \end{aligned} \tag{2.1}$$

for all vector fields $X, Y, Z, W \in T\mathcal{N}$.

Let us consider a local orthonormal tangent frame $\{E_1, \ldots, E_n\}$ of the tangent bundle $T\mathcal{N}$ of $\mathcal{N}$ and a local orthonormal normal frame $\{E_{n+1}, \ldots, E_m\}$ of the normal bundle $T^{\perp}\mathcal{N}$ of $\mathcal{N}$ in $\overline{\mathcal{N}}$. Then, at any point $p \in \mathcal{N}$, the scalar curvature τ is given by

$$\tau = \sum_{i \le i < j \le n} R(E_i, E_j, E_j, E_i)$$

and the normalized scalar curvature ρ of $\mathcal{N}$ is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

The mean curvature vector denoted by $\mathcal{H}$ of $\mathcal{N}$ is given by

$$\mathcal{H} = \sum_{i=1}^{n} \frac{1}{n} h(E_i, E_i).$$

Conveniently, let us put

$$h^r_{ij} = g(h(E_i, E_j), E_r)$$

for $i, j = \{1, \ldots, n\}$ and $r = \{n+1, \ldots, m\}$. Then the squared norm of mean curvature vector of $\mathcal{N}$ is defined as

$$||\mathcal{H}||^2 = \frac{1}{n^2} \sum_{r=n+1}^{m} \left\{ \sum_{i=1}^{n} h^r_{ii} \right\}^2.$$

and the squared norm of second fundamental form h is denoted by

$$\mathcal{C} = \frac{1}{n} ||h||^2, \tag{2.2}$$

where

$$||h||^2 = \sum_{r=n+1}^{m} \sum_{i,j=1}^{n} \left(h^r_{ij}\right)^2.$$

It is known as the *Casorati curvature* $\mathcal{C}$ of $\mathcal{N}$.

Let us assume that $\mathcal{L}$ be a s-dimensional subspace of $T\mathcal{N}$, $s \ge 2$, and $\{E_1, \ldots, E_s\}$ be an orthonormal basis of $\mathcal{L}$, then the scalar curvature of the s-plane section $\mathcal{L}$ is given by

$$\tau(\mathcal{L}) = \sum_{i \le i < j \le s} R(E_i, E_j, E_j, E_i)$$

and the Casorati curvature of the subspace $\mathcal{L}$ is as follows

$$\mathcal{C}(\mathcal{L}) = \frac{1}{s}\sum_{r=n+1}^{m}\sum_{i,j=1}^{s}\left(h_{ij}^{r}\right)^{2}$$

The normalized δ-Casorati curvatures $\delta_c(n)$ and $\widehat{\delta}_c(n)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n(n-1)}\inf\{\mathcal{C}(\mathcal{L})|\mathcal{L} : \text{a hyperplane of } T_p\mathcal{N}\}$$

and

$$[\widehat{\delta}_c(n-1)]_p = 2\mathcal{C}_p - \frac{2n-1}{2n}\sup\{\mathcal{C}(\mathcal{L})|\mathcal{L} : \text{a hyperplane of } T_p\mathcal{N}\}.$$

The generalized normalized δ-Casorati curvatures $\delta_C(r;n-1)$ and $\widehat{\delta}_C(r;n-1)$ of the submanifold $\mathcal{N}^n$ are defined for any positive real number $r \neq n(n-1)$ as

$$\begin{aligned}[\delta_c(r;n-1)]_p &= r\mathcal{C}_p \\ &+ \frac{(n-1)(n+r)(n^2-n-r)}{rn}\inf\{\mathcal{C}(\mathcal{L})|\mathcal{L} : \text{a hyperplane of } T_p\mathcal{N}\}\end{aligned}$$

if $0 < r < n^2 - n$, and

$$\begin{aligned}[\widehat{\delta}_c(r;n-1)]_p &= r\mathcal{C}_p \\ &- \frac{(n-1)(n+r)(r-n^2+n)}{rn}\sup\{\mathcal{C}(\mathcal{L})|\mathcal{L} : \text{a hyperplane of } T_p\mathcal{N}\}\end{aligned}$$

if $r > n^2 - n$.

A point $p \in \mathcal{N}$ is said to be an *invariantly quasi-umbilical point* if there exist $m-n$ orthogonal unit normal vectors $\{E_{n+1}, \ldots, E_m\}$ such that the shape operator with respect to all directions E_r have an eigenvalue of multiplicity $n-1$ and that for each E_r the distinguished eigendirection is the same. The submanifold $\mathcal{N}$ is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point [2].

2.2 Golden Riemannian manifolds

Let $(\overline{\mathcal{N}}, \overline{g})$ be $(n+m)$-dimensional Riemannian manifold and let F be a $(1,1)$-tensor field on $\overline{\mathcal{N}}$. If F satisfies the following equation

$$L(X) = X^n + a_n X^{n-1} + ... + a_2 X + a_1 I = 0,$$

where I is the identity transformation and (for $X = F$) $F^{n-1}(p), F^{n-2}(p), ..., F(p), I$ are linearly independent at every point $p \in \overline{\mathcal{N}}$. Then the polynomial $L(X)$ is called the structure polynomial. If we select the structure polynomial $L(X) = X^2 + I$ (or $L(X) = X^2 - I$) we get an almost complex structure (or an almost product structure) [7, 10, 1] .

Let $(\overline{\mathcal{N}}, \overline{g})$ be $(n+m)$-dimensional Riemannian manifold and let φ be a $(1,1)$-tensor field on $\overline{\mathcal{N}}$. If φ satisfies the following equation

$$\varphi^2 - \varphi - I = 0,$$

where I is the identity transformation. Then the tensor field φ is called a golden structure on $\overline{\mathcal{N}}$. If the Riemannian metric $\overline{g}$ is φ compatible, then $(\overline{\mathcal{N}}, \overline{g}, \varphi)$ is called a Golden Riemannian manifold [10, 13, 1]. We have the following relation for φ-compatible metric

$$\overline{g}(\varphi X, Y) = \overline{g}(X, \varphi Y)$$

$\forall X, Y \in \Gamma(T\overline{\mathcal{N}})$, where $\Gamma(T\overline{\mathcal{N}})$ is the set of all vector fields on $\overline{\mathcal{N}}$. If we interchange X by φX in above equation, we get

$$\overline{g}(\varphi X, \varphi Y) = \overline{g}(\varphi^2 X, Y) = \overline{g}(\varphi X, Y) + \overline{g}(X, Y)$$

Let $\overline{\mathcal{N}}$ be an $(n+m)$-dimensional differentiable manifold with a tensor field F of type $(1,1)$ on $\overline{\mathcal{N}}$ such that $F^2 = I,\ F \neq \pm I$. Then F is called an almost product structure. If an almost product structure F admits a Riemannian metric $\overline{g}$ such that

$$\overline{g}(FX,Y) = \overline{g}(X,FY), \forall X,Y \in \Gamma(T\overline{\mathcal{N}}),$$

then $(\overline{\mathcal{N}},\overline{g})$ is called almost product Riemannian manifold.
An almost product structure F induces a Golden structure as follows

$$\varphi = \frac{1}{2}(I + \sqrt{5}F)$$

Conversely, if φ is a golden structure then

$$F = \frac{1}{\sqrt{5}}(2\varphi - I)$$

is an almost product structure [7, 1].

Example 1 [1, 12] Consider the Euclidean 4-space R^4 with standard coordinates (x_1, x_2, x_3, x_4). Let φ be an $(1,1)$ tensor field on R^4 defined by

$$\varphi(x_1,x_2,x_3,x_4) = (\psi x_1, \psi x_2, (1-\psi)x_3, (1-\psi)x_4)$$

for any vector field $(x_1,x_2,x_3,x_4) \in R^4$, where $\psi = \frac{1+\sqrt{5}}{2}$ and $1-\psi = \frac{1-\sqrt{5}}{2}$ are the roots of the equation $x^2 = x+1$. Then we obtain

$$\begin{aligned}\varphi^2(x_1,x_2,x_3,x_4) &= (\psi^2 x_1, \psi^2 x_2, (1-\psi)^2 x_3, (1-\psi)^2 x_4)\\ &= (\psi x_1, \psi x_2, (1-\psi)x_3, (1-\psi)x_4) + (x_1,x_2,x_3,x_4).\end{aligned}$$

Thus, we have $\varphi^2 - \varphi - I = 0$. Moreover, we get

$$< \varphi(x_1,x_2,x_3,x_4), (y_1,y_2,y_3,y_4) >=< (x_1,x_2,x_3,x_4), \varphi(y_1,y_2,y_3,y_4) >$$

for each vector fields $(x_1,x_2,x_3,x_4), (y_1,y_2,y_3,y_4) \in R^4$, where $<,>$ is the standard metric on R^4. Hence, $(R^4, <,>, \varphi)$ is a Golden Riemannian manifold.

Let $(\mathcal{N}, g)$ be a submanifold of a Golden Riemannian manifold $(\overline{\mathcal{N}}, \overline{g}, \varphi)$, where g is the induced metric on $\mathcal{N}$. Then, for any $X \in \Gamma(T\mathcal{N})$ we can write

$$\varphi X = PX + QX,$$

where P and Q are the projections of $T\overline{\mathcal{N}}$ onto $T\mathcal{N}$ and $trT\mathcal{N}$, respectively, that is, PX and QX are tangent and transversal components of φX. We can also write

$$g(PX,Y) = g(X,PY).$$

For each nonzero vector X tangent to $\mathcal{N}$ at p, let $\theta(X)$ be the angle between $T\mathcal{N}$ and φX. If $\theta(X)$ is independent of the choice of $p \in \mathcal{N}$ and $X \in T_p\mathcal{N}$ then $\mathcal{N}$ is called a slant submanifold. If the slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, then $\mathcal{N}$ is an φ-invariant and φ-anti-invariant submanifold, respectively. A slant submanifold which is neither invariant nor anti-invariant is called proper slant (or θ-slant proper) submanifold.

Inspired by the characterization given in [5, 6], we give the following characterization for slant submanifolds of Golden Riemannian manifolds.

Theorem 2.1. *[1] Let $(\mathcal{N}, g)$ be a submanifold of a Golden Riemannian manifold $(\overline{\mathcal{N}}, \overline{g}, \varphi)$. Then, $\mathcal{N}$ is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(\varphi + I).$$

Furthermore, if θ is slant angle of $\mathcal{N}$, then $\lambda = cos^2\theta$.

Theorem 2.2. *[1] Let $(\mathcal{N}, g)$ be a slant submanifold of a Golden Riemannian manifold $(\overline{\mathcal{N}}, \overline{g}, \varphi)$. Then, for any $X, Y \in \Gamma(T\mathcal{N})$, we have*

$$g(PX, PY) = cos^2\theta(g(X,Y) + g(X, PY)).$$

Now, let us suppose that $\mathcal{N}_p$ and $\mathcal{N}_q$ be two real-space forms with constant sectional curvatures c_p and c_q, respectively. Then, the Riemannian curvature tensor R of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$ is given by [20]:

$$\begin{aligned} R(X,Y)Z &= \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\{g(Y,Z)X - g(X,Z)Y + g(\varphi Y, Z)\varphi X \\ &- g(\varphi X, Z)\varphi Y\} + \Big(-\frac{(1-\psi)c_p + \psi c_q}{4}\Big)\{g(\varphi Y, Z)X \\ &- g(\varphi X, Z)Y + g(Y,Z)\varphi X - g(X,Z)\varphi Y\} \end{aligned} \tag{2.3}$$

3 Main results

We prove sharp inequalities involving the generalized normalized δ-Casorati curvature for slant submanifold of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$.

Theorem 3.1. *Let $\mathcal{N}$ be an n-dimensional slant submanifold of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(r; n-1)$ satisfies

$$\begin{aligned} \rho \leq\ & \frac{\delta_c(r;n-1)}{n(n-1)} \\ &+\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{1 + \frac{1}{n(n-1)}tr^2\varphi - cos^2\theta\Big\{\frac{1}{n-1} + \frac{1}{n(n-1)}trP\Big\}\Big\} \\ &+\Big(-\frac{(1-\psi)c_p + \psi c_q}{4}\Big)\frac{2}{n}tr\varphi \end{aligned} \tag{3.1}$$

for any real number r such that $0 < r < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(r; n-1)$ satisfies

$$\begin{aligned} \rho \leq\ & \frac{\widehat{\delta}_c(r;n-1)}{n(n-1)} \\ &+\Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}}\Big)\Big\{1 + \frac{1}{n(n-1)}tr^2\varphi - cos^2\theta\Big\{\frac{1}{n-1} + \frac{1}{n(n-1)}trP\Big\}\Big\} \\ &+\Big(-\frac{(1-\psi)c_p + \psi c_q}{4}\Big)\frac{2}{n}tr\varphi \end{aligned} \tag{3.2}$$

for any real number $r > n(n-1)$.

Moreover, the equalities hold in the relations (3.1) and (3.2) if and only if $\mathcal{N}^n$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{N}}$, such that with respect to some orthonormal tangent frame $\{E_1, \ldots, E_n\}$ and orthonormal normal frame $\{E_{n+1}, \ldots, E_{n+m}\}$, the shape operators $S_r, r \in \{n+1, \ldots, n+m\}$, take the following forms:

$$S_{n+1} = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{r}b \end{pmatrix}, \qquad S_{n+2} = \cdots = S_{n+m} = 0. \tag{3.3}$$

Proof. (i) Since $\overline{\mathcal{N}}$ is a locally golden product space form, from (2.3) and Gauss equation, we have

$$\begin{aligned} 2\tau(p) &= \Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{n(n-1)+tr^2\varphi\Big\} \\ &\quad -\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)cos^2\theta(n+trP) \\ &\quad +\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)2(n-1)tr\varphi+n^2||\mathcal{H}||^2-n\mathcal{C}, \end{aligned} \tag{3.4}$$

where we have used (2.2).

Now, let $\mathcal{L}$ be a hyperplane of $T_p\mathcal{N}$ and $\mathcal{Q}$ be a quadratic polynomial in the components of the second fundamental form, defined as:

$$\begin{aligned} \mathcal{Q} &= r\mathcal{C}+\frac{(n-1)(n+r)(n^2-n-r)}{rn}\mathcal{C}(\mathcal{L})-2\tau(p) \\ &\quad +\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{n(n-1)+tr^2\varphi\Big\} \\ &\quad -\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)cos^2\theta(n+trP) \\ &\quad +\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)2(n-1)tr\varphi. \end{aligned} \tag{3.5}$$

We can assume without loss of generality that $\mathcal{L}$ is spanned by $\{E_1,\dots,E_{n-1}\}$. Then we have

$$\begin{aligned} \mathcal{Q} &= \frac{r}{n}\sum_{\alpha=n+1}^{n+m}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2+\frac{(n+r)(n^2-n-r)}{rn}\sum_{\alpha=n+1}^{n+m}\sum_{i,j=1}^{n-1}(h_{ij}^\alpha)^2-2\tau(p) \\ &\quad +\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)\Big\{n(n-1)+tr^2\varphi\Big\} \\ &\quad -\Big(-\frac{(1-\psi)c_p-\psi c_q}{2\sqrt{5}}\Big)cos^2\theta(n+trP) \\ &\quad +\Big(-\frac{(1-\psi)c_p+\psi c_q}{4}\Big)2(n-1)tr\varphi. \end{aligned} \tag{3.6}$$

Taking into accounts (3.4) and (3.6), we obtain

$$\begin{aligned} \mathcal{Q} &= \frac{n+r}{n}\sum_{\alpha=n+1}^{n+m}\sum_{i,j=1}^{n}(h_{ij}^\alpha)^2+\frac{(n+r)(n^2-n-r)}{rn}\sum_{\alpha=n+1}^{n+m}\sum_{i,j=1}^{n-1}(h_{ij}^\alpha)^2 \\ &\quad -\sum_{\alpha=n+1}^{n+m}\Big(\sum_{i=1}^{n}h_{ii}^\alpha\Big)^2. \end{aligned}$$

One can easily derive that

$$\begin{aligned} \mathcal{Q} &= \sum_{\alpha=n+1}^{n+m}\sum_{i=1}^{n-1}\Big[\frac{n^2+n(r-1)-2r}{r}(h_{ii}^\alpha)^2+\frac{2(n+r)}{n}(h_{in}^\alpha)^2\Big] \\ &\quad +\sum_{\alpha=n+1}^{n+m}\Big[\frac{2(n+r)(n-1)}{r}\sum_{i<j=1}^{n-1}(h_{ij}^\alpha)^2-2\sum_{i<j=1}^{n}h_{ii}^\alpha h_{jj}^\alpha \\ &\quad +\frac{r}{n}(h_{nn}^\alpha)^2\Big]. \end{aligned} \tag{3.7}$$

From (3.7), we can find the critical points

$$h^c=(h_{11}^{n+1},h_{12}^{n+1},\dots,h_{nn}^{n+1},\dots,h_{11}^{n+m},\dots,h_{nn}^{n+m})$$

of $\mathcal{Q}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{aligned}
\frac{\partial \mathcal{Q}}{\partial h^{\alpha}_{ii}} &= \frac{2(n+r)(n-1)}{r} h^{\alpha}_{ii} - 2\sum_{l=1}^{n} h^{\alpha}_{ll} = 0, \\
\frac{\partial \mathcal{Q}}{\partial h^{\alpha}_{nn}} &= \frac{2r}{n} h^{\alpha}_{nn} - 2\sum_{l=1}^{n-1} h^{\alpha}_{ll} = 0, \\
\frac{\partial \mathcal{Q}}{\partial h^{\alpha}_{ij}} &= \frac{4(n+r)(n-1)}{r} h^{\alpha}_{ij} = 0, \\
\frac{\partial \mathcal{Q}}{\partial h^{\alpha}_{in}} &= \frac{4(n+r)}{n} h^{\alpha}_{in} = 0,
\end{aligned} \tag{3.8}$$

where $i, j = \{1, 2, \ldots, n-1\}, i \neq j$, and $\alpha \in \{n+1, \ldots, n+m\}$.

Hence, every solution h^c has $h^r_{ij} = 0$ for $i \neq j$ and the corresponding determinant to the first two sets of equations of the above system (3.8) is zero (there exist solutions for non-totally geodesic submanifolds). Moreover, we find that the Hessian matrix $H(\mathcal{Q})$ has the following eigenvalues:

$$\lambda_{11} = 0,\ \lambda_{22} = \frac{2(n^3 - n^2 + r^2)}{rn},\ \lambda_{33} = \cdots = \lambda_{nn} = \frac{2(n+r)(n-1)}{r},$$

$$\lambda_{ij} = \frac{4(n+r)(n-1)}{r},\ \lambda_{in} = \frac{4(n+1)}{n},\ \forall\, i, j \in \{1, 2, \ldots, n-1\},\ i \neq j.$$

Thus, it follows know that $\mathcal{Q}$ is parabolic and reaches a minimum $\mathcal{Q}(h^c) = 0$ for the solution h^c of the system (3.8). It implies that $\mathcal{Q} \geq 0$ and hence we have

$$\begin{aligned}
2\tau(p) \leq\ & r\mathcal{C} + \frac{(n-1)(n+r)(n^2-n-r)}{rn}\mathcal{C}(\mathcal{L}) \\
&+ \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ n(n-1) + tr^2\varphi \Big\} \\
&- \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) cos^2\theta (n + trP) \\
&+ \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) 2(n-1) tr\varphi,
\end{aligned}$$

whereby, we obtain

$$\begin{aligned}
\rho \leq\ & \frac{r}{n(n-1)}\mathcal{C} + \frac{(n+r)(n^2-n-r)}{rn^2}\mathcal{C}(\mathcal{L}) \\
& \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2\varphi \Big\} \\
& - \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) cos^2\theta \Big\{ \frac{1}{n-1} + \frac{1}{n(n-1)} trP \Big\} \\
& + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi
\end{aligned}$$

for every tangent hyperplane $\mathcal{L}$ of $T_p\mathcal{N}$. If we take the infimum over all tangent hyperplanes $\mathcal{L}$, the result trivially follows. Moreover, the equality sign holds if and only if

$$h^{\alpha}_{ij} = 0,\ \forall\, i, j \in \{1, \ldots, n\},\ i \neq j \tag{3.9}$$

and

$$h^{\alpha}_{nn} = \frac{n(n-1)}{r} h^{\alpha}_{11} = \frac{n(n-1)}{r} h^{\alpha}_{22} \cdots = \frac{n(n-1)}{r} h^{\alpha}_{n-1n-1}, \quad \forall\, \alpha \in \{n+1, \ldots, n+m\}. \tag{3.10}$$

In the light of (3.9) and (3.10), we conclude that the equality sign holds in the inequality (3.1) if and only if the submanifold $\mathcal{N}$ is invariantly quasi-umbilical with trivial normal connection in $\mathcal{N}$, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the form of (3.3).

(ii) In the same manner, we can establish an inequality in the second part of the theorem. □

Next, We give sharp inequalities involving the normalized δ-Casorati curvature for slant submanifold of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$.

Theorem 3.2. *Let $\mathcal{N}$ be an n-dimensional slant submanifold of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\begin{aligned}\rho \quad \leq \quad & \delta_c(n-1) + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2\varphi \Big\} \\ & - \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) cos^2\theta \Big\{ \frac{1}{n-1} + \frac{1}{n(n-1)} trP \Big\} \\ & + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \end{aligned} \tag{3.11}$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\begin{aligned}\rho \quad \leq \quad & \widehat{\delta}_c(n-1) + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2\varphi \Big\} \\ & - \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) cos^2\theta \Big\{ \frac{1}{n-1} + \frac{1}{n(n-1)} trP \Big\} \\ & + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \end{aligned} \tag{3.12}$$

Moreover, the equalities hold in the relations (3.11) and (3.12) if and only if $\mathcal{N}^n$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{N}}$, such that with respect to some orthonormal tangent frame $\{E_1, \ldots, E_n\}$ and orthonormal normal frame $\{E_{n+1}, \ldots, E_{n+m}\}$, the shape operators $S_r, r \in \{n+1, \ldots, n+m\}$, take the following forms:

$$S_{n+1} = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2b \end{pmatrix}, \qquad S_{n+2} = \cdots = S_{n+m} = 0. \tag{3.13}$$

and

$$S_{n+1} = \begin{pmatrix} 2b & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2b & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2b & 0 \\ 0 & 0 & 0 & \ldots & 0 & b \end{pmatrix}, \qquad S_{n+2} = \cdots = S_{n+m} = 0. \tag{3.14}$$

As a consequence of theorem 3.1, we give sharp inequalities that involve generalized normalized δ-Casorati curvature for invariant and anti-invariant submanifolds in golden Riemannian space forms. We know that invariant submanifolds are slant submanifolds with $\theta = 0$. We have the following result.

Corollary 3.3. *Let $\mathcal{N}$ be an n-dimensional invariant submanifold of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(r; n-1)$ satisfies

$$\begin{aligned}\rho \;\leq\; & \frac{\delta_c(r;n-1)}{n(n-1)} + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2\varphi \Big\} \\ & - \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ \frac{1}{n-1} + \frac{1}{n(n-1)} trP \Big\} \\ & + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \end{aligned} \tag{3.15}$$

for any real number r such that $0 < r < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(r; n-1)$ satisfies

$$\begin{aligned}\rho \;\leq\; & \frac{\widehat{\delta}_c(r;n-1)}{n(n-1)} + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2\varphi \Big\} \\ & - \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ \frac{1}{n-1} + \frac{1}{n(n-1)} trP \Big\} \\ & + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \end{aligned} \tag{3.16}$$

for any real number $r > n(n-1)$.

Moreover, the equalities hold in the relations (3.15) and (3.16) if and only if $\mathcal{N}^n$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{N}}$, such that with respect to some orthonormal tangent frame $\{E_1, \ldots, E_n\}$ and orthonormal normal frame $\{E_{n+1}, \ldots, E_{n+m}\}$, the shape operators $S_r, r \in \{n+1, \ldots, n+m\}$, take the following forms:

$$S_{n+1} = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{r} b \end{pmatrix}, \qquad S_{n+2} = \cdots = S_{n+m} = 0. \tag{3.17}$$

Anti-invariant submanifolds are slant submanifolds with $\theta = \frac{\pi}{2}$, we have the following result for anti-invariant submanifolds in a locally golden product space form.

Corollary 3.4. *Let $\mathcal{N}$ be an n-dimensional anti-invariant submanifold of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(r; n-1)$ satisfies

$$\begin{aligned}\rho \;\leq\; & \frac{\delta_c(r;n-1)}{n(n-1)} + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2\varphi \Big\} \\ & + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \end{aligned} \tag{3.18}$$

for any real number r such that $0 < r < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(r; n-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta}_c(r;n-1)}{n(n-1)} + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2 \varphi \Big\} + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \tag{3.19}$$

for any real number $r > n(n-1)$.

Moreover, the equalities hold in the relations (3.18) and (3.19) if and only if M^n is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{N}}$, such that with respect to some orthonormal tangent frame $\{E_1, \ldots, E_n\}$ and orthonormal normal frame $\{E_{n+1}, \ldots, E_{n+m}\}$, the shape operators $S_r, r \in \{n+1, \ldots, n+m\}$, take the following forms:

$$S_{n+1} = \begin{pmatrix} b & 0 & 0 & \ldots & 0 & 0 \\ 0 & b & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{r} b \end{pmatrix}, \qquad S_{n+2} = \cdots = S_{n+m} = 0. \tag{3.20}$$

Now, we give sharp inequalities that involve normalized δ-Casorati curvature for invariant and anti-invariant submanifolds in golden Riemannian space forms as special cases of Theorem 3.2. We have the following results.

Corollary 3.5. *Let $\mathcal{N}$ be an n-dimensional invariant submanifold of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2 \varphi \Big\} - \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ \frac{1}{n-1} + \frac{1}{n(n-1)} trP \Big\} + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \tag{3.21}$$

for any real number r such that $0 < r < n(n-1)$.

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) + \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2 \varphi \Big\} - \Big(- \frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ \frac{1}{n-1} + \frac{1}{n(n-1)} trP \Big\} + \Big(- \frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \tag{3.22}$$

for any real number $r > n(n-1)$.

Moreover, the equalities hold in the relations (3.21) and (3.22) if and only if $\mathcal{N}^n$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{N}}$, such that with respect to some orthonormal tangent frame $\{E_1, \ldots, E_n\}$ and orthonormal normal frame $\{E_{n+1}, \ldots, E_{n+m}\}$, the shape operators $S_r, r \in \{n+1, \ldots, n+m\}$, take the following forms:

$$S_{n+1} = \begin{pmatrix} b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & 0 & \dots & 0 & 0 \\ 0 & 0 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{r}b \end{pmatrix}, \qquad S_{n+2} = \dots = S_{n+m} = 0 \tag{3.23}$$

and

$$S_{n+1} = \begin{pmatrix} 2b & 0 & 0 & \dots & 0 & 0 \\ 0 & 2b & 0 & \dots & 0 & 0 \\ 0 & 0 & 2b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2b & 0 \\ 0 & 0 & 0 & \dots & 0 & b \end{pmatrix}, \qquad S_{n+2} = \dots = S_{n+m} = 0. \tag{3.24}$$

For, anti-invariant submanifolds, we have

Corollary 3.6. *Let $\mathcal{N}$ be an n-dimensional anti-invariant submanifold of a locally golden product space form $(\overline{\mathcal{N}} = \mathcal{N}_p(c_p) \times \mathcal{N}_q(c_q), g, \varphi)$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\begin{aligned} \rho \quad \leq \quad & \delta_c(n-1) + \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2\varphi \Big\} \\ & + \Big(-\frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \end{aligned} \tag{3.25}$$

for any real number r such that $0 < r < n(n-1)$.

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\begin{aligned} \rho \quad \leq \quad & \widehat{\delta}_c(n-1) + \Big(-\frac{(1-\psi)c_p - \psi c_q}{2\sqrt{5}} \Big) \Big\{ 1 + \frac{1}{n(n-1)} tr^2\varphi \Big\} \\ & + \Big(-\frac{(1-\psi)c_p + \psi c_q}{4} \Big) \frac{2}{n} tr\varphi \end{aligned} \tag{3.26}$$

for any real number $r > n(n-1)$.

Moreover, the equalities hold in the relations (3.25) and (3.26) if and only if $\mathcal{N}^n$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{\mathcal{N}}$, such that with respect to some orthonormal tangent frame $\{E_1, \dots, E_n\}$ and orthonormal normal frame $\{E_{n+1}, \dots, E_{n+m}\}$, the shape operators $S_r, r \in \{n+1, \dots, n+m\}$, take the following forms:

$$S_{n+1} = \begin{pmatrix} b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & 0 & \dots & 0 & 0 \\ 0 & 0 & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{r}b \end{pmatrix}, \qquad S_{n+2} = \dots = S_{n+m} = 0 \tag{3.27}$$

and

$$S_{n+1} = \begin{pmatrix} 2b & 0 & 0 & \dots & 0 & 0 \\ 0 & 2b & 0 & \dots & 0 & 0 \\ 0 & 0 & 2b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2b & 0 \\ 0 & 0 & 0 & \dots & 0 & b \end{pmatrix}, \qquad S_{n+2} = \dots = S_{n+m} = 0. \tag{3.28}$$

Remark : The proofs of the theorem 3.3 and theorem 3.4 are similar to theorem 3.1. In fact, using theorem 3.1 we can obtain these results by putting $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. On the similar way, we can prove theorem 3.5 and theorem 3.6.

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