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QUASI HEMI-SLANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLD

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Abstract In this paper, we study quasi hemi-slant (QHS) submanifolds of trans-Sasakian manifold. We obtain various results satisfied by these submanifolds. Further, we obtain necessary and sufficient conditions for integrability of distributions related to these submanifolds, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. Moreover, we conclude the necessary and sufficient condition for a QHS submanifold of a trans-Sasakian manifold to be a local product Riemannian manifold. At last, we construct an example of a QHS submanifold of a trans-Sasakian manifold.

1 Introduction

The notion of slant submanifold was introduced by B. Y. Chen in 1990 [5] as a generalization of holomorphic and totally real immersions. Later he collected many consequent results in his book [2]. Further slant submanifold was generalized as semi-slant, pseudo-slant, bi-slant and hemi-slant submanifolds etc. in different types of differentiable manifolds.

Many geometers studied invariant [8], anti-invariant [17], semi-invariant [19], slant [16], semi-slant [9], pseudo-slant [7] and bi-slant [18] submanifolds of trans-Sasakian manifolds in different times.

The concept of quasi hemi-slant submanifold was introduced recently by R. Prasad et al. in 2020 [13] as a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant and semi-slant submanifolds. Later in 2020-2021, R. Prasad along with some other researchers discussed this submanifold in various types of manifolds ([11], [12], [14]).

Motivated from the works mentioned above, in this paper, we study quasi hemi-slant (QHS) submanifolds of trans-Sasakian manifold. This paper consists of four sections. After introduction, in the second section, we mention some definitions and properties related to the main topic. The third section deals with some results satisfied by a QHS submanifold of a trans-Sasakian manifold. In the fourth section, we obtain necessary and sufficient conditions for integrability of distributions related to this submanifold, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. At the end of the fourth section, we conclude the necessary and sufficient condition for a QHS submanifold of a trans-Sasakian manifold to be local product Riemannian manifold and also make two other conclusions after observing the results. Finally, at last, we construct an example of a QHS submanifold of a trans-Sasakian manifold.

2 Preliminaries

In this section, we mention some definitions and properties related to quasi hemi-slant submanifolds of trans-Sasakian manifold.

Let $\tilde{M}$ be an odd dimensional differentiable manifold equipped with a metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the following relations−

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0,\ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y),\ \eta(X) = g(X, \xi)\ \ \forall X, Y \in \chi(\tilde{M}), \tag{2.3}$$

then $\tilde{M}$ is called *almost contact metric manifold* [6].

An odd dimensional almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$ is called *trans-Sasakian manifold of type* (α, β) (α, β are smooth functions on $\tilde{M}$) if [6] $\forall X, Y \in \chi(\tilde{M})$

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{2.5}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$.

Let φ be a differentiable map from a manifold M into a manifold $\tilde{M}$ and let the dimensions of $M,\ \tilde{M}$ be $n,\ m$ respectively. If at each point p of M, $(\varphi_*)_p$ is a 1-1 map i.e., if rank(φ)=n, then φ is called an *immersion* of M into $\tilde{M}$.

If an immersion φ is one-one i.e., if $\varphi(p) \neq \varphi(q)$ for $p \neq q$, then φ is called an *imbedding* of M into $\tilde{M}$.

If the manifolds $M, \tilde{M}$ satisfy the following two conditions, then M is called a *submanifold* of $\tilde{M}$−
(i) $M \subset \tilde{M}$,
(ii) the inclusion map i from M into $\tilde{M}$ is an imbedding of M into $\tilde{M}$.

Let M be a submanifold of $\tilde{M}$ and $A,\ h$ denote the *shape operator*, *second fundamental form* respectively of the immersion of M into $\tilde{M}$, then the Gauss and Weingarten formulae of M into $\tilde{M}$ are given by [3]

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.6}$$

$$\tilde{\nabla}_X V = A_V X + \nabla^{\perp}_X V \tag{2.7}$$

$\forall X, Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$, where ∇ is the induced connection on M, $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$ of M and A_V is the shape operator of M with respect to the normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, A_V and h are related by the following equation−

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.8}$$

The *mean curvature vector* is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $\dim(M) = n$ and $\{e_i\}_{i=1}^n$ is an orthonormal basis of the tangent space of M.

We have $\forall X \in \Gamma(TM)$,

$$\phi X = TX + NX, \tag{2.9}$$

where $TX,\ NX$ are the tangential and normal components of ϕX on M respectively.

Similarly, we have $\forall V \in \Gamma(T^\perp M)$,

$$\phi V = tV + nV, \tag{2.10}$$

where $tV,\ nV$ are the tangential and normal components of ϕV on M respectively.

A submanifold M is called−

(i) *totally umbilical* if

$$h(X,Y) = g(X,Y)H \ \ \forall X, Y \in \Gamma(TM); \tag{2.11}$$

(ii) *totally geodesic* if [4]

$$h(X,Y) = 0 \ \ \forall X, Y \in \Gamma(TM); \tag{2.12}$$

(iii) *minimal* if

$$H = 0. \tag{2.13}$$

The covariant derivatives of the tangential and normal components given in the equations (2.9), (2.10) are given by $\forall X, Y \in \Gamma(TM), V \in \Gamma(T^\perp M)$,

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.14}$$

$$(\tilde{\nabla}_X N)Y = \nabla_X^\perp NY - N\nabla_X Y, \tag{2.15}$$

$$(\tilde{\nabla}_X t)V = \nabla_X tV - t\nabla_X^\perp V, \tag{2.16}$$

$$(\tilde{\nabla}_X n)V = \nabla_X^\perp nV - n\nabla_X^\perp V. \tag{2.17}$$

Quasi hemi-slant submanifold M of a trans-Sasakian manifold $\tilde{M}$ is a submanifold that admits three orthogonal complementary distributions $D,\ D_\theta,\ D^\perp$ such that [13]

(i) TM admits the orthogonal direct decomposition

$$TM = D \oplus D_\theta \oplus D^\perp \oplus <\xi>, \tag{2.18}$$

(ii) the distribution D is invariant i.e., $\phi D = D$,
(iii) the distribution D_θ is slant with constant angle θ and hence θ is called *slant angle*,
(iv) the distribution $D^\perp$ is ϕ anti-invariant i.e., $\phi D^\perp \subseteq T^\perp M$.

In the above case, θ is called the *quasi hemi-slant angle* of M, and M is called *proper* [13] if $D \neq \{0\},\ D_\theta \neq \{0\},\ D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Let the dimensions of the distributions $D,\ D_\theta,\ D^\perp$ be $n_1,\ n_2,\ n_3$ respectively, then we obtain the following particular cases [13]−
(i) if $n_1 = 0$, then M is a hemi-slant submanifold,
(ii) if $n_2 = 0$, then M is a semi-invariant submanifold,

(iii) if $n_3 = 0$, then M is a semi-slant submanifold.

Now, it can be concluded from the definitions of invariant [8], anti-invariant [17], semi-invariant [1], slant [16], hemi-slant [15] and semi-slant [9] submanifolds that, quasi hemi-slant submanifold is a generalization of all these kinds of submanifolds [13].

From the definition of quasi hemi-slant submanifold given above, it is clear that if $D \neq \{0\},\ D_\theta \neq \{0\},\ D^\perp \neq \{0\}$, then $\dim(D) \geq 2$, $\dim(D_\theta) \geq 2$ and $\dim(D^\perp) \geq 1$. Thus, we have the following remark [13]−

Remark 2.1. *For a proper quasi hemi-slant submanifold M, dim$(M) \geq 6$.*

Note. From now on, in this paper, we will write the term *quasi hemi-slant* in its abbreviated form i.e., *QHS*.

Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ and the projections of $X \in \Gamma(TM)$ on the distributions $D,\ D_\theta,\ D^\perp$ be $P,\ Q,\ R$ respectively, then we have $\forall X \in \Gamma(TM)$,

$$X = PX + QX + RX + \eta(X)\xi. \tag{2.19}$$

Using (2.9) in (2.19) we get

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since $\phi D = D,\ \phi D^\perp \subseteq T^\perp M$, we have $NPX = 0,\ TRX = 0$ and hence we obtain

$$\phi X = TPX + TQX + NQX + NRX. \tag{2.20}$$

Comparing (2.20) with (2.9) we have

$$TX = TPX + TQX, \tag{2.21}$$

$$NX = NQX + NRX. \tag{2.22}$$

From (2.20) we have the following decomposition−

$$\phi(TM) = TD \oplus TD_\theta \oplus ND_\theta \oplus ND^\perp. \tag{2.23}$$

Again, since $ND_\theta \subseteq \Gamma(T^\perp M),\ ND^\perp \subseteq \Gamma(T^\perp M)$, we have another decomposition−

$$T^\perp M = ND_\theta \oplus ND^\perp \oplus \mu, \tag{2.24}$$

where μ is the orthogonal complement of $ND_\theta \oplus ND^\perp$ in $\Gamma(T^\perp M)$ and it is anti-invariant with respect to ϕ [13].

3 QHS submanifolds of trans-Sasakian manifold

This section deals with some results satisfied by a QHS submanifold of a trans-Sasakian manifold.

Theorem 3.1. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X, Y \in \Gamma(TM)$,*

$$\nabla_X TY - A_{NY}X - T(\nabla_X Y) - th(X,Y) = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)TX], \tag{3.1}$$

$$h(X,TY) + \nabla^\perp_X NY - N(\nabla_X Y) - nh(X,Y) = -\beta\eta(Y)NX. \tag{3.2}$$

Proof. Using (2.9) in (2.4) we get

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)(TX + NX)]. \tag{3.3}$$

Again, using (2.6), (2.7), (2.9) and (2.10) in $(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y)$ we obtain

$$(\tilde{\nabla}_X \phi)Y = \nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY - th(X,Y) - nh(X,Y) - T(\nabla_X Y) - N(\nabla_X Y). \tag{3.4}$$

Equating tangential and normal components of (3.3), (3.4) we obtain (3.1) and (3.2) respectively. □

Using (2.14) and (2.15) respectively in (3.1) and (3.2), we can conclude the following–

Corollary 3.1. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then we have $\forall X, Y \in \Gamma(TM)$,*

$$(\tilde{\nabla}_X T)Y = A_{NY}X + th(X,Y) + \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)TX], \tag{3.5}$$

$$(\tilde{\nabla}_X N)Y = -h(X,TY) + nh(X,Y) - \beta\eta(Y)NX. \tag{3.6}$$

Next, we state the following theorem [13]–

Theorem 3.2. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then we have*

$$TD = D,\ TD_\theta = D_\theta,\ TD^\perp = \{0\},\ tND_\theta = D_\theta,\ tND^\perp = D^\perp.$$

Now, using (2.9) and (2.10) on $\phi^2 = -I + \eta \otimes \xi$ we get the following theorem–

Theorem 3.3. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then we get*
(i) $T^2 + nN = -I + \eta \otimes \xi$ *on* TM,
(ii) $NT + tN = 0$ *on* TM,
(iii) $Tt + n^2 = -I$ *on* $T^\perp M$,
(iv) $Nt + tn = 0$ *on* $T^\perp M$,
where I is the identity operator.

Next, we have the following theorem [10]–

Theorem 3.4. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then we have $\forall X, Y \in \Gamma(D_\theta)$,*
(i) $T^2X = -(cos^2\theta)X$,
(ii) $g(TX,TY) = (cos^2\theta)g(X,Y)$,
(iii) $g(NX,NY) = (sin^2\theta)g(X,Y)$.

Theorem 3.5. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then we have $\forall X \in \Gamma(TM), V \in \Gamma(T^\perp M)$,*

$$\nabla_X tV - A_{nV}X + T(A_V X) - t\nabla^\perp_X V = \beta g(NX,V)\xi, \tag{3.7}$$

$$h(X,tV) + \nabla^\perp_X nV + N(A_V X) - n\nabla^\perp_X V = 0. \tag{3.8}$$

Proof. Using (2.6), (2.7), (2.9) and (2.10) in $(\tilde{\nabla}_X \phi)V = \tilde{\nabla}_X \phi V - \phi(\tilde{\nabla}_X V)$ we get

$$(\tilde{\nabla}_X \phi)V = \nabla_X tV + h(X,tV) - A_{nV}X + \nabla^\perp_X nV + T(A_V X) + N(A_V X) - t\nabla^\perp_X V - n\nabla^\perp_X V.$$

Again, applying (2.4) and then (2.9) in the left hand side of the above equation we obtain

$$\beta[g(NX,V)\xi] = \nabla_X tV + h(X,tV) - A_{nV}X + \nabla^\perp_X nV + T(A_V X) + N(A_V X) - t(\nabla^\perp_X V) - n(\nabla^\perp_X V). \tag{3.9}$$

Equating tangential and normal components from both sides of (3.9) we get (3.7) and (3.8) respectively. □

Now, using (2.16) and (2.17) in (3.7) and (3.8) respectively we conclude the following–

Corollary 3.2. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we get $\forall X \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,*

$$(\tilde{\nabla}_X t)V = A_{nV}X - T(A_V X) + \beta g(NX, V)\xi, \tag{3.10}$$

$$(\tilde{\nabla}_X n)V = -h(X, tV) - N(A_V X). \tag{3.11}$$

Theorem 3.6. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X \in \Gamma(TM)$,*

$$\nabla_X \xi = -\alpha TX - \beta T^2 X, \tag{3.12}$$

$$h(X, \xi) = -\alpha NX - \beta nNX. \tag{3.13}$$

Proof. Using (2.6), (2.9) and Theorem 3.3.(i) in (2.5) we obtain

$$\nabla_X \xi + h(X, \xi) = -\alpha(TX + NX) + \beta[-T^2 - nN]X.$$

Equating tangential and normal components from both sides of the above equation we get (3.12) and (3.13) respectively. □

Theorem 3.7. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X, Y \in \Gamma(D^{\perp})$,*

$$A_{\phi X}Y = A_{\phi Y}X \; if \; and \; only \; if \; \phi[X, Y] = 2\beta g(X, \phi Y)\xi. \tag{3.14}$$

Proof. Replacing V by ϕY in (2.7) and then applying (2.4), (2.6) and the fact that $Y \in \Gamma(D^{\perp})$ we get

$$\alpha g(X, Y)\xi + \beta g(\phi X, Y)\xi + \phi(\nabla_X Y) + \phi h(X, Y) = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y.$$

Equating tangential components from both sides of the above equation we obtain

$$A_{\phi Y}X = -\alpha g(X, Y)\xi - \beta g(\phi X, Y)\xi - \phi(\nabla_X Y). \tag{3.15}$$

Interchanging X, Y in (3.15) and then subtracting (3.15) from the resultant equation we have

$$A_{\phi X}Y - A_{\phi Y}X = \phi[X, Y] - 2\beta g(X, \phi Y)\xi \tag{3.16}$$

from which we get (3.14). □

Theorem 3.8. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X, Y \in \Gamma(D \oplus D_\theta \oplus D^{\perp})$,*

$$g([X, Y], \xi) = 2\alpha g(TX, Y), \tag{3.17}$$

$$g(\tilde{\nabla}_X Y, \xi) = \alpha g(TX, Y) - \beta cos^2\theta g(X, Y). \tag{3.18}$$

Proof. Applying (3.12) and Theorem 3.4.(i) on the following equation

$$g([X, Y], \xi) = g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) = -g(Y, \nabla_X \xi) + g(X, \nabla_Y \xi)$$

and after simplifying we obtain (3.17).

Again, using (2.6) we have

$$g(\tilde{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) + h(X, Y)\eta(\xi) = -g(Y, \nabla_X \xi) + h(X, Y).$$

Now, applying (3.12) and Theorem 3.4.(i) on the above equation we get (3.18).

Thus the proof is completed. □

4 Integrability of distributions and decomposition theorems

In this section, we obtain necessary and sufficient conditions for integrability of distributions related to the QHS submanifolds of a trans-Sasakian manifold, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. At the end, we make three conclusions after observing the results.

Theorem 4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the invariant distribution D is not integrable.*

Proof. Let $X, Y \in \Gamma(D)$, then using (2.6), $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$ and then (2.5), $g(\phi X, Y) = -g(X, \phi Y)$ in the following equation

$$g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi)$$

we get on simplifying,

$$g([X, Y], \xi) = 2\alpha g(\phi X, Y). \tag{4.1}$$

Applying (2.19), (2.20) on (4.1) we obtain $g([X, Y], \xi) = 2\alpha g(TPX, PY) \neq 0$. Thus, D is not integrable. □

Theorem 4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D\oplus < \xi >$ is integrable if and only if $\forall X, Y \in \Gamma(D\oplus < \xi >), Z \in \Gamma(D_\theta \oplus D^\perp)$,*

$$g(T\nabla_X Y - T\nabla_Y X, TQZ) + g(nh(X, Y) - nh(Y, X), NQZ + NRZ) = 0. \tag{4.2}$$

Proof. Using (2.2) in $g([X, Y], Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X, Y], Z) = g(\phi\tilde{\nabla}_X Y, \phi Z) - g(\phi\tilde{\nabla}_Y X, \phi Z)$$

on which applying (2.6), (2.9), (2.10), (2.20) and after simplifying we get

$$g([X, Y], Z) = g(T\nabla_X Y - T\nabla_Y X, TQZ) + g(nh(X, Y) - nh(Y, X), NQZ + NRZ).$$

Hence $g([X, Y], Z) = 0$ if and only if (4.2) holds and thus the proof is completed. □

Theorem 4.3. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the slant distribution D_θ is not integrable.*

Proof. Let $X, Y \in \Gamma(D_\theta)$. Applying (2.19) and (2.20) in (4.1) we have $g([X, Y], \xi) = 2\alpha g(TQX + NQX, QY) \neq 0$ and hence the proof is completed. □

Theorem 4.4. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D_\theta\oplus < \xi >$ is integrable if and only if $\forall X, Y \in \Gamma(D_\theta\oplus < \xi >), Z \in \Gamma(D \oplus D^\perp)$,*

$$g(n(\nabla^\perp_X Y) - n(\nabla^\perp_Y X), NRZ) + cos^2\theta g(A_X Y - A_Y X, PZ) = 0. \tag{4.3}$$

Proof. Using (2.2) in $g([X, Y], Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X,Y],Z) = g(\phi\tilde{\nabla}_X Y, \phi Z) - g(\phi\tilde{\nabla}_Y X, \phi Z)$$

on which applying (2.7), (2.9), (2.10), (2.20), Theorem 3.4.(ii) and after simplifying we get

$$g([X,Y],Z = cos^2\theta g(A_X Y - A_Y X, PZ) + g(n(\nabla^{\perp}_X Y) - n(\nabla^{\perp}_Y X), NRZ).$$

Therefore, $g([X,Y],Z) = 0$ if and only if (4.3) holds and hence the proof is completed. □

From the above theorem, using (2.18) and (2.24) respectively we conclude the following–

Corollary 4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the distribution $D_\theta \oplus <\xi>$ is integrable if $\forall X,Y \in \Gamma(D_\theta \oplus <\xi>)$,*

$$A_X Y - A_Y X \in \Gamma(D_\theta \oplus D^{\perp}), \tag{4.4}$$

$$n(\nabla^{\perp}_X Y) - n(\nabla^{\perp}_Y X) \in \Gamma(ND_\theta \oplus \mu). \tag{4.5}$$

Theorem 4.5. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the anti-invariant distribution $D^{\perp}$ is integrable if and only if $\forall X,Y \in \Gamma(D^{\perp}), Z \in \Gamma(D \oplus D_\theta)$*

$$g(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X, NQZ) = 0. \tag{4.6}$$

Proof. Using (2.2) in $g([X,Y],Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X,Y],Z) = g(\tilde{\nabla}_X \phi Y, \phi Z) - g(\tilde{\nabla}_Y \phi X, \phi Z)$$

on which applying (2.7), (2.8), (2.20) and Theorem 3.2 we get after simplification,

$$g([X,Y],Z) = g(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X, NQZ).$$

Thus $g([X,Y],Z) = 0$ if and only if (4.6) holds and hence the proof is completed. □

Using (2.24) in the above theorem we conclude the following–

Corollary 4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the anti-invariant distribution $D^{\perp}$ is integrable if $\forall X,Y \in \Gamma(D^{\perp})$, $\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X \in \Gamma(ND^{\perp} \oplus \mu)$.*

Theorem 4.6. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then M is totally geodesic if and only if $\forall X,Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,*

$$g(\nabla_X TY - A_{NY}X, tV) + g(h(X,TY) + \nabla^{\perp}_X NY, nV) = 0. \tag{4.7}$$

Proof. Applying (2.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$.

Further, using (2.6), (2.7), (2.9), (2.10) in the above equation we obtain on simplifying,

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TY - A_{NY}X, tV) + g(h(X,TY) + \nabla^{\perp}_X NY, nV). \tag{4.8}$$

Now, M is totally geodesic $<=> h = 0 <=> \forall X,Y \in \Gamma(TM), \tilde{\nabla}_X Y = \nabla_X Y$ (from (2.6))$<=> g(\tilde{\nabla}_X Y, V) = 0 \; \forall V \in \Gamma(T^{\perp}M)$. Hence from (4.8) we have, M is totally geodesic if and only if (4.7) holds. Thus the proof is completed. □

Theorem 4.7. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the invariant distribution D does not define a totally geodesic foliation on M.*

Proof. Let $X, Y \in \Gamma(D)$. Using (2.5) and the fact that $X \in \Gamma(D)$ in $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$ we get $g(\tilde{\nabla}_X Y, \xi) = -\beta g(X, Y) + \alpha g(Y, \phi X) \neq 0$, and hence the proof is completed. □

Theorem 4.8. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D \oplus < \xi >$ defines a totally geodesic foliation on M if and only if $\forall X, Y \in \Gamma(D), Z \in \Gamma(D_\theta \oplus D^\perp), V \in \Gamma(T^\perp M)$,*

$$g(\nabla_X TY, TQZ) = -g(h(X, TY), NZ), \tag{4.9}$$

$$g(\nabla_X TY, tV) = -g(h(X, TY), nV). \tag{4.10}$$

Proof. Applying (2.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (2.6) and (2.20) we get

$$g(\tilde{\nabla}_X Y, Z) = g(\nabla_X TY, TQZ) + g(h(X, TY), NZ)$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (4.9) holds.

Again, applying (2.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (2.6), (2.10) and (2.20) we obtain

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TY, tV) + g(h(X, TY), nV).$$

Hence we have $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (4.10) holds.

Thus the proof is completed. □

Theorem 4.9. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the slant distribution D_θ does not define a totally geodesic foliation on M.*

Proof. Let $X, Y \in \Gamma(D_\theta)$. Applying (2.5) and the fact that $X \in \Gamma(D_\theta)$ on $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$ we get $g(\tilde{\nabla}_X Y, \xi) = -\beta g(X, Y) + \alpha g(\phi X, Y) \neq 0$.

Hence the proof is completed. □

Theorem 4.10. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D_\theta \oplus < \xi >$ defines a totally geodesic foliation on M if and only if $\forall X, Y \in \Gamma(D_\theta \oplus < \xi >), Z \in \Gamma(D \oplus D^\perp), V \in \Gamma(T^\perp M)$,*

$$g(\nabla_X TQY - A_{NQY} X, TPZ) + g(h(X, TQY) + \nabla^\perp_X NQY, NRZ) = 0, \tag{4.11}$$

$$g(\nabla_X TQY - A_{NQY} X, tV) + g(h(X, TQY) + \nabla^\perp_X NQY, nV) = 0. \tag{4.12}$$

Proof. Applying (2.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (2.6), (2.7) and (2.20) we get

$$g(\tilde{\nabla}_X Y, Z) = g(\nabla_X TQY - A_{NQY} X, TPZ) + g(h(X, TQY) + \nabla^\perp_X NQY, NRZ)$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (4.11) holds.

Again, applying (2.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (2.6), (2.7), (2.10) and (2.20) we obtain

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TQY - A_{NQY} X, tV) + g(h(X, TQY) + \nabla^\perp_X NQY, nV),$$

which implies that $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (4.12) holds.

Thus the proof is completed. □

Theorem 4.11. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the anti-invariant distribution $D^{\perp}$ defines a totally geodesic foliation on M if and only if* $\forall X, Y \in \Gamma(D^{\perp}), Z \in \Gamma(D \oplus D_{\theta}), V \in \Gamma(T^{\perp}M)$,

$$g(A_{NY}X, TZ) = g(\nabla^{\perp}_{X}NY, NQZ), \tag{4.13}$$

$$g(A_{NY}X, tV) = g(\nabla^{\perp}_{X}NY, nV). \tag{4.14}$$

Proof. Applying (2.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (2.7) and (2.20) we obtain

$$g(\tilde{\nabla}_X Y, Z) = -g(A_{NY}X, TZ) + g(\nabla^{\perp}_{X}NY, NQZ)$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (4.13) holds.

Now, applying (2.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (2.7), (2.10) and (2.20) we get

$$g(\tilde{\nabla}_X Y, V) = -g(A_{NY}X, tV) + g(\nabla^{\perp}_{X}NY, nV)$$

which implies that $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (4.14) holds.

Thus the proof is completed. □

From theorems 4.8, 4.10 and 4.11, we reach to the following conclusion−

Conclusion 4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then M is a local product Riemannian manifold of the form $M_D \times M_{D_\theta} \times M_{D^{\perp}}$ if and only if equations (4.9)-(4.14) hold, where $M_D,\ M_{D_\theta},\ M_{D^{\perp}}$ are leaves of the distributions $D,\ D_\theta,\ D^{\perp}$ respectively.*

Next, theorems 4.1 and 4.3 give us the following conclusion−

Conclusion 4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then both of the invariant distribution D and the slant distribution D_θ are not integrable.*

Again, observing theorems 4.7 and 4.9 we can conclude the following−

Conclusion 4.3. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then both of the invariant distribution D and the slant distribution D_θ do not define a totally geodesic foliation on M.*

Example. Now, we construct an example of a QHS submanifold of a trans-Sasakian manifold.

Let $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$-dimensional Euclidean space endowed with the almost contact metric structure (ϕ, ξ, η, g) defined by

$$\phi(x^1, x^2, ..., x^{2n}, t) = (-x^{n+1}, -x^{n+2}, ..., -x^{2n}, x^1, x^2, ..., x^n, 0),$$

$$\xi = e^t \frac{\partial}{\partial t},\ \eta = e^{-t}dt,\ g = e^{-2t}k,$$

where $(x^1, x^2, ..., x^{2n}, t)$ are cartesian coordinates and k is the Euclidean Riemannian metric on $\mathbb{R}^{2n+1}$. Then (ϕ, ξ, η, g) is a trans-Sasakian structure on $\mathbb{R}^{2n+1}$ which is neither cosymplectic nor Sasakian.

For $\theta \in \left(0, \frac{\pi}{2}\right)$, we have, the map given by

$$x(u_1,u_2,u_3,u_4,u_5,u_6,u_7) = (u_1, u_2\cos\theta, 0, u_2\sin\theta, u_3, u_4, u_5, u_6, 0, 0, u_7)$$

defines a 7-dimensional submanifold M of $\mathbb{R}^{11}$ with the trans-Sasakian structure described above. Further, let

$$E_1 = e^t\frac{\partial}{\partial x^1},\ E_2 = e^t\frac{\partial}{\partial x^6},$$

$$E_3 = e^t\big(\cos\theta\frac{\partial}{\partial x^2} + \sin\theta\frac{\partial}{\partial x^4}\big),\ E_4 = e^t\frac{\partial}{\partial x^7},$$

$$E_5 = e^t\frac{\partial}{\partial x^5},\ E_6 = e^t\frac{\partial}{\partial x^8},\ E_7 = e^t\frac{\partial}{\partial t} = \xi,$$

then $\{E_i\}_{i=1}^7$ is an orthonormal frame of TM.

If we define the distributions as

$$D =< E_1, E_2 >,\ D_\theta =< E_3, E_4 >,\ D^\perp =< E_5, E_6 >,$$

then it is clear that

$$TM = D \oplus D_\theta \oplus D^\perp \oplus < \xi >$$

and D is an invariant distribution since $\phi E_1 = E_2$ and $\phi E_2 = -E_1$, D_θ is a slant distribution with slant angle $\theta \in \big(0, \frac{\pi}{2}\big)$ since $g(\phi E_3, E_4) = \cos\theta = -g(E_3, \phi E_4)$, $D^\perp$ is an anti-invariant distribution since $\phi E_5 = e^t\frac{\partial}{\partial x^{10}}$ and $\phi E_6 = -e^t\frac{\partial}{\partial x^3}$.

Therefore, M is a QHS submanifold of the trans-Sasakian manifold $(\mathbb{R}^{11}, \phi, \xi, \eta, g)$.

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