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FURTHER BEREZIN RADIUS INEQUALITIES

Verda Gürdal, Hamdullah Başaran and Mualla B. Huban

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Abstract We study some new inequalities by using bounded function $\widetilde{A}$, involving Berezin radius inequalities and the Berezin norm for operators acting on the reproducing kernel Hilbert space. In particular, it is proved for arbitrary bounded linear operator A that

$$\text{ber}^4(A) \leq \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{8}\left\| |A|^2 + |A^*|^2 \right\|_{\text{ber}} \text{ber}\left(A^2\right),$$

where $\text{ber}(\cdot)$ is the Berezin radius of operator A.

1 Introduction

Let $\mathcal{L}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle .,. \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$. Throughout this paper we work in reproducing kernel Hilbert space (RKHS). These spaces are complete inner-product spaces comprised of complex-valued functions defined on a set Ω, where point evaluation is bounded. Formally, that is, if Ω is a set and $\mathcal{H} = \mathcal{H}(\Omega)$ is a subset of all functions $\Omega \to \mathbb{C}$, then $\mathcal{H}$ is an RKHS on Ω if it is a complete inner product space and point evaluation at each $\lambda \in \Omega$ is a bounded linear functional on $\mathcal{H}$. Via the classical Riesz representation theorem, we know if $\mathcal{H}$ is an RKHS on Ω, there is a unique element $k_\lambda \in \mathcal{H}$ such that $h(\lambda) = \langle h, k_\lambda \rangle_{\mathcal{H}}$ for every $\lambda \in \Omega$ and all $h \in \mathcal{H}$. The element k_λ is called the reproducing kernel at λ. Further, we will denote the normalized reproducing kernel at λ as $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$.

The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions.

Definition 1.1. Let $\mathcal{H}$ be an RKHS on a set Ω and let A be a bounded linear operator on $\mathcal{H}$.

(i) For $\lambda \in \Omega$, the Berezin transform of A at λ (or Berezin symbol of A) is

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}}.$$

(ii) The Berezin range of A (or Berezin set of A) is

$$\text{Ber}(A) := \text{Range}(\widetilde{A}) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\}.$$

(iii) The Berezin radius of A (or Berezin number of A) is

$$\text{ber}(A) := \sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right|.$$

We also define the following so-called Berezin norm of operators $A \in \mathcal{L}(\mathcal{H})$:

$$\|A\|_{\text{Ber}} := \sup_{\lambda \in \Omega} \left\| A\widehat{k}_\lambda \right\|.$$

It is easy to see that actually $\|A\|_{\text{Ber}}$ determines a new operator norm in $\mathcal{L}(\mathcal{H}(\Omega))$ (since the set of reproducing kernels $\{k_\lambda : \lambda \in \Omega\}$ span the space $\mathcal{H}(\Omega)$). It is also trivial that $\text{ber}(A) \leq \|A\|_{\text{Ber}} \leq \|A\|$.

For each bounded operator A on $\mathcal{H}$, the Berezin transform $\widetilde{A}$ is a bounded real-analytic function on Ω. Properties of the operator A are often reflected in properties of the Berezin transform $\widetilde{A}$. The Berezin transform itself was introduced by F. Berezin in [8] and has proven to be a critical tool in operator theory, as many foundational properties of important operators are encoded in their Berezin transforms. The Berezin set and number, also denoted by Ber(A) and ber(A), respectively, were purportedly first formally introduced by Karaev in [22].

An important inequality for ber(A) is the power inequality stating that

$$\text{ber}(A^n) \leq \text{ber}(A)^n \tag{1.1}$$

for $n = 1, 2, ...$; more generally, if A is not nilpotent, then

$$C_1 \text{ber}(A)^n \leq \text{ber}(A^n) \leq C_1 \text{ber}(A)^n$$

for some constants $C_1, C_2 > 0$.

In an RKHS, the Berezin range of an operator A is a subset of the numerical range of A,

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H}(\Omega) \text{ and } \|x\| = 1\}.$$

Hence

$$\text{ber}(A) \leq w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}(\Omega) \text{ and } \|x\| = 1\}$$

(the numerical radius of operator A). The numerical range of an operator has some interesting properties. For example, it is well known that the spectrum of an operator is contained in the closure of its numerical range. For basic properties of the numerical radius, we refer to [1, 17, 18, 24, 25, 26, 27, 33].

Berezin range and Berezin radius of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [22]. For the basic properties and facts on these new concepts, see [3, 4, 5, 15, 16, 23, 28, 29].

It is well-known that

$$\text{ber}(T) \leq w(T) \leq \|T\| \tag{1.2}$$

for any $T \in \mathcal{L}(\mathcal{H}(\Omega))$.

In [7], Başaran et al. obtained the following result.

$$\text{ber}(A) \leq \frac{1}{2}\left(\|A\|_{\text{ber}} + \|A^2\|_{\text{ber}}^{1/2}\right). \tag{1.3}$$

It has been shown in [20] and [21], respectively, that if $A \in \mathcal{L}(\mathcal{H}(\Omega))$, then

$$\frac{1}{4}\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} \leq \text{ber}^2(A) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|_{\text{ber}} \tag{1.4}$$

where $|A| = (A^*A)^{1/2}$ is the absolute value of A, and

$$\text{ber}^{2r}(A) \leq \frac{1}{2}\left\||A|^{2r} + |A^*|^{2r}\right\|_{\text{ber}} \tag{1.5}$$

where $r \geq 1$.

The purpose of this paper is to establish several refinements of the above Berezin radius inequalities for reproducing kernel Hilbert space operators. Some other related questions are also discussed.

2 New estimates for the Berezin radius

2.1 Lemmas

In order to achieve our goal, we need the following sequence of corollaries.

Recall that an operator $A \in \mathcal{L}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. In this case we will write $A \geq 0$. The classical operator Jensen inequality for the positive operators $A \in \mathcal{B}(\mathcal{H})$ is

$$\langle Ax, x\rangle^r \leq (\geq) \langle A^r x, x\rangle, \ r \geq 1 \ (0 \leq r \leq 1) \tag{2.1}$$

for any unit vector $x \in \mathcal{H}$.

The following lemma is known as Cauchy-Buzano inequality (see [9]).

Lemma 2.1. *Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\| \|y\|). \tag{2.2}$$

Next lemma is a corollary of Cauchy-Buzano inequality (see [27, Lemma 2.2]).

Lemma 2.2. *Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{2}\sqrt{3\|x\|^2\|y\|^2 + \|x\|\|y\||\langle x, y\rangle|}. \tag{2.3}$$

The following lemma which can be found in [2, Theorem 2.3] gives a norm inequality involving convex function of positive operators.

Lemma 2.3. *Let f be a non-negative non-decreasing convex function on $[0, \infty)$ and $A, B \in \mathcal{L}(\mathcal{H})$. Then*

$$\left\| f\left(\frac{A+B}{2}\right)\right\| \leq \left\| \frac{f(A) + f(B)}{2}\right\|. \tag{2.4}$$

In particular,

$$\left\| \left(\frac{A+B}{2}\right)^r\right\| \leq \left\| \frac{A^r + B^r}{2}\right\|, \ (r \geq 1).$$

The next lemma is found in [18].

Lemma 2.4. *Let $A \in \mathcal{L}(\mathcal{H})$ be a positive operator. Then*

$$|\langle Ax, x\rangle| \leq \sqrt{\langle |A| x, x\rangle \langle |A^*| x, x\rangle} \tag{2.5}$$

for any unit vector $x \in \mathcal{H}$.

2.2 Main Results

Now, we are ready to present the main results of this section.

Theorem 2.5. *Let $A \in \mathcal{L}(\mathcal{H}(\Omega))$. Then*

$$\text{ber}^4(A) \leq \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{8}\left\| |A|^2 + |A^*|^2 \right\|_{\text{ber}} \text{ber}(A^2). \tag{2.6}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. By setting $e = \widehat{k}_\lambda$, $x = A\widehat{k}_\lambda$ and $y = A^*\widehat{k}_\lambda$ in (2.3), we have

$$\begin{aligned}\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^4 &\leq \frac{3}{4}\left\|A\widehat{k}_\lambda\right\|^2\left\|A^*\widehat{k}_\lambda\right\|^2 + \frac{1}{4}\left\|A\widehat{k}_\lambda\right\|\left\|A^*\widehat{k}_\lambda\right\|\left|\left\langle A\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle\right| \\ &= \frac{3}{4}\left\langle A\widehat{k}_\lambda, A\widehat{k}_\lambda\right\rangle\left\langle A^*\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle + \frac{1}{4}\sqrt{\left\langle A\widehat{k}_\lambda, A\widehat{k}_\lambda\right\rangle\left\langle A^*\widehat{k}_\lambda, A^*\widehat{k}_\lambda\right\rangle}\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{4}\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \frac{1}{4}\sqrt{\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} \left|\left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| \\
&\le \frac{3}{8}\left(\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 + \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2\right) \\
&+ \frac{1}{8}\left(\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) \left|\left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| \\
&\text{(by the AM-GM inequality)} \\
&\le \frac{3}{8}\left(\left\langle |A|^4 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |A^*|^4 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) \\
&+ \frac{1}{8}\left(\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |A^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) \left|\left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| \\
&\text{(by the inequalities (2.1))} \\
&= \frac{3}{8}\left\langle \left(|A|^4 + |A^*|^4\right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \frac{1}{8}\left\langle \left(|A|^2 + |A^*|^2\right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left|\left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|.
\end{aligned}
$$

By taking the supremum over all $\lambda \in \Omega$, we have

$$
\begin{aligned}
\sup_{\lambda\in\Omega} \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^4 &\le \sup_{\lambda\in\Omega} \frac{3}{8}\left\langle \left(|A|^4 + |A^*|^4\right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&+ \sup_{\lambda\in\Omega} \frac{1}{8}\left\langle \left(|A|^2 + |A^*|^2\right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left|\left\langle A^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|
\end{aligned}
$$

and

$$
\mathrm{ber}^4(A) \le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\mathrm{ber}} + \frac{1}{8}\left\| |A|^2 + |A^*|^2 \right\|_{\mathrm{ber}} \mathrm{ber}\left(A^2\right),
$$

which proves the theorem. □

Next corollary, based on Lemma 2.3, is refinement of the inequality (1.5) when $r = 2$.

Corollary 2.6. *Let $A \in \mathcal{L}(\mathcal{H}(\Omega))$. Then*

$$
\begin{aligned}
\mathrm{ber}^4(A) &\le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\mathrm{ber}} + \frac{1}{8}\left\| |A|^2 + |A^*|^2 \right\|_{\mathrm{ber}} \mathrm{ber}\left(A^2\right) \\
&\le \frac{1}{2}\left\| |A|^4 + |A^*|^4 \right\|_{\mathrm{ber}}.
\end{aligned}
$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Notice that if $T \in \mathcal{L}(\mathcal{H}(\Omega))$ and f is a non-negative increasing function $[0, \infty)$, then $\|f(|T|)\|_{\mathrm{ber}} = f(\|T\|_{\mathrm{ber}})$. Thus, we have

$$
\begin{aligned}
\mathrm{ber}^4(A) &\le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\mathrm{ber}} + \frac{1}{8}\left\| |A|^2 + |A^*|^2 \right\|_{\mathrm{ber}} \mathrm{ber}\left(A^2\right) \\
&\le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\mathrm{ber}} + \frac{1}{8}\left\| |A|^2 + |A^*|^2 \right\|_{\mathrm{ber}} \mathrm{ber}^2(A) \\
&\text{(by the inequality (1.1))} \\
&\le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\mathrm{ber}} + \frac{1}{16}\left\| |A|^2 + |A^*|^2 \right\|_{\mathrm{ber}}^2 \\
&\text{(by the inequality (1.4))}
\end{aligned}
$$

$$= \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{16}\left\| \left(|A|^2 + |A^*|^2 \right)^2 \right\|_{\text{ber}}$$

(by the inequality (2.4))

$$= \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{16}\left\| \left(\frac{2|A|^2 + 2|A^*|^2}{2} \right)^2 \right\|_{\text{ber}}$$

$$\leq \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}}$$

$$= \frac{1}{2}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}}.$$

Hence we get

$$\begin{aligned} \text{ber}^4(A) &\leq \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{8}\left\| |A|^2 + |A^*|^2 \right\|_{\text{ber}} \text{ber}\left(A^2\right) \\ &\leq \frac{1}{2}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}}, \end{aligned}$$

and the proof is complete. □

By applying the inequality (2.3), we obtain the following refinement.

Theorem 2.7. *Let* $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$. *Then for any* $\lambda \in \Omega$, *we have*

$$\left| \widetilde{A}(\lambda)\, \widetilde{B}(\lambda) \right|^2 \leq \frac{3}{8}\left(\widetilde{|A|^4 + |B^*|^4} \right)(\lambda) + \frac{1}{8}\left(\widetilde{|A|^2 + |B^*|^2} \right)(\lambda) \left| \widetilde{BA}(\lambda) \right|. \tag{2.7}$$

Proof. Let $\lambda \in \Omega$ be arbitrary. Putting $e = \widehat{k}_\lambda$, $x = A\widehat{k}_\lambda$ and $y = A^*\widehat{k}_\lambda$ in the inequality (2.3) we have

$$\begin{aligned} \left| \widetilde{A}(\lambda)\, \widetilde{B}(\lambda) \right|^2 &= \left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \\ &= \left| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle \widehat{k}_\lambda, B^*\widehat{k}_\lambda \right\rangle \right|^2 \\ &\leq \frac{3}{4}\left\| A\widehat{k}_\lambda \right\|^2 \left\| B^*\widehat{k}_\lambda \right\|^2 + \frac{1}{4}\left\| A\widehat{k}_\lambda \right\| \left\| B^*\widehat{k}_\lambda \right\| \left| \left\langle A\widehat{k}_\lambda, B^*\widehat{k}_\lambda \right\rangle \right| \\ &= \frac{3}{4}\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |B^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \frac{1}{4}\sqrt{\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle |B^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} \left| \left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \\ &\leq \frac{3}{8}\left(\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 + \left\langle |B^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 \right) \\ &\quad + \frac{1}{8}\left(\left\langle |A|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |B^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right) \left| \left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \end{aligned}$$

(by the AM-GM inequality)

$$\begin{aligned} &\leq \frac{3}{8}\left(\left\langle |A|^4 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |B^*|^4 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right) + \frac{1}{8}\left\langle \left(|A|^2 + |B^*|^2 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left| \widetilde{BA}(\lambda) \right| \\ &= \frac{3}{8}\left(\left\langle \left(|A|^4 + |B^*|^4 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right) + \frac{1}{8}\left(\widetilde{|A|^2 + |B^*|^2} \right)(\lambda) \left| \widetilde{BA}(\lambda) \right|. \end{aligned}$$

Thus, we have

$$\left| \widetilde{A}(\lambda)\, \widetilde{B}(\lambda) \right|^2 \leq \frac{3}{8}\left(\widetilde{|A|^4 + |B^*|^4} \right)(\lambda) + \frac{1}{4}\left(\widetilde{|A|^2 + |B^*|^2} \right)(\lambda) \left| \widetilde{BA}(\lambda) \right|.$$

This completes the proof. □

The following corollary is an easy consequence of Theorem 2.7.

Corollary 2.8. *Let $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$. Then*

$$\mathrm{ber}^4(B^*A) \le \frac{3}{8}\left\||A|^8 + |B|^8\right\|_{\mathrm{ber}} + \frac{1}{8}\left\||A|^4 + |B|^4\right\|_{\mathrm{ber}} \mathrm{ber}\left(|B|^2|A|^2\right). \tag{2.8}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. Considering $A = |A|^2$ and $B = |B|^2$ in the inequality (2.7), we have that

$$\begin{aligned}&\left(\left\langle |A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle |B|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right)^2\\ &\le \frac{3}{8}\left\langle\left(|A|^8 + |B|^8\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \frac{1}{8}\left\langle\left(|A|^4 + |B|^4\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left|\left\langle |B|^2|A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|. \end{aligned}\tag{2.9}$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}\left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^4 &= \left|\left\langle A\widehat{k}_\lambda, B\widehat{k}_\lambda\right\rangle\right|^4\\ &\le \left\|A\widehat{k}_\lambda\right\|^4\left\|B^*\widehat{k}_\lambda\right\|^4\\ &= \left(\left\langle A\widehat{k}_\lambda, A\widehat{k}_\lambda\right\rangle\left\langle B\widehat{k}_\lambda, B\widehat{k}_\lambda\right\rangle\right)^2\\ &= \left(\left\langle |A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle |B|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right)^2. \end{aligned}\tag{2.10}$$

Now, on making use of the inequalities (2.9) and (2.10), we get the inequality

$$\begin{aligned}\left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^4 \le{}& \frac{3}{8}\left(\left\langle |A|^8\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \left\langle |B|^8\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right)\\ &+ \frac{1}{8}\left\langle\left(|A|^4 + |B|^4\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left|\left\langle |B|^2|A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|.\end{aligned}$$

Therefore, taking the supremum over $\lambda \in \Omega$ we deduce

$$\begin{aligned}\sup_{\lambda\in\Omega}\left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^4 \le{}& \frac{3}{8}\sup_{\lambda\in\Omega}\left\langle\left(|A|^8 + |B|^8\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\\ &+ \frac{1}{8}\sup_{\lambda\in\Omega}\left\langle\left(|A|^4 + |B|^4\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left|\left\langle |B|^2|A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\end{aligned}$$

and

$$\mathrm{ber}^4(B^*A) \le \frac{3}{8}\left\||A|^8 + |B|^8\right\|_{\mathrm{ber}} + \frac{1}{8}\left\||A|^4 + |B|^4\right\|_{\mathrm{ber}} \mathrm{ber}\left(|B|^2|A|^2\right),$$

as desired. □

It follows from Theorem 3.11 in [20] that if $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$ and $r \ge 1$, then

$$\mathrm{ber}^r(B^*A) \le \frac{1}{2}\left\||A|^{2r} + |B|^{2r}\right\|_{\mathrm{ber}}. \tag{2.11}$$

Remark 2.9. If $r = 4$ is taken in the inequality (2.11) in particular, then Corollary 2.8 improves the inequality

$$\mathrm{ber}^4(B^*A) \le \frac{1}{2}\left\||A|^8 + |B|^8\right\|_{\mathrm{ber}}.$$

Next we prove the following inequality.

Corollary 2.10. *Let $A \in \mathcal{L}(\mathcal{H}(\Omega))$, then*

$$\mathrm{ber}^4(A) \le \frac{3}{8}\left\||A|^4 + |A^*|^4\right\|_{\mathrm{ber}} + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}} \mathrm{ber}\left(|A^*||A|\right). \tag{2.12}$$

Proof. Let $\lambda \in \Omega$ be arbitrary. Replacing $A = |A|$ and $B = |A^*|$ in Theorem 2.7, we have

$$\begin{aligned}&\left(\left\langle |A|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle |A^*|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right)^2 \\ &\leq \frac{3}{8}\left\langle \left(|A|^4 + |A^*|^4\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \frac{1}{8}\left\langle \left(|A|^2 + |A^*|^2\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left|\left\langle |A^*|\,|A|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|. \end{aligned} \tag{2.13}$$

On the other hand, it follows from Lemma 2.4 that

$$\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^4 \leq \left(\left\langle |A|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle |A^*|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right)^2. \tag{2.14}$$

Now, on making use of the inequalities (2.13) and (2.14), we get the inequality

$$\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^4 \leq \frac{3}{8}\left\langle \left(|A|^4 + |A^*|^4\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \frac{1}{8}\left\langle \left(|A|^2 + |A^*|^2\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left|\left\langle |A^*|\,|A|_\lambda, \widehat{k}_\lambda\right\rangle\right|.$$

Taking the supremum over $\lambda \in \Omega$ in the above inequality, we obtain

$$\begin{aligned}\sup_{\lambda\in\Omega}\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^4 &\leq \frac{3}{8}\sup_{\lambda\in\Omega}\left(\left\langle \left(|A|^4 + |A^*|^4\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right) \\ &\quad + \frac{1}{8}\sup_{\lambda\in\Omega}\left(\left\langle \left(|A|^2 + |A^*|^2\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left|\left\langle |A^*|\,|A|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right).\end{aligned}$$

and

$$\mathrm{ber}^4(A) \leq \frac{3}{8}\left\||A|^4 + |A^*|^4\right\|_{\mathrm{ber}} + \frac{1}{8}\left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}} \mathrm{ber}\left(|A^*|\,|A|\right).$$

We deduce the desired inequality (2.12). □

If we choose of A a normal operator we get on both on (2.12) the same quantify $\|A\|^4$.

Corollary 2.11. *For any $A \in \mathcal{L}(\mathcal{H}(\Omega))$, we have the inequality*

$$\mathrm{ber}\left(|A^*|\,|A|\right) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}}.$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel. By the Schwarz inequality in the Hilbert space we have

$$\begin{aligned}\left|\left\langle |A^*|\,|A|\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| &= \left|\left\langle |A|\widehat{k}_\lambda, |A^*|\widehat{k}_\lambda\right\rangle\right| \leq \left\||A|^2\widehat{k}_\lambda\right\| \left\||A^*|^2\widehat{k}_\lambda\right\| \\ &= \sqrt{\left\langle |A|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle |A^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle} \\ &\leq \frac{1}{2}\left\langle \left(|A|^2 + |A^*|^2\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle.\end{aligned}$$

Taking the supremum $\lambda \in \Omega$ in the above inequality, we have

$$\mathrm{ber}\left(|A^*|\,|A|\right) \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|_{\mathrm{ber}}. \tag{2.15}$$

Now, on making use of the inequalities (2.12), we deduce that

$$\begin{aligned}
\text{ber}^4(A) &\le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{8}\left\| |A|^2 + |A^*|^2 \right\|_{\text{ber}} \text{ber}(|A^*||A|) \\
&\le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{16}\left\| |A|^2 + |A^*|^2 \right\|^2_{\text{ber}} \\
&\text{(by the inequality (2.15))} \\
&\le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{16}\left\| \left(\frac{2|A|^2 + 2|A^*|^2}{2} \right)^2 \right\|_{\text{ber}} \\
&\le \frac{3}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} + \frac{1}{8}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}} \\
&\text{(by the inequality (2.4))} \\
&= \frac{1}{2}\left\| |A|^4 + |A^*|^4 \right\|_{\text{ber}}.
\end{aligned}$$

□

So, our inequality (2.12) improves the inequality (1.5) when $r = 2$.

Now, we present Berezin norm inequalities and a related Berezin radius inequality for the sum of two operators.

Theorem 2.12. *Let* $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$. *Then*

$$\text{ber}^2(A+B) \le \text{ber}^2(A) + \text{ber}^2(B) + \|A\|_{\text{Ber}} \|B\|_{\text{Ber}} + \text{ber}(A).$$

Proof. Let $\lambda \in \Omega$ be arbitrary. Then we have

$$\begin{aligned}
\left| \left\langle (A+B)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 &\le \left(\left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| + \left| \left\langle B\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \right)^2 \\
&\text{(by the triangle inequality)} \\
&= \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \left| \left\langle B\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + 2\left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \left| \left\langle B\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \\
&= \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \left| \left\langle B\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + 2\left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \left| \left\langle \hat{k}_\lambda, B\hat{k}_\lambda \right\rangle \right| \\
&\le \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \left| \left\langle B\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \left\| A\hat{k}_\lambda \right\| \left\| B\hat{k}_\lambda \right\| + \left| \left\langle A\hat{k}_\lambda, B\hat{k}_\lambda \right\rangle \right| \\
&\text{(by the inequality (2.2))} \\
&= \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \left| \left\langle B\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \left\| A\hat{k}_\lambda \right\| \left\| B\hat{k}_\lambda \right\| + \left| \left\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|.
\end{aligned}$$

This implies that

$$\left| \left\langle (A+B)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 \le \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \left| \left\langle B\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \left\| A\hat{k}_\lambda \right\| \left\| B\hat{k}_\lambda \right\| + \left| \left\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|.$$

By taking the supremum over all $\lambda \in \Omega$ in the above inequality, we get

$$\begin{aligned}
\sup_{\lambda\in\Omega} \left| \left\langle (A+B)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 \le \sup_{\lambda\in\Omega} \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \sup_{\lambda\in\Omega} \left| \left\langle B\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 + \sup_{\lambda\in\Omega} \left\| A\hat{k}_\lambda \right\| \left\| B\hat{k}_\lambda \right\| \\
+ \sup_{\lambda\in\Omega} \left| \left\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|.
\end{aligned}$$

Therefore,

$$\text{ber}^2(A+B) \le \text{ber}^2(A) + \text{ber}^2(B) + \|A\|_{\text{Ber}} \|B\|_{\text{Ber}} + \text{ber}(B^*A).$$

This completes the proof. □

The following theorem gives a Berezin radius inequality for the sum of two operators.

Theorem 2.13. *Let* $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$. *Then*

$$\text{ber}^2(A+B) = \text{ber}^2(A) + \text{ber}^2(B) + \frac{1}{2}\left\| |A|^2 + |B^*|^2 \right\|_{\text{ber}} + \text{ber}(BA). \tag{2.16}$$

Proof. Let $\lambda \in \Omega$ be arbitrary. By using the triangle inequality and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
\left|\left\langle (A+B)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 &\le \left(\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| + \left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|\right)^2 \\
&= \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + 2\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|\left|\left\langle \widehat{k}_\lambda, B^*\widehat{k}_\lambda \right\rangle\right| \\
&\le \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \left\|A\widehat{k}_\lambda\right\|\left\|B^*\widehat{k}_\lambda\right\| + \left|\left\langle A\widehat{k}_\lambda, B^*\widehat{k}_\lambda \right\rangle\right| \\
&\quad \text{(by the inequality (2.2))} \\
&\le \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \sqrt{\left\langle |A|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\left\langle |B^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} \\
&\quad + \left|\left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| \\
&\le \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \frac{1}{2}\left(\left\langle |A|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left\langle |B^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right) \\
&\quad + \left|\left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| \\
&= \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \frac{1}{2}\left\langle |A|^2 + |B^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \left|\left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|.
\end{aligned}$$

Taking the supremum over $\lambda \in \Omega$, we have

$$\begin{aligned}
\sup_{\lambda\in\Omega}\left|\left\langle (A+B)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 &\le \sup_{\lambda\in\Omega}\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \sup_{\lambda\in\Omega}\left|\left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|^2 + \frac{1}{2}\sup_{\lambda\in\Omega}\left\langle |A|^2 + |B^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\quad + \sup_{\lambda\in\Omega}\left|\left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|
\end{aligned}$$

and

$$\text{ber}^2(A+B) \le \text{ber}^2(A) + \text{ber}^2(B) + \frac{1}{2}\left\| |A|^2 + |B^*|^2 \right\|_{\text{ber}} + \text{ber}(BA),$$

hence, we get (2.16) as required. □

Now, we are ready to present our new improvement of the inequality in $\text{ber}(A) \le \|A\|$.

Corollary 2.14. *Let* $A \in \mathcal{L}(\mathcal{H}(\Omega))$. *Then*

$$\text{ber}^2(A) \le \frac{1}{4}\left\| |A|^2 + |A^*|^2 \right\|_{\text{ber}} + \frac{1}{2}\text{ber}\left(A^2\right). \tag{2.17}$$

Proof. Let $\widehat{k}_\lambda$ be a normalized reproducing kernel of space $\mathcal{H}(\Omega)$ and $\theta \in \mathbb{R}$. First of all, we note that

$$\text{ber}(A) = \sup_{\theta\in\mathbb{R}}\left\|\text{Re}\left(e^{i\theta}A\right)\right\|_{\text{ber}}, \tag{2.18}$$

since

$$\sup_{\theta\in\mathbb{R}}\text{Re}\left\{e^{i\theta}\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right\} = \left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right|$$

and

$$\sup_{\theta\in\mathbb{R}}\left\|\text{Re}\left(e^{i\theta}A\right)\right\|_{\text{ber}} = \sup_{\theta\in\mathbb{R}}\text{ber}\left(\text{Re}\left(e^{i\theta}A\right)\right) = \text{ber}(A).$$

Then by (2.18), we have

$$\begin{aligned}
\operatorname{ber}(A) &= \sup_{\theta\in\mathbb{R}} \left\| \operatorname{Re}\left(e^{i\theta}A\right)\right\|_{\text{ber}} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| e^{i\theta}A + e^{-i\theta}A^*\right\|_{\text{ber}} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| \left(e^{i\theta}A + e^{-i\theta}A^*\right)^2\right\|_{\text{ber}}^{1/2} \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\| |A|^2 + |A^*|^2 + 2\operatorname{Re}\left(e^{2i\theta}A^2\right)\right\|_{\text{ber}}^{1/2} \\
&\leq \frac{1}{2}\sqrt{\left\| |A|^2 + |A^*|^2\right\|_{\text{ber}} + 2\sup_{\theta\in\mathbb{R}} \left\|\operatorname{Re}\left(e^{2i\theta}A^2\right)\right\|_{\text{ber}}} \\
&= \frac{1}{2}\sqrt{\left\| |A|^2 + |A^*|^2\right\|_{\text{ber}} + 2\operatorname{ber}\left(A^2\right)},
\end{aligned}$$

which proves the inequality in (2.17). Hence we have the desired inequality. □

For more recent results concerning Berezin radius inequalities for operators and other related results, we suggest [6, 11, 12, 13, 14, 15, 16, 28, 30, 31, 32].

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Author information

Verda Gürdal, Department of Mathematics, Süleyman Demirel University, Isparta, TURKEY.
E-mail: verdagurdal@icloud.com

Hamdullah Başaran, Department of Mathematics, Süleyman Demirel University, Isparta, TURKEY.
E-mail: 07hamdullahbasaran@gmail.com

Mualla B. Huban, Isparta University of Applied Sciences, Isparta, TURKEY.
E-mail: muallahuban@isparta.edu.tr

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