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A NOVEL ITERATIVE ALGORITHM FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract This article introduces a novel iterative algorithm by combining an exponential type kernel integral transform, namely, the Shehu transform, with the Variational Iteration Method to solve linear and nonlinear Fractional differential equations(FDEs). The proposed algorithm suggests to identify an optimal lagrange multiplier by implementing the Shehu transform and gives a series solution which converges rapidly to the exact solution. The applicability and the efficiency of the proposed algorithm are illustrated by some numerical examples.

1 Introduction

Fractional calculus is a generalization of ordinary calculus to an arbitrary order. In recent years, several fractional models have drawn much attention in diverse disciplines of science and technology, such as Viscoelasticity, Diffusion, Control Theory, Electromagnetism, Electrochemistry, Biosciences, Bio Engineering, Fluid Mechanics, Chaotic systems, Non-linear Dynamical systems and so on.

However, the analytical solutions do not exist for the most of FDEs. Due to this fact, many numerical schemes have been suggested to find approximate solutions of FDEs, namely, Variational Iteration Method, Separation of variables, Adomian Decomposition Method, Finite Difference Method, Homotopy Analysis Method, Homotopy Perturbation Method and so on. Apart from these numerical techniques, notable interest is initiated for reforming numerical algorithms by combining integral transforms with the semi analytical methods. Wu and Baleanu[25] combined the variational iteration method with the Laplace transform to identify the lagrange multipliers without tedious calculation. In[2], variational iteration method is improved with Aboodh transform method for solving fractional partial differential equations(FPDEs). A.K. Alomari et al.[1] implemented the Homotopy Sumudu transform method to solve FDEs. In[23], K.M. Saad et al. employed the Homotopy analysis method with the Laplace decomposition method to solve some time fractional order differential equations. Khalouta et al.[11] solved a certain class of nonlinear FPDEs by combining the Shehu transform with the reduced differential transform method.

The proposed algorithm is based upon one of the semi analytical methods, namely, variational iteration method which was developed by Ji-Huan He and one of exponential type kernel integral transforms, namely, Shehu transform which was introduced by S. Maitama et al.[13]. The main advantage of the proposed algorithm is to obtain the lagrange multipliers in a more straight forward way by implementing the Shehu transform.

We present this article as follows. In section 2, we introduce some necessary definitions and properties of fractional calculus. In section 3, the Shehu transform and some of its essential properties are discussed. Section 4 is devoted to demonstrate the proposed iterative algorithm, namely, the Shehu variational iteration method. In section 5, some numerical examples and absolute errors are presented to demonstrate the applicability, the efficiency and the accuracy of the proposed algorithm. Finally, we conclude our work in section 6.

2 Preliminaries

We here discuss some essential definitions and preliminary mathematical facts of fractional calculus.

Definition 2.1. [3, 14, 20, 21]The Riemann-Liouville fractional integration with order $\mu \geq 0$, of the function $h(x) \in L^1([0,\infty))$, is given by

$$J^{\mu}h(x) = \begin{cases} \frac{1}{\Gamma(\mu)}\int_0^x (x-\upsilon)^{\mu-1}h(\upsilon)d\upsilon, & \mu > 0. \\ h(x), & \mu = 0. \end{cases}$$

Let $h(x)$, $g(x) \in L^1([0,\infty))$, λ, $\gamma \in \mathbb{R}$, $\nu > -1$, $\mu \geq 0$.
Then

(i) $J^{\mu}\left(\lambda h(x) + \gamma g(x)\right) = \lambda J^{\mu}h(x) + \gamma J^{\mu}g(x)$.

(ii) $J^{\mu}x^{\nu} = \dfrac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)}x^{\mu+\nu}$.

Definition 2.2. [3, 14, 20, 21]The fractional derivative with order $\mu > 0$, of the function $h(x) \in L^1([0,\infty))$, in Caputo sense, is given by

$${}_0^cD_x^{\mu}h(x) = \begin{cases} \frac{1}{\Gamma(m-\mu)}\int_0^x \frac{h^{(m)}(\upsilon)}{(x-\upsilon)^{\mu-m+1}}d\upsilon, & m-1 < \mu < m,\ m \in \mathbb{N}. \\ \frac{d^m}{dx^m}h(x), & \mu = m \in \mathbb{N}. \end{cases}$$

Let $h(x)$, $g(x) \in L^1([0,\infty))$, λ, $\gamma \in \mathbb{R}$, $\mu \geq 0$.
Then

(i) ${}_0^cD_x^{\mu}k = 0, \quad \textit{where } k \textit{ is a constant}.$

(ii) ${}_0^cD_x^{\mu}\left(\lambda h(x) + \gamma g(x)\right) = \lambda\left({}_0^cD_x^{\mu}h(x)\right) + \gamma\left({}_0^cD_x^{\mu}g(x)\right)$.

(iii) ${}_0^cD_x^{\mu}\left(J^{\mu}h(x)\right) = h(x)$.

(iv) $J^{\mu}\left({}_0^cD_x^{\mu}h(x)\right) = h(x) - \displaystyle\sum_{k=0}^{m-1} h^k(0^+)\frac{x^k}{k!}, \qquad m-1 < \mu \leq m,$

where m is a positive integer, $x > 0$ and $h^k(0^+) := \lim_{x\to 0^+} D^k h(x)$, $k = 0,1,2,\ldots,m-1$.

Definition 2.3. The Mittag-Leffler function is named after a Swediss mathematician who defined and studied it in 1903 [15]. The function is a direct generalization of the exponential function e^x and it plays a major role in fractional calculus. The one parameter representation of the Mittag-Leffler function is defined as

$$E_{\mu}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\mu+1)},\ \mu \in \mathbb{C},\ Re(\mu) > 0.$$

The two parameter representation of the Mittage-Leffler function is defined as

$$E_{\mu,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\mu+\beta)},\ \mu,\beta \in \mathbb{C},\ Re(\mu) > 0,\ Re(\beta) > 0.$$

Moreover, $E_{\mu,\beta}(x) = \frac{1}{\Gamma(\beta)} + xE_{\mu,\mu+\beta}(x)$ and $E_{\mu,\beta}(x) = \beta E_{\mu,\beta+1}(x) + \mu x\frac{d}{dx}E_{\mu,\beta+1}(x)$.

3 The Shehu transform and its some properties

The Shehu transform, introduced by S. Maitama et al.[13], is an exponential type kernel integral transform which also generalizes the Laplace and Sumudu integral transforms.

Definition 3.1. [13] For any exponential order time domain function $h(\tau)$ belonging to the class
$A = \left\{ h(\tau) \in L^1(\mathbb{R}); \exists N, \eta_1, \eta_2 > 0, |h(\tau)| < Ne^{\left(\frac{|\tau|}{\eta_j}\right)},\ if\ \tau \in (-1)^j \times [0,\infty),\ j = 1,2 \right\}$,
the Shehu transform of $h(\tau)$ is defined by

$$\mathbb{S}[h(\tau)] = H(s,u) = \int_0^\infty e^{\left(\frac{-s\tau}{u}\right)} h(\tau) d\tau, \quad \tau > 0,$$

where s and u are the Shehu transform variables.

Definition 3.2. [9, 22] The Shehu transform of Caputo fractional derivative with order $\mu > 0$, of the function $h(x) \in L^1([0,\infty))$, is given by

$$\mathbb{S}[{}_0^cD_x^\mu h(x)] = \left(\frac{s}{u}\right)^\mu \mathbb{S}[h(x)] - \sum_{k=0}^{m-1} \left(\frac{s}{u}\right)^{\mu-(k+1)} h^{(k)}(0), \quad m-1 < \mu \le m,\ m \in \mathbb{N}.$$

Definition 3.3. The Shehu transform of the time-fractional derivative with order $\mu > 0$, of the function $h(x,\tau) \in L^1([0,\infty) \times [0,\infty))$, in the Caputo sense, is defined as

$$\mathbb{S}\left[{}_0^cD_\tau^\mu h(x,\tau)\right] = \left(\frac{s}{u}\right)^\mu \mathbb{S}[h(x,\tau)] - \sum_{k=0}^{m-1} \left(\frac{s}{u}\right)^{\mu-(k+1)} h^{(k)}(x,0), \quad m-1 < \mu \le m,\ m \in \mathbb{N}.$$

Let $h(\tau), g(\tau) \in A$, $\lambda, \gamma \in \mathbb{R} - \{0\}$.
Then

(i) $\mathbb{S}[\lambda h(\tau) + \gamma g(\tau)] = \lambda \mathbb{S}[h(\tau)] + \gamma[g(\tau)]$, *a linear property.*

(ii) $\mathbb{S}\left[\frac{\tau^m}{m!}\right] = \left(\frac{u}{s}\right)^{m+1}, \quad m = 0,1,2,\ldots$

(iii) $\mathbb{S}\left[\frac{\tau^\mu}{\Gamma(\mu+1)}\right] = \left(\frac{u}{s}\right)^{\mu+1}, \quad \mu \ge 0.$

(iv) $\mathbb{S}[h * g] = \mathbb{S}[h]\,\mathbb{S}[g]$, *a convolution property.*

4 Analysis of the Shehu variational iteration method

We here explore the basic idea of the proposed iterative algorithm. Let us consider the general non-linear fractional partial differential equation

$${}_0^cD_\tau^\mu h(x,\tau) + R[h(x,\tau)] + N[h(x,\tau)] = g(x,\tau), \qquad m-1 < \mu \le m, \tag{4.1}$$

with the initial conditions,

$$\frac{\partial^k h(x,0)}{\partial \tau^k} = h^{(k)}(x,0), \qquad k = 0,1,2,\ldots,m-1, \tag{4.2}$$

where ${}_0^cD_\tau^\mu$ is a Caputo fractional differential operator with respect to 'τ', R and N are linear and non-linear operators respectively, $g(x,\tau)$ is a known function. According to the variational iteration method[5, 6, 7, 17, 24, 25], the correction functional of (4.1) is given as,

$$h_{m+1}(x,\tau) = h_m(x,\tau) + \int_0^\tau \lambda(\tau-\varepsilon)\left[{}_0^cD_\varepsilon^\mu h_m + R\{h_m\} + N\{h_m\} - g(x,\varepsilon)\right] d\varepsilon, \tag{4.3}$$

where $\lambda(\tau-\varepsilon)$ is a general lagrange multiplier, the subscript $m \ge 0$ denotes the m^{th} approximation.
Taking the Shehu transform and using convolution property in (4.3), we attain

$$\begin{aligned}\mathbb{S}[h_{m+1}(x,\tau)] &= \mathbb{S}[h_m(x,\tau)] + \mathbb{S}[\lambda(\tau)]\mathbb{S}\left[{}_0^cD_\tau^\mu h_m + R\{h_m\} + N\{h_m\} - g(x,\tau)\right] \\ &= \mathbb{S}[h_m(x,\tau)] + \mathbb{S}[\lambda(\tau)]\Bigg\{\left(\frac{s}{u}\right)^\mu \mathbb{S}[h_m] - \sum_{k=0}^{m-1}\left(\frac{s}{u}\right)^{\mu-(k+1)} h^{(k)}(x,0) \\ &\qquad + \mathbb{S}\left[R\{h_m\} + N\{h_m\} - g(x,\tau)\right]\Bigg\}.\end{aligned} \tag{4.4}$$

The optimal value of λ can be identified by making (4.4) stationary with respect to h_m.

$$\delta\Big(\mathbb{S}[h_{m+1}(x,\tau)]\Big) = \delta\Big(\mathbb{S}[h_m(x,\tau)]\Big) + \delta\Bigg(\mathbb{S}[\lambda(\tau)]\Bigg\{\left(\frac{s}{u}\right)^{\mu}\mathbb{S}[h_m] - \sum_{k=0}^{m-1}\left(\frac{s}{u}\right)^{\mu-(k+1)} h^{(k)}(x,0) + \mathbb{S}\Big[R\{h_m\}+N\{h_m\}-g(x,\tau)\Big]\Bigg\}\Bigg). \tag{4.5}$$

Considering $\mathbb{S}[R\{h_m\}+N\{h_m\}]$ as restricted variation, i.e., $\delta\Big(\mathbb{S}[R\{h_m\}+N\{h_m\}]\Big)=0$, we have

$$\left(1+\left(\frac{s}{u}\right)^{\mu}\mathbb{S}[\lambda(\tau)]\right)=0, \text{ and so } \mathbb{S}[\lambda(\tau)] = -\left(\frac{u}{s}\right)^{\mu}. \tag{4.6}$$

Using (4.6) in (4.4) and taking the inverse Shehu transform , we attain a new correction functional

$$h_{m+1}(x,\tau) = \sum_{k=0}^{m-1}\frac{\tau^k}{k!}h^{(k)}(x,0) - \mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathbb{S}\Big[R\{h_m\}+N\{h_m\}-g(x,\tau)\Big]\right], \tag{4.7}$$

where the initial iteration $h_0(x,\tau)$ can be determined by

$$h_0(x,\tau) = \sum_{k=0}^{m-1}\frac{\tau^k}{k!}h^{(k)}(x,0).$$

The successive approximations rapidly converge to the exact solution of (4.1) as $m\to\infty$, that is, $h(x,\tau)=\lim_{m\to\infty}h_m(x,\tau)$.

5 Numerical Examples

We here discuss some fractional differential equations to elucidate the solution process and the simplicity of the proposed iterative algorithm.

Example 5.1. Consider the fractional relaxation oscillator equation[12]

$${}_0^cD_x^{\mu}h(x) + \omega^{\mu}h(x) = 0, \tag{5.1}$$

where $x\ge 0$, $1<\mu\le 2$, $\omega>0$, with the initial conditions, $h(0)=1$, $h'(0)=0$.
The exact solution of (5.1) is $E_\mu[(-\omega x)^\mu]$, where $E_\mu[(-\omega x)^\mu]$ denotes the Mittag-Leffler function. Using (4.7), we have the iteration formula,

$$h_{m+1}(x) = 1-\mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathbb{S}[\omega^{\mu}h_m(x)]\right], \tag{5.2}$$

where the initial iteration $h_0(x)$ is given by $h(0)+xh'(0)=1$. (5.3)

We then arrive the following iterations by (5.2) and (5.3).

$$\begin{aligned}
h_1(x) &= 1-\mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^{\mu}\mathbb{S}[\omega^{\mu}h_0(x)]\right] = 1-\omega^{\mu}\frac{x^{\mu}}{\Gamma(\mu+1)},\\
h_2(x) &= 1-\omega^{\mu}\frac{x^{\mu}}{\Gamma(\mu+1)}+\omega^{2\mu}\frac{x^{2\mu}}{\Gamma(2\mu+1)},\\
h_3(x) &= 1-\omega^{\mu}\frac{x^{\mu}}{\Gamma(\mu+1)}+\omega^{2\mu}\frac{x^{2\mu}}{\Gamma(2\mu+1)}-\omega^{3\mu}\frac{x^{3\mu}}{\Gamma(3\mu+1)},\\
&\vdots\\
h_m(x) &= \sum_{k=0}^{m}\frac{(-\omega x)^{k\mu}}{\Gamma(k\mu+1)},\\
&\vdots
\end{aligned}$$

The exact solution of (5.1) is attained by

$$h(x) = \lim_{m\to\infty} h_m(x) = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{(-\omega x)^{k\mu}}{\Gamma(k\mu+1)} = E_\mu\left[(-\omega x)^\mu\right].$$

The absolute errors incurred by the proposed strategy, Generalized Taylor Collocation Method (GTCM)[8] and Generalized Taylor Matrix Method (GTMM)[4] are exhibited in table 1. It is also proved in table 1 that the proposed iterative method reaches a higher degree of accuracy than GTCM and GTMM. Hence the numerical solutions obtained by the proposed strategy are in good agreement with the exact solutions.

Table 1. Absolute errors of Example 5.1 for $\mu = 1.5$ *and* $\omega = 1$

x	GTCM	GTMM	The proposed method$(m = 20)$
0.0	0.00000e-0	0.000000e-0	0.000000e-0
0.2	0.44805e-7	0.370000e-14	0.000000e-0
0.4	0.60316e-7	0.739000e-11	0.000000e-0
0.6	0.67627e-7	0.151700e-9	0.000000e-0
0.8	0.69751e-7	0.130000e-8	3.330669e-14
1.0	0.66570e-7	0.685800e-7	1.149080e-15

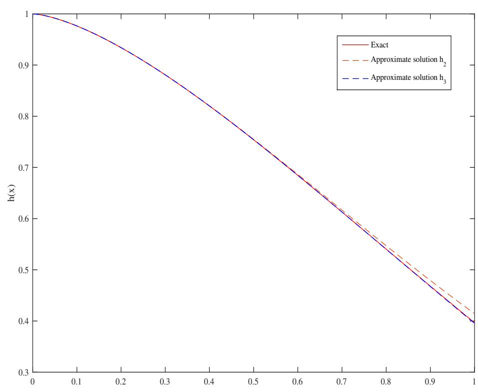

Figure 1. Comparison of the approximate solutions $h_2(x)$, $h_3(x)$ with the exact solution of Example 5.1 for $\mu = 1.5$ *and* $\omega = 1$

Example 5.2. Consider the non-homogeneous fractional differential equation,

$$ {}_0^cD_x^\mu h(x) - h(x) = 1, \tag{5.4}$$

where $x \geq 0$, $0 < \mu \leq 1$, with the initial condition, $h(0) = 0$.
The exact solution of (5.4) is $E_\mu(x^\mu) - 1$, where $E_\mu(x^\mu)$ denotes the Mittag-Leffler function. Using (4.7), we have the iteration formula,

$$h_{m+1}(x) = -\mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^\mu \mathbb{S}\left[-h_m(x) - 1\right]\right], \tag{5.5}$$

where the initial iteration $h_0(x)$ is given by $h(0) = 0$. (5.6)

We then arrive the following iterations by (5.5) and (5.6).

$$h_1(x) = \frac{x^\mu}{\Gamma(\mu+1)},$$
$$h_2(x) = \frac{x^\mu}{\Gamma(\mu+1)} + \frac{x^{2\mu}}{\Gamma(2\mu+1)},$$
$$h_3(x) = \frac{x^\mu}{\Gamma(\mu+1)} + \frac{x^{2\mu}}{\Gamma(2\mu+1)} + \frac{x^{3\mu}}{\Gamma(3\mu+1)},$$
$$\vdots$$
$$h_m(x) = \sum_{k=1}^{m} \frac{x^{k\mu}}{\Gamma(k\mu+1)},$$
$$\vdots$$

Table 2. Absolute errors of Example 5.2 attained by the proposed iterative method and HPM for $\mu = 0.5$ and $m = 4$

x	HPM	The proposed method
0.1	1.8697e-01	1.1503e-03
0.3	3.0482e-01	2.1053e-02
0.5	4.7315e-01	8.5440e-02
0.7	7.1827e-01	2.2117e-01
0.9	1.0624e+00	4.5943e-01

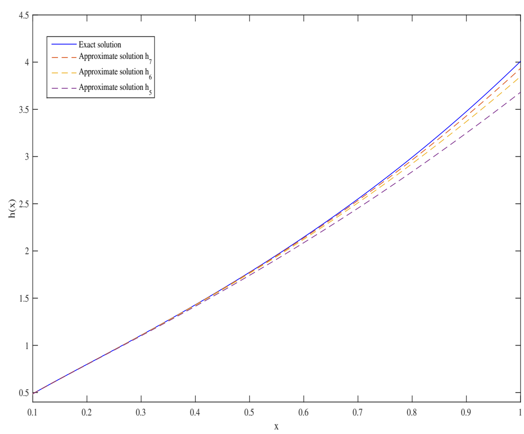

Figure 2. Comparison of the approximate solutions $h_5(x)$, $h_6(x)$, $h_7(x)$ with the exact solution of Example 5.2 for $\mu = 0.5$.

The exact solution of (5.4) is attained by

$$h(x) = \lim_{m\to\infty} h_m(x) = \lim_{m\to\infty} \sum_{k=1}^{m} \frac{x^{k\mu}}{\Gamma(k\mu+1)} = E_\mu(x^\mu) - 1.$$

The absolute errors incurred by the proposed strategy and Homotopy Perturbation Method (HPM)[16] for $\mu = 0.5$ are exhibited in table 2. We clearly see from table 2 that the proposed strategy is superior to HPM.

Example 5.3. Consider the non-homogeneous time fractional Burger equation[17]

$$ {}_0^cD_\tau^\mu h(x,\tau) + h_x(x,\tau) - h_{xx}(x,\tau) = \frac{2\tau^{2-\mu}}{\Gamma(3-\mu)} + 2x - 2, \tag{5.7}$$

where $\tau \geq 0$, $x \in \mathbb{R}$, $0 < \mu \leq 1$, with the initial condition, $h(x,0) = x^2$. The exact solution of (5.7) is $x^2 + \tau^2$.
Using (4.7), we have the iteration formula,

$$h_{m+1}(x,\tau) = x^2 - \mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathbb{S}\left[\frac{\partial h_m}{\partial x} - \frac{\partial^2 h_m}{\partial x^2} - \frac{2\tau^{2-\mu}}{\Gamma(3-\mu)} - 2x + 2\right]\right], \tag{5.8}$$

where the initial iteration $h_0(x,\tau)$ is given by $h(x,0) = x^2$. (5.9)

We then arrive the following iterations by (5.8) and (5.9).

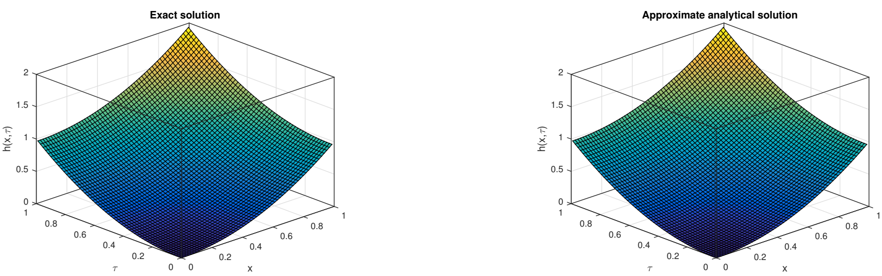

Figure 3. Comparison of the approximate solution $h_m(x,\tau)$ with the exact solution of Example 5.3 for any $\mu \in (0,1]$.

$$\begin{aligned}
h_1(x,\tau) &= x^2 + \tau^2, \\
h_2(x,\tau) &= x^2 + \tau^2, \\
h_3(x,\tau) &= x^2 + \tau^2, \\
&\vdots \\
h_m(x,\tau) &= x^2 + \tau^2, \\
&\vdots
\end{aligned}$$

The exact solution of (5.7) is attained by

$$h(x,\tau) = \lim_{m\to\infty} h_m(x,\tau) = x^2 + \tau^2.$$

In this example, in the first iteration itself, we attain the exact solution of (5.7).

Example 5.4. Consider the non-linear time fractional partial differential equation,

$$ {}_0^c D_\tau^\mu h(x,\tau) = h(x,\tau) h_{xx}(x,\tau) - \frac{2}{x} h(x,\tau) h_x(x,\tau) + \frac{2}{x^2} h^2(x,\tau) + h(x,\tau), \tag{5.10}$$

where $\tau \geq 0$, $x \in \mathbb{R}$, $0 < \mu \leq 1$, with the initial condition, $h(x,0) = x^2$. The exact solution of (5.10) is $x^2 E_\mu(\tau^\mu)$, where $E_\mu(\tau^\mu)$ denotes the Mittag-Leffler function.
Using (4.7), we have the iteration formula,

$$h_{m+1}(x,\tau) = x^2 - \mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^{\mu} \mathbb{S}\left[-h_m \frac{\partial^2 h_m}{\partial x^2} + 2\frac{h_m}{x}\frac{\partial h_m}{\partial x} - 2\frac{h_m^2}{x^2} - h_m\right]\right], \tag{5.11}$$

where the initial iteration $h_0(x,\tau)$ is given by $h(x,0) = x^2$. (5.12)

We then arrive the following iterations by (5.11) and (5.12).

$$
\begin{aligned}
h_1(x,\tau) &= x^2\left(1+\frac{\tau^{\mu}}{\Gamma(\mu+1)}\right),\\
h_2(x,\tau) &= x^2\left(1+\frac{\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\tau^{2\mu}}{\Gamma(2\mu+1)}\right),\\
h_3(x,\tau) &= x^2\left(1+\frac{\tau^{\mu}}{\Gamma(\mu+1)}+\frac{\tau^{2\mu}}{\Gamma(2\mu+1)}+\frac{\tau^{3\mu}}{\Gamma(3\mu+1)}\right),\\
&\ \ \vdots\\
h_m(x,\tau) &= x^2\sum_{k=0}^{m}\frac{\tau^{k\mu}}{\Gamma(k\mu+1)},\\
&\ \ \vdots
\end{aligned}
$$

The exact solution of (5.10) is attained by

$$
h(x,\tau)=\lim_{m\to\infty}h_m(x,\tau)=x^2E_{\mu}(\tau^{\mu}).
$$

Table 3. Absolute errors of Example 5.4 attained by the proposed iterative method for $m = 20$ and $\mu = 0.75$

x	τ	Absolute errors
0.1	0.1	1.7475e-31
0.2	0.2	3.9109e-26
0.3	0.3	5.2934e-23
0.4	0.4	8.8475e-21
0.5	0.5	4.7003e-19
0.6	0.6	1.2093e-17
0.7	0.7	1.8865e-16
0.8	0.8	2.0403e-15
0.9	0.9	1.6681e-14
1.0	1.0	1.0937e-13

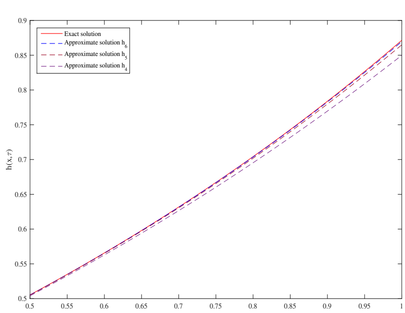

Figure 4. Comparison of the approximate solutions $h_4(x,\tau)$, $h_5(x,\tau)$, $h_6(x,\tau)$ with the exact solution of Example 5.4 for $\mu = 0.75$ at $x = 0.5$.

The absolute errors incurred by the proposed strategy for $\mu = 0.75$ are exhibited in table 3. We also see from table 3 that iterative solutions converge rapidly to the exact solution.

6 Conclusion

In this article, the Shehu transform combined with the variational iteration method was successfully employed to attain approximate solutions of linear and nonlinear fractional differential equations. The major advantage of the proposed iterative algorithm is its rapid convergence to the exact solution and so its high degree of accuracy. Numerical examples and absolute errors elucidated the simplicity, the efficiency and the high degree accuracy of the proposed strategy.

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