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Sufficient Conditions for Univalence

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Abstract Let $\mathcal{A}$ denote the class of all functions f defined and analytic in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. In the present paper, we obtain sufficient conditions for f to be starlike, Bazilevic or bounded turning of some order $\beta, 0 < \beta < 1$ in $\mathbb{E}$. Our result extends an earlier such result which is available only for the range $[1/2, 1)$ of the parameter β.

1 Introduction

Let $\mathcal{H}$ denote the class of functions which are analytic in the open unit disc $\mathbb{E} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. For some $a \in \mathbb{C}$ and $n \in \mathbb{N}$ (the set of positive integers), let $\mathcal{H}(a, n)$ represents family of all functions f in $\mathcal{H}$ which are of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

We denote by $\mathcal{A}$ the class of all those functions f in $\mathcal{H}$ which are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, we also let S denote the subclass of all univalent functions in $\mathcal{A}$. For a real number $\beta, 0 \leq \beta < 1$, let

$$\mathcal{R}(\beta) = \{f \in \mathcal{A} : \Re f'(z) > \beta, z \in E\},$$

$$S^*(\beta) = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \beta, z \in E \right\}$$

and

$$\mathcal{K}(\beta) = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta, z \in E \right\}.$$

Functions in the class $\mathcal{R}(\beta), S^*(\beta)$ and $\mathcal{K}(\beta)$ are called, respectively, functions of bounded turning of order β, starlike functions of order β and convex functions of order β. It is well known that functions in $\mathcal{R}(\beta), S^*(\beta)$ and $\mathcal{K}(\beta)$ are univalent and $\mathcal{K}(\beta) \subseteq S^*(\beta), 0 \leq \beta < 1$. Further note that $S^*(0) := S^*$ and $\mathcal{K}(0) := \mathcal{K}$ are the usual classes of starlike (with respect to the origin) functions and convex functions in S, respectively. For more details on these classes of functions we refer the reader to [3].

Following Babalola [1], we say that a function $f \in \mathcal{A}$ is Bazilevic function of order $\beta, 0 \leq \beta < 1$ and type $\lambda + 1,\ \lambda \geq -1$, if it satisfies the condition

$$\Re \left\{ f'(z) \left(\frac{f(z)}{z} \right)^{\lambda} \right\} > \beta, z \in E. \tag{1.1}$$

It is known that functions satisfying (1.1) are univalent for $\lambda \geq -1$ in E [6]. Note that functions of bounded turning of order β are Bazilevic functions of order β and type 1, whereas starlike functions of order β are Bazilevic functions of order β and type 0. Babalola [1] proved the following sufficient condition for $f \in \mathcal{A}$ to be Bazilevic function of order β, $\frac{1}{2} \leq \beta < 1$ and type $\lambda + 1$.

Theorem 1.1. *If $f \in \mathcal{A}$ satisfies,*

$$\Re\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)+\lambda\frac{zf'(z)}{f(z)}\right\} > \lambda+\frac{3\beta-1}{2\beta}, z \in E,$$

then

$$\Re\left\{f'(z)\left(\frac{f(z)}{z}\right)^{\lambda}\right\} > \beta, \frac{1}{2} \le \beta < 1.$$

The main objective of the present article is to extend Theorem 1.1 by including the range $(0, \frac{1}{2})$ in the set of values taken by the parameter β.

Before we state and prove our main result, we inform the reader that, in 2017, a general class of functions which contains starlike functions of order β, functions of bounded turning of order β and Bazilevic functions of order β is defined and characterized by Jimoh and Babalola [4] and a recent paper of Babalola and Jimoh [2] constitutes another interesting reading related to this general class.

2 Main Result

To prove our main result, we shall need the following lemma:

Lemma 2.1 (Miller and Mocanu [5]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that ϕ is a mapping from $C^2 \times \mathbb{E}$ to $\mathbb{C}$ which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in \mathbb{E}$ and for all real x, y such that $y \le -\frac{n(1+x^2)}{2}$. If the function $p \in \mathcal{H}[1, n]$ and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{E}$, then $\Re\{p(z)\} > 0$ in $\mathbb{E}$.*

Theorem 2.2. *Let $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ in $\mathbb{E}$ and*

$$\alpha(\beta) = \begin{cases} \frac{2-3\beta}{2(1-\beta)} & \text{for } 0 < \beta < \frac{1}{2} \\ \frac{3\beta-1}{2\beta} & \text{for } \frac{1}{2} \le \beta < 1. \end{cases}$$

If

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}+\mu\left(1-\frac{zf'(z)}{f(z)}\right)\right\} > \alpha(\beta), z \in \mathbb{E}, \tag{2.1}$$

then

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right\} > \beta,$$

where $0 < \beta < 1$.

Proof. Define a function p by

$$f'(z)\left(\frac{z}{f(z)}\right)^{\mu} = \beta + (1-\beta)p(z), \tag{2.2}$$

Then, clearly, p is analytic in $\mathbb{E}$ and $p \in \mathcal{H}[1, 1]$.
A simple calculation yields

$$\frac{zf''(z)}{f'(z)}+\mu\left(1-\frac{zf'(z)}{f(z)}\right) = \frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}$$

$$1+\frac{zf''(z)}{f'(z)}+\mu\left(1-\frac{zf'(z)}{f(z)}\right) = 1+\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}$$

$$= \psi(p(z), zp'(z); z)$$

where $\psi(u,v;z) = 1 + \dfrac{(1-\beta)v}{\beta+(1-\beta)u}$ is continuous in $\mathbb{D} = \left[\mathbb{C}\setminus\left(\dfrac{-\beta}{1-\beta}\right)\right]\times\mathbb{C}\times\mathbb{E}$.

Let $u = u_1 + iu_2$, $v = v_1 + iv_2$, where u_1, u_2, v_1, v_2 are reals with $v_1 \le -\dfrac{1+u_2^2}{2}$. Then

$$\begin{aligned}\Re\,\psi(iu_2, v_1; z) &= 1 + \frac{(1-\beta)\beta v_1}{\beta^2+(1-\beta)^2u_2^2}\\ &\le 1 - \frac{(1-\beta)\beta(1+u_2^2)}{2[\beta^2+(1-\beta)^2u_2^2]}\\ &= \phi(u_2)(say)\\ &\le \max\phi(u_2).\end{aligned}$$

Define

$$\Omega = \{w : \Re\, w > \alpha(\beta)\}.$$

From (2.1), $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{E}$. Further, since the function $\phi(u_2)$ is symmetrical with respect to u_2, so it is sufficient to consider its behaviour for $u_2 \ge 0$. We can readily see that

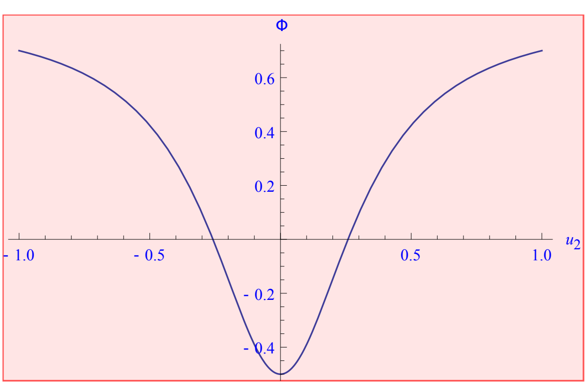
$$\phi'(u_2) = \frac{\beta(1-\beta)(1-2\beta)u_2}{[\beta^2+(1-\beta)^2u_2^2]^2}.$$

As $0 < \beta < 1$, so $\phi'(u_2) > 0$ for $0 < \beta < \frac{1}{2}$ and $\phi'(u_2) \le 0$ for $\frac{1}{2} \le \beta < 1$. Thus $\phi(u_2)$ is an increasing function of u_2 for $0 < \beta < \frac{1}{2}$ and decreasing function of u_2 for $\frac{1}{2} \le \beta < 1$. Behaviour of $\phi(u_2)$ for $\beta = 0.25$ and $\beta = 0.75$ is depicted in Figure 1 and Figure 2, respectively (using MATHEMATICA version 12.0).
Therefore,

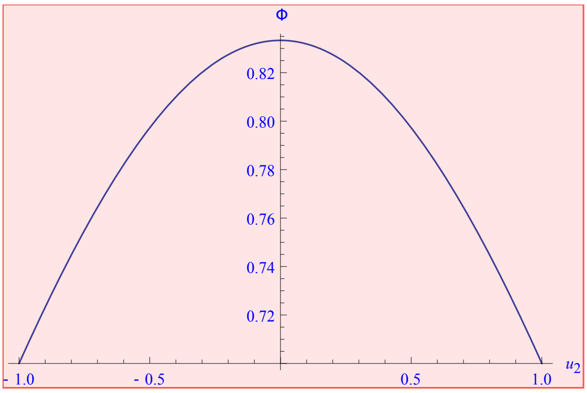
$$\max\phi(u_2) = \begin{cases} \lim_{u_2\to\infty}\phi(u_2) = \frac{2-3\beta}{2(1-\beta)} & \text{for } 0 < \beta < \frac{1}{2}\\ \phi(0) = \frac{3\beta-1}{2\beta} & \text{for } \frac{1}{2} \le \beta < 1. \end{cases} \tag{2.3}$$

Thus $\Re\,\psi(iu_2, v_1; z) \le \alpha(\beta)$, where $\alpha(\beta)$ is given by (2.3). Hence, $\psi(iu_2, v_1; z) \notin \Omega$. Therefore, by Lemma 2.1, $\Re\, p(z) > 0$, $z \in \mathbb{E}$. Finally, by (2.2),

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right\} > \beta,\ 0 < \beta < 1, z \in \mathbb{E}.$$



Graph of $\phi(u_2)$ at $\beta = 0.25$
Figure 1



Graph of $\phi(u_2)$ at $\beta = 0.75$
Figure 2

Remark 2.3. Theorem1.1 corresponds to $\mu = -\lambda$ and $\frac{1}{2} \leq \beta < 1$ in Theorem 2.2.

Selecting $\mu = 0$ in Theorem 2.2, we obtain the following sufficient condition for $f \in \mathcal{A}$ to be in $\mathcal{R}(\beta)$ for $0 < \beta < 1$. (Compare with Corollary 2.5, [1]).

Corollary 2.4. *Let $f \in \mathcal{A}$ and*

$$\alpha(\beta) = \begin{cases} \frac{2-3\beta}{2(1-\beta)} & \text{for } 0 < \beta < \frac{1}{2} \\ \frac{3\beta-1}{2\beta} & \text{for } \frac{1}{2} \leq \beta < 1. \end{cases}$$

If

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha(\beta),$$

then

$$\Re(f'(z)) > \beta, 0 < \beta < 1, z \in \mathbb{E}.$$

Taking $\mu = 1$ in Theorem 2.2, we obtain the following sufficient condition for $f \in \mathcal{A}$ to be in $\mathcal{S}^*(\beta)$ for $0 < \beta < 1$. (Compare with Corollary 2.2, [1]).

Corollary 2.5. *Let $f \in \mathcal{A}$ and*

$$\alpha(\beta) = \begin{cases} \frac{2-3\beta}{2(1-\beta)} & \text{for } 0 < \beta < \frac{1}{2} \\ \frac{3\beta-1}{2\beta} & \text{for } \frac{1}{2} \leq \beta < 1. \end{cases}$$

If

$$\Re\left\{2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > \alpha(\beta),$$

then

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, 0 < \beta < 1, z \in \mathbb{E}.$$

Putting $\mu = -1$ in Theorem 2.2, we obtain the following result.

Corollary 2.6. *Let $f \in \mathcal{A}$ and*

$$\alpha(\beta) = \begin{cases} \frac{2-3\beta}{2(1-\beta)} & \text{for } 0 < \beta < \frac{1}{2} \\ \frac{3\beta-1}{2\beta} & \text{for } \frac{1}{2} \leq \beta < 1. \end{cases}$$

If

$$\Re\left\{\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}\right\}>\alpha(\beta),$$

then

$$\Re\left\{\frac{f'(z)f(z)}{z}\right\}>\beta, 0<\beta<1.$$

Therefore, f is Bazilevic function of order β and type 2 in $\mathbb{E}$.

Choosing $\mu=\dfrac{1}{2}$ in Theorem 2.2, we obtain the following. (Compare with Corollary 2.3, [1]).

Corollary 2.7. *Let $f\in\mathcal{A}$ and*

$$\alpha(\beta)=\begin{cases}\frac{2-3\beta}{2(1-\beta)} & \text{for } 0<\beta<\frac{1}{2}\\ \frac{3\beta-1}{2\beta} & \text{for } \frac{1}{2}\leq\beta<1.\end{cases}$$

If

$$\Re\left[2\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(1-\frac{zf'(z)}{f(z)}\right)\right]>2\alpha(\beta), z\in\mathbb{E},$$

then

$$\Re\frac{z^{\frac{1}{2}}f'(z)}{f^{\frac{1}{2}}(z)}>\beta.$$

Therefore, f is Bazilevic of order β and type $\dfrac{1}{2}$ in $\mathbb{E}$.

Example 2.8. If we take $\beta=\dfrac{1}{4}$ in Corollary 2.7, we have the following result.
If

$$\Re\left[2\left(1+\frac{zf''(z)}{f'(z)}\right)-\frac{zf'(z)}{f(z)}\right]>\frac{2}{3}, z\in\mathbb{E},$$

then

$$\Re\left(\frac{z^{1/2}f'(z)}{f^{1/2}(z)}\right)>\frac{1}{4}.$$

Remark 2.9. We are unable to include the value 0 in the set of values taken by β in Theorem 2.2 as in that case $\alpha(0)=1$ and this violates the normalization condition in inequality (2.1).

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