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EXISTENCE AND UNIQUENESS SOLUTION FOR FRACTIONAL VOLTERRA EQUATION WITH FRACTIONAL ANTI-PERIODIC BOUNDARY CONDITIONS

Faraj Y. Ishak

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Abstract In this paper some existence results for a first kind Volterra differential equation of fractional order with fractional anti-periodic boundary conditions are presented. The main tool of the study is Leray-schauder degree theory. Some illustrative examples are discussed.

1 Introduction

The subject of fractional calculus has recently gained much momentum and a variety of problems involving differential equations and inclusions of fractional order have been addressed by several researchers. Fractional differential equations appear naturally in a number of fields such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, Bode analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, etc.[2]-[4].

For some recent work on fractional differential equations and inclusions, anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have recently received considerable attention. For examples and details of anti-periodic boundary conditions, see [5]-[15].

In this paper we study the existence and uniqueness solution for the following anti-periodic fractional boundary value problem:

$$^{c}D^{q}u(t)=\int_{-\infty}^{t}K(t,s)\phi(t,s,u(s))ds,\quad t\in[0,T],\quad 1<q\leq 2$$

$$u(0)=-u(T)\quad,\quad D^{(q-1)/2}u(0)=-D^{(q-1)/2}u(T)\tag{1.1}$$

where $^{c}D^{q}$ denotes the Caputo fractional derivative of order $q, K(t,s)$ is a continuous given kernel,$\phi:[0,T]\times[0,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function and T is a fixed positive constant .

2 Preliminaries

We need the following definitions:

Definition 2.1. diffinition 2.1: For a continuous function $g:[0,\infty)\to\mathbb{R}$ the Caputo derivative of fractional order q is defined as:

$$^{c}D^{q}g(t)=\frac{1}{\Gamma(n-q)}\int_{0}^{t}(t-s)^{n-q-1}g^{(n)}(s)ds$$

$n-1<q<n, n=[q]+1$ where $[q]$ denotes the integer part of the real number q [2].

Definition 2.2. diffinition 2.2: The Riemann–Liouville fractional integral of order q is defined as:

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \;\; q > 0$$

provided the integral exists [2] .

Definition 2.3. diffinition 2.3: The Riemann–Liouville fractional derivative of order q for a function $g(t)$ is defined by:

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dn}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q] + 1$$

provided the right-hand side is pointwise defined on$(0, \infty)$ [7] .

Theorem 2.4. *Let X be a Banach space, assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $T : \overline{\Omega} \to X$ be a completely continuous operator such that:*

$$\|Tu\| \leq \|u\|, \forall u \in \partial\Omega$$

then T has a fixed point in Ω [1] .

Lemma 2.5. *For $q > 0$ the general solution of the fractional differential equation $D^q x(t) = 0$ is given by :*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, (n = [q] + 1)$ [1] .

In view of lemma 2.2, it follows that

$$I^q \, {}^c D^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} \tag{2.1}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, (n = [q] + 1)$.

Rewrite problem (1.1) as :

$${}^c D^q u(t) = \sigma(t, u(t)) \quad 0 < t < T, \quad 1 < q \leq 2$$

$$u(0) = -u(T) \quad , \quad D^{(q-1)/2} u(0) = -D^{(q-1)/2} u(T) \tag{2.2}$$

Where

$$\sigma(t, u(t)) = \int_{-\infty}^t K(t, s) \phi(t, s, u(s)) ds$$

To study the nonlinear problem (2.2), we need the following lemma :

Lemma 2.6. *For any $\sigma \in C[0, T]$ there exists exactly one solution u for problem (2.2), moreover a function u is a solution for problem (2.2) if and only if:*

$$u(t) = \int_0^T G(t, s) \sigma(t, s, u(s)) ds$$

where $G(t, s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \frac{t(3-q)-2T}{T(1+q)} + \frac{(3-q)\Gamma(3-q/2)(T-2t)}{(1+q)T^{3-q/2}}, & if\; 0 \leq t < s \leq T \\ \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t(3-q)-2T}{T(1+q)} + \frac{(3-q)\Gamma(3-q/2)(T-2t)}{(1+q)T^{3-q/2}}, & if\; 0 \leq s \leq t \leq T \end{cases}$$

Proof. Using (2.1) we have :

$$u(t) = I^q\sigma(t) - b_0 - b_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - b_0 - b_1 t$$

for arbitrary constants b_0 and b_1 , in view of the relations ${}^cD^qI^qu(t) = u(t)$ and $I^qI^pu(t) = I^{q+p}u(t)$ for $q, p > 0$, u$\in C[0,T]$ and from the boundary conditions of problem (1.1) we obtain:

$$b_0 = \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \frac{b_1 T}{2} \tag{2.3}$$

$$b_1 = \frac{\Gamma(5-q/2)}{T^{3-q/2}}\int_0^T \frac{(T-s)^{q-1/2)}}{\Gamma(q+1/2)}\sigma(s)ds - \frac{b_0\Gamma(5-q/2)}{T} \tag{2.4}$$

Solving (2.3) , (2.4) for b_0 and b_1 we find:

$$b_0 = \frac{2}{1+q}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \frac{(3-q)\Gamma(3-q/2)}{T^{1-q/2}(1+q)}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\sigma(s)ds$$

$$b_1 = \frac{2(3-q)\Gamma(3-q/2)}{T^{3-q/2}(1+q)}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\sigma(s)ds - \frac{(3-q)}{T(1+q)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds$$

Thus the unique solution of (2.2) is:

$$u(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \frac{t(3-q)-2T}{T(1+q)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds$$
$$+\frac{(3-q)\Gamma(3-q/2)(T-2t)}{(1+q)T^{3-q/2}}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\sigma(s)ds = \int_0^T G(t,s)\sigma(s)ds$$

3 Existence results

Let $\omega = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with the norm defined by :

$$\|u\| = \sup\{|u(t)|, t \in [0,T]\}$$

Define an operator $\Psi : \omega \to \omega$ as :

$$(\Psi u)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \frac{t(3-q)-2T}{T(1+q)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\sigma(s)ds$$
$$+\frac{(3-q)\Gamma(3-q/2)(T-2t)}{(1+q)T^{3-q/2}}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\sigma(s)ds \quad ,t \in [0,T] \tag{3.1}$$

Observe that problem (1.1) has a solution if and only if the operator Ψ has a fixed point.

Lemma 3.1. *The operator $\Psi : \omega \to \omega$ is completely continuous.*

Proof. Let $\Omega \subset \omega$ be bounded then $\forall t \in [0,T], u \in \Omega$ there exist a positive constant M such that $|\phi(t,s,u(t))| \leq M$ *and* $\lambda, \delta \in \mathbb{R}$, such that $|K(s,t)| \leq \delta e^{-\lambda(t-s)}$, thus we have :

$$|(\Psi u)(t)| \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^t |K(t,s)||\phi(t,s,u(s))|ds(s)\right)ds$$
$$+\frac{|t(3-q)-2T|}{|T(1+q)|}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^T |K(t,s)||\phi(t,s,u(s))|ds(s)\right)ds$$
$$+\frac{|(3-q)\Gamma(3-q/2)(T-2t)|}{|(1+q)T^{3-q/2}|}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\left(\int_{-\infty}^T |K(t,s)|\phi(t,s,u(s)) \mid ds(s)\right)ds$$

$$\leq \frac{\delta M}{\lambda}\left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \frac{|t(3-q)-2T|}{|T(1+q)|}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}ds \right.$$
$$\left. + \frac{|(3-q)\Gamma(3-q/2)(T-2t)|}{|(1+q)T^{3-q/2}|}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}ds\right]$$
$$\leq \frac{\delta M}{\lambda}\left[\frac{t^q}{\Gamma(q+1)} + \frac{(t(3-q)-2T)T^q}{T(1+q)\Gamma(q+1)} + \frac{(3-q)\Gamma(3-q/2)(T-2t)T^{q+1/2}}{(1+q)T^{3-q/2}\Gamma(q+3/2)}\right]$$
$$\leq \frac{\delta M}{\lambda}\left(\frac{2T^q}{(1+q)\Gamma(q+1)} - \frac{(3-q)\Gamma(3-q/2)T^q}{(1+q)\Gamma(q+3/2)}\right) \leq M_1$$

Which implies that $\|\Psi u\| \leq M_1$, furthermore for $t_1, t_2 \in [0,T]$ we have :

$$\begin{aligned}
&|(\Psi u)(t_2) - (\Psi u)(t_1)| \\
&\leq \int_0^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^{t_2} |K(t,s)||\phi(t,s,u(s))|ds(s)\right)ds \\
&+ \frac{|(t_2(3-q)-2T)|}{T(1+q)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^{t_2} |K(t,s)||\phi(t,s,u(s))|ds(s)\right)ds \\
&+ \frac{|(3-q)\Gamma(3-q/2)(T-2t_2)|}{(1+q)T^{3-q/2}}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\left(\int_{-\infty}^{t_2} |K(t,s)\|\phi(t,s,u(s))|ds(s)\right)ds \\
&- \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^{t_1} |K(t,s)\|\phi(t,s,u(s))|ds(s)\right)ds \\
&- \frac{|(t_1(3-q)-2T)|}{T(1+q)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^{t_1} |K(t,s)\|\phi(t,s,u(s))|ds(s)\right)ds \\
&- \frac{|(3-q)\Gamma(3-q/2)(T-2t_1)|}{(1+q)T^{3-q/2}}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\left(\int_{-\infty}^{t_1} |K(t,s)||\phi(t,s,u(s))|ds(s)\right)ds
\end{aligned}$$

$$\leq \frac{\delta M(t_2^q - t_1^q)}{\lambda\Gamma(q+1)} + \frac{\delta M T^{q-1}(3-q)(t_2-t_1)}{\lambda(q+1)\Gamma(q+1)} + \frac{2\delta M T^{q-1}(3-q)\Gamma(3-q/2)(t_1-t_2)}{\lambda\Gamma(q+3/2)(q+1)}$$

Obviously the right-hand side of the above inequality tends to zero independently of $u \in \Omega$ as $t_2 \to t_1$, therefore it follows by the Arzelá-Ascoli theorem that $\Psi : \omega \to \omega$ is completely continuous.

Theorem 3.2. *Let* $\phi : [0,T]\times[0,T]\times\mathbb{R} \to \mathbb{R}$ *and*

$$\lim_{u\to 0}\frac{\phi(t,s,u(t))}{u} = 0$$

then problem (1.1) has at least one solution .

Proof. Since $\lim_{u\to 0}\frac{\phi(t,s,u(t))}{u} = 0$, there exists a constant $\epsilon > 0$ such that $|\phi(t,s,u)| \leq \xi|u|$ for $0 < |u| < \epsilon$, where $\xi > 0$ is :

$$\max_{t\in[0,T]} \frac{\delta}{\lambda}\left[\frac{|t^q|}{\Gamma(q+1)} + \frac{|t(3-q)-2T|T^q}{T(1+q)\Gamma(q+1)} + \frac{|(3-q)\Gamma(3-q/2)(T-2t)|T^{q-1}}{(1+q)\Gamma(q+3/2)}\right]\xi \leq 1 \quad (3.2)$$

define $\Omega_1 = \{u \in \omega : \|u\| < \epsilon\}$ and take $u \in \omega$ such that $\|u\| = \epsilon$ that is $u \in \partial\Omega_1$, by lamma (3.1) we know that Ψ is completely continuous and :

$$|(\Psi u)(t)| \leq \max_{t\in[0,T]} \frac{\delta}{\lambda}\left[\frac{|t^q|}{\Gamma(q+1)} + \frac{|t(3-q)-2T|T^q}{T(1+q)\Gamma(q+1)} + \frac{|(3-q)\Gamma(3-q/2)(T-2t)|T^{q-1}}{(1+q)\Gamma(q+3/2)}\right]\xi\|u\|$$

Thus in view of (3.2) we obtain $\|\Psi(u)\| \leq \|u\|, u \in \partial\Omega_1$, hence by theorem (2.1) the operator Ψ has at least one fixed point which in turn implies that problem (1.1) has at least one solution .

Theorem 3.3. *Assume that* $\phi : [0,T]\times[0,T]\times X \to X$ *is a jointly continuous function satisfying the condition :*

$$\|\phi(u)-\phi(v)\| \leq L\|u-v\|, \quad \forall t\in[0,T], \quad u,v\in X$$

with

$$L \leq \frac{\lambda(q+1)\Gamma(q+1)\Gamma(q+3/2)}{2T^q\delta((q+1)\Gamma(q+3/2)+(1-q)\Gamma(q+3/2)-(3-q)\Gamma(3-q/2)\Gamma(q+1))}$$

Then the anti-periodic boundary value problem (1.1) has a unique solution.

Proof. Setting $\sup_{t\in[0,T]}\phi(t,s,0)=M, |K(s,t)|\leq \delta e^{-\lambda(t-s)}$ and selecting:

$$r \geq \frac{2\left(T^q M\delta((q+1)\Gamma(q+3/2)+(1-q)\Gamma(q+3/2)-(3-q)\Gamma(3-q/2)\Gamma(q+1))\right)}{\lambda(q+1)\Gamma(q+1)\Gamma(q+3/2)}$$

We show that $\Psi B_r \subset B_r$ where $B_r=\{u\in C[0,T] : \|u\|\leq r\}$, for $u\in B_r$ we have :

$$\begin{aligned}
&|(\Psi u)(t)| \\
&\leq \max_{t\in[0,T]}\left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^t |K(t,s)||\phi(t,s,u(s))|ds(s)\right)ds\right. \\
&+\frac{|t(3-q)-2T|}{T(q+1)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^T |K(t,s)||\phi(t,s,u(s))|ds(s)\right)ds \\
&\left.+\frac{|(3-q)\Gamma(3-q/2)(T-2t)|}{(q+1)T^{3-q/2}}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\left(\int_{-\infty}^T |K(t,s)||\phi(t,s,u(s))|ds(s)\right)ds\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{t\in[0,T]}\left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^t |K(t,s)||\phi(t,s,u(s))-\phi(t,s,0)+\phi(t,s,0)|ds(s)\right)ds\right. \\
&+\frac{|t(3-q)-2T|}{T(q+1)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\left(\int_{-\infty}^T |K(t,s)||\phi(t,s,u(s))-\phi(t,s,0)+\phi(t,s,0)|ds(s)\right)ds \\
&+\frac{|(3-q)\Gamma(3-q/2)(T-2t)|}{(q+1)T^{3-q/2}}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}\left(\int_{-\infty}^T |K(t,s)|\mid\phi(t,s,u(s))-\phi(t,s,0)\right. \\
&+\phi(t,s,0)\mid ds(s))ds]
\end{aligned}$$

$$\begin{aligned}
\leq \frac{\delta(Lr+M)}{\lambda}\max_{t\in[0,T]}&\left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \frac{|t(3-q)-2T|}{T(q+1)}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}ds\right. \\
&\left.+\frac{|(3-q)\Gamma(3-q/2)(T-2t)|}{(q+1)T^{3-q/2}}\int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)}ds\right]
\end{aligned}$$

$$\leq \frac{\delta T^q(Lr+M)}{\lambda}\left[\frac{2((q+1)\Gamma(q+3/2)+(1-q)\Gamma(q+3/2)-(3-q)\Gamma(3-q/2)\Gamma(q+1))}{2(q+1)\Gamma(q+1)\Gamma(q+3/2)}\right] \leq r$$

for $u, v \in C[0,1]$ and for $t \in [0,T]$ we obtain :

$$\|(\Psi u)(t) - (\Psi v)(t)\|$$

$$\leq \max_{t\in[0,T]} \left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left(\int_{-\infty}^t \|K(t,s)\| \|\phi(t,s,u(s)) - \phi(t,s,v(s))\| ds(s) \right) ds \right.$$
$$+ \frac{|(t(3-q)-2T)|}{T(q+1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \left(\int_{-\infty}^T \|K(t,s)\| \|\phi(t,s,u(s)) - \phi(t,s,v(s))\| ds(s) \right) ds$$
$$+ \frac{|(3-q)\Gamma(3-q/2)(T-2t)|}{(q+1)T^{3-q/2}} \int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)} \int_{-\infty}^t \|K(t,s)\| \|\phi(t,s,u(s))$$
$$- \phi(t,s,v(s))\| ds(s) ds]$$

$$\leq \frac{\delta L \|u-v\|}{\lambda} \left[\frac{T^q}{\Gamma(q+1)} + \frac{(1-q)T^q}{(q+1)\Gamma(q+1)} + \frac{(3-q)\Gamma(3-q/2)T^q}{(q+1)\Gamma(q+3/2)} \right]$$

$$\leq \Lambda_{L_r,q,\delta,\lambda} \|u-v\|$$

where

$$\Lambda_{L,T,q,\delta,\lambda} = \frac{\delta L}{\lambda} \left[\frac{T^q}{\Gamma(q+1)} + \frac{(1-q)T^q}{(q+1)\Gamma(q+1)} + \frac{(3-q)\Gamma(3-q/2)T^q}{(q+1)\Gamma(q+3/2)} \right]$$

Which depends only on the parameters involved in the theorem. As $\Lambda_{L,T,q,\delta,\lambda} < 1, \Psi$ is contraction. Thus the conclusion of the theorem follows by contraction mapping principle (Banach fixed point theorem).

Theorem 3.4. *Let $\phi : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$, assume that there exist constant $0 \leq k < \frac{1}{\xi}$ where*

$$\xi = \frac{\delta T^q((q+1)\Gamma(q+3/2) + (1-q)\Gamma(q+3/2) + (3-q)\Gamma(3-q/2)\Gamma(q+1))}{\lambda(q+1)\Gamma(q+1)\Gamma(q+3/2)}$$

and $M > 0$ such that $|\phi(t,s,u(t))| \leq k|u| + M$ for all $t \in [0,T], u \in \mathbb{R}$ problem (1.1) has as least one solution .

Proof. Let us define a fixed point problem by :

$$u = \Psi u \tag{3.3}$$

Where Ψ is define by (3.1) then we just need to prove the existence of at least one solution $u \in [0,T]$ satisfying (3.3) . Define a suitable ball $B_R \subset C[0,T]$ with radius $R > 0$ as:

$$B_R = \left\{ u \in C[0,T] : \max_{t\in[0,T]} |u(t)| < R \right\}$$

Where R well be fixed later , then it's sufficient to show that $\Psi u : \overline{B_R} \to C[0,T]$ satisfies:

$$u \neq \lambda \Psi u, \forall u \in \partial B_R \; and \; \forall \lambda \in [0,T] \tag{3.4}$$

Let us define $H(\lambda, u) = \lambda \Psi u, u \in C(\mathbb{R}), \lambda \in [0,T]$,then by Arzesla'-Ascoli theorem $h_\lambda(u) = u - H(\lambda,u) = u - \lambda \Psi u$ is completely continuous if (3.4) is true then the following Leray-Schauder degree are well define and by the homotopy invariance of topological degree it follows that:

$\deg(h_\lambda, B_R, 0) = \deg(\mathrm{I} - \lambda\Psi \mathrm{u}, B_R, 0) = \deg(h_1, B_R, 0) = \deg(h_0, B_R, 0) = \deg(\mathrm{I}, B_R, 0) = 1 \neq 0, 0 \in B_R$.

Where I denotes the unit operator , by non zero property of the Leray -Schauder degree $h_1(u) =$

$u - \lambda\Psi u = 0$ for at least one $u \in B_R$ in order to prove (3.4) we assume that $u = \lambda\Psi u$ for some $\lambda \in [0, T]$ and for all $t \in [0, T]$ so that :

$$\begin{aligned}
|u(t)| &= |\lambda\Psi u(t)| \\
&\leq \Bigg[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left(\int_{-\infty}^t |K(t,s)||\phi(t,s,u(s))|ds(s)\right) ds \\
&+ \frac{|(t(3-q)-2T)|}{T(q+1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \left(\int_{-\infty}^T |K(t,s)||\phi(t,s,u(s))|ds(s)\right) ds \\
&+ \frac{|(3-q)\Gamma(3-q/2)(T-2t)|}{(q+1)T^{3-q/2}} \int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)} \int_{-\infty}^t |K(t,s)||\phi(t,s,u(s))|ds(s)ds \\
&\leq \frac{\delta(k|u|+M)}{\lambda} \Bigg(\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{|(t(3-q)-2T)|}{T(q+1)} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \\
&\qquad + \frac{|(3-q)\Gamma(3-q/2)(T-2t)|}{(q+1)T^{3-q/2}} \int_0^T \frac{(T-s)^{q-1/2}}{\Gamma(q+1/2)} ds\Bigg) \\
&\leq \frac{\delta(k|u|+M)}{\lambda} \left[\frac{T^q}{\Gamma(q+1)} + \frac{(1-q)T^q}{T(q+1)\Gamma(q+1)} + \frac{(3-q)\Gamma(3-q/2)T^q}{(q+1)\Gamma(q+3/2)}\right] \\
\leq (k|u|+M) &\left[\frac{\delta T^q((q+1)\Gamma(q+3/2) + (1-q)\Gamma(q+3/2) + (3-q)\Gamma(3-q/2)\Gamma(q+1))}{\lambda(q+1)\Gamma(q+1)\Gamma(q+3/2)}\right] \\
&\leq (k|u|+M)\xi
\end{aligned}$$

Which on taking norm ($\sup_{t\in[0,T]} |u| = \|u\|$) and solving for$\|u\|$ yields :

$$\|u\| \leq \frac{M\xi}{1-k\xi}$$

letting $R = \frac{M\xi}{1-k\xi} + 1$ (3.4) hold , this complete the proof .

Example 1: Consider the following anti-periodic fractional boundary value problem

$$D^{7/4}u(t) = \int_{-\infty}^t (t-5)(t+u(s))ds$$

$$u(0) = -u(1) \qquad D^{3/8}u(0) = -D^{3/8}u(1) \tag{3.5}$$

Where $q = 7/4,\ T = 1\ and\ L = 2\ as\ \|\phi(t,u) - \phi(t,v)\| \leq 2\|u - v\|$, further for $\delta = 3.25, \lambda = -1$ we have :

$$\frac{\lambda(q+1)\Gamma(q+1)\Gamma(q+3/2)}{2T^q\delta((q+1)\Gamma(q+3/2) + (1-q)\Gamma(q+3/2) - (3-q)\Gamma(3-q/2)\Gamma(q+1))} = 1.8923 < 2$$

Thus all the assumptions of theorem (3.3) are satisfied hence the fractional boundary value problem (3.5) has a unique solution on [0, 1].

Example 2: Consider the following anti-periodic fractional boundary value problem:

$$D^{3/2}u(t) = \int_{-\infty}^t e^{-2t}(0.75u(s) - cos(t))ds$$

$$u(0) = -u(2) \qquad D^{1/4}u(0) = -D^{1/4}u(2) \tag{3.6}$$

Where $q = 3/2, T = 2$, for $\delta = 1, \lambda = 2$ we have $e^{-2t} \leq \delta e^{-\lambda(t-s)}$, further :

$$\xi = \frac{\delta T^q((q+1)\Gamma(q+3/2) + (1-q)\Gamma(q+3/2) + (3-q)\Gamma(3-q/2)\Gamma(q+1))}{\lambda(q+1)\Gamma(q+1)\Gamma(q+3/2)} \leq 1.313$$

for $0 \leq k = 0.75 < 1/\xi$ and $M = cos(t)$ we have $|0.75u(t) - cos(t)| \leq 0.75|u| + M$, thus all the assumptions of theorem (3.4) are satisfied, hence the fractional boundary value problem (3.6) has at least one solution on [0,2] .

Conclusion: In this research paper, we have proven the existence and uniqueness of solution for the first kind of fractional Volterra equation with anti-periodic boundary value conditions by selecting $1 < q \leq 2$. The boundary value conditions have been chosen to be fractional as we have shown in (1.1) for which have never been used before in any article as far as we know. Existence of solutions have been shown by Leray–Schauder degree theory, and uniqueness solutions have been investigated by Banach's fixed-point theorem.

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Author information

Faraj Y. Ishak, Department of Statistics, University of Duhok, Iraq.
E-mail: `faraj.ishak@uod.ac`

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