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ABOUT SCHAUDER FRAMES AND BESSELIAN SCHAUDER FRAMES OF BANACH SPACES

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Abstract: In this paper, we introduce, for a separable Banach space, a new notion of besselian paires and of besselian Schauder frames for which we obtain some fundamental results some of which will allow us to get :

(i) a theorem providing necessary and sufficient conditions for the existence of a Schauder frame for a separable Banach space;

(ii) a theorem providing necessary conditions for the existence of a besselian Schauder frame for a weakly sequentially complete separable Banach space.

From the conditions in (ii) we deduce that the Banach space $L_1([0,1])$ doesn't have a besselian Schauder frame. Finally, we will show the existence of two universal Banach spaces in terms of which conditions in (i) and in (ii) can be expressed.

1 Introduction

In 1946, Gabor [15] performed a new method for signal reconstruction from elementary signals. In 1952, Duffin and Schaeffer [12] developed, in the field of nonharmonic series, a similar tool and introduced frame theory for Hilbert spaces. For more than thirty years, the results of Duffin and Schaeffer have not received from the mathematical community, the interest they deserve, until the publication of the work of Young [31] where the author studied frames in abstract Hilbert spaces. In 1986, the work of Daubechies, Grossmann, and Meyer gave frame theory the momentum it lacked and allowed it to be widely studied. This contributed, among other things, to the wider development of wavelet theory. The concept of atomic decompositions was introduced, in 1988, by Feichtinger and Gröchenig [14], in order to extend the definition of frames from the setting of Hilbert spaces to that of general separable Banach spaces. In 1991, Gröchenig [16], presented a generalization of the notions of atomic decomposition and of synthesis operator and introduced the definition of Banach frames. In 2001, Aldroubi, Sun and Tang [1] introduced the concepts of p-frames. In 2003, Christensen and Stoeva [7] extended the definition of p-frames, by replacing the sequence space l^p with a more general scalar sequence space X_d. By "getting rid" of the sequence spaces X_d in the definition of atomic decompositions, Casazza, Han, and Larson in 1999 [18] and in 2000, Han and Larson [19], generalized the notion of atomic decompositions by introducing the new notion of Schauder frames. One of the peculiarities of Schauder frames is that they constitute a natural extension of the concept of Schauder basis.

On the other hand, we derive from the definitions of frames and p-frames, a new classes of sequences, the class of Bessel sequences and the class of p-Bessel sequences respectively, for which there exists an extensive literature with various lines of research :

In 2006, Jia [17] studied Bessel sequences in Sobolev spaces. In 2008, Casazza and Leonhard [6] proved that any Bessel sequence in a infinite-dimensional space can be extended to a tight frame. This result was extended, in 2009, by Li and Sun [20], to finite-dimensional spaces. In the same year, Christensen, H.O.Kim and R.Y.Kim [8] proved that it is possible, in any separable Hilbert space, to extend each pair of Bessel sequences to a pair of mutually dual frames. In 2010, Rahimi and Balazs [28] investigated multipliers for p-Bessel sequences in Banach spaces. In 2015, Bakic and Beric [3] studied finite extensions of Bessel sequences in infinite dimensional Hilbert spaces. In 2015, Arias, Corach and Pacheco [2] studied the set of all Bessel sequences for a separable Hilbert as a Banach space. In 2015, Dehgan and Mesbah [11] endowed the set of all Bessel sequences for a separable Hilbert with a structure of C^* algebra.

In this paper, we introduce a new notion of besselian paires and of besselian Schauder frames of a Banach space.

Let us explain briefly how we "discovered" theses notions :

- The starting point of our reasoning was the following observation concerning Bessel sequences of separable Hilbert spaces : we can prove (see lemma 4.2. in section 4., with $p = 2$) that a sequence $(x_n)_{n\in\mathbb{N}^*}$ of a separable Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Bessel sequence if and only if there exists a constant $C > 0$ such that :

$$\sum_{n=1}^{+\infty} |x_n^*(x)|\,|y^*(x_n)| \leq C\,\|x\|_{\mathcal{H}}\,\|y^*\|_{\mathcal{H}^*},\ x\in\mathcal{H},\ y^*\in\mathcal{H}^* \tag{1.1}$$

  where $\mathcal{H}^*$ denotes the dual space of $\mathcal{H}$, and x_n^* is for every $n \in \mathbb{N}^*$ the element of $\mathcal{H}^*$ defined, for each $x \in \mathcal{H}$, by the relation $x_n^*(x) = \langle x, x_n\rangle$.

- The following steps in our approach was :

(i) To replace the separable Hilbert space by a general separable banach space X.

(ii) To "get rid" of the relation, in the assertion (1.1), between the vector x_n and the linear form x_n^* and to consider instead of a sequence of vectors $(x_n)_{n\in\mathbb{N}^*}$ a sequence $((x_n, y_n^*))_{n\in\mathbb{N}^*}$ of elements of $X \times X^*$ that we called a paire of X. Hence considering a paire $\mathcal{F} := ((x_n, y_n^*))_{n\in\mathbb{N}^*}$ of a Banach space X, we say that $\mathcal{F}$ is a besselian paire of X if the following condition holds :

$$\sum_{n=1}^{+\infty} |y_n^*(x)|\,|y^*(x_n)| \leq C\,\|x\|_{\mathcal{H}}\,\|y^*\|_{\mathcal{H}^*},\ x\in X,\ y^*\in X^* \tag{1.2}$$

where $C > 0$ is a constant.

(iii) Since Schauder frames on the Banach space X are also paires of X, we say that a Schauder frame $\mathcal{F}$ is a besselian Schauder frame of X if it is a besselian paire in the sens of definition 1.2.

The paper is structured as follows. After stating, in section 2, the main necessary definitions and notations, we devote section 3 to some examples of classical separable banach spaces (mostly scalar sequence spaces) with besselian Schauder frames. In section 4, we prove for besselian Schauder frame some fundamental results, some of which will allow us to obtain in section 5 :

(i) a theorem providing necessary and sufficient conditions for the existence of a Schauder frame for a separable Banach space;

(ii) a theorem providing necessary conditions for the existence of a besselian Schauder frame for a weakly sequentially complete separable Banach space. As a corollary of the second theorem, we show that the Banach space $L_1([0,1])$ don't have a besselian Schauder frame. Finally, we will show in section 5, the existence of two universal separable Banach spaces in terms of which conditions (i) and (ii) can be expressed.

Let us point out that we were inspired, in the proof of the theorems of section 5, by the work [27] of A. Pełczyński and that the proof of lemma 4.2. is a generalisation of the proof of a part of proposition 4 in [32, page 59].

2 Main definitions and notations

Let X be a sparable Banach space on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, X^* its dual, $\mathbb{I}$ the set $\mathbb{N}^*$or a set of the form $\{1, ..., r\}$ where $r \in \mathbb{N}^*$and $((x_n, y_n^*))_{n\in\mathbb{I}}$ a sequence of elements of $X \times X^*$.

(i) We make the convention that each sum indexed by an expression of the form $j \in \emptyset$, is equal to zero.

(ii) For each $q \in]1, +\infty[$, we set $q^* = \dfrac{q}{q-1}$.

(iii) For each $(j,k) \in \mathbb{N}^2$, we denote by δ_{jk} the so-called Kroneker symbole defined by

$$\delta_{jk} := \begin{cases} 1 \text{ if } j = k \\ 0 \text{ else} \end{cases}$$

(iv) We denote by $\mathcal{P}_{\mathbb{N}^*}$ (resp. $\mathcal{D}_{\mathbb{N}^*}$) the set of all the nonempty subsets (resp. all the nonempty finite subsets) of $\mathbb{N}^*$. For each $G \in \mathcal{P}_{\mathbb{N}^*}$ we denote by $\mathcal{P}_G$ the set of all the nonempty subsets of G.

(v) For each $A \in \mathcal{P}_{\mathbb{N}^*}$, we denote by $\min(A)$ the minimal element of A.We denote by $\mathfrak{S}_{\mathbb{N}^*}$ the set of all the permutations of $\mathbb{N}^*$.

(vi) For each mapping $F : G \to H$ between nonempty sets, and for each nonempty subset L of G, we denote by $F_{|L}$ the restriction of F to L.

(vii) We denote by 0_X the null vector of X.

(viii) Let $(x_n)_{n\in\mathbb{I}}$ be a sequence of elements of E. If $\mathbb{I}$ is of the form $\mathbb{I} = \{1, ..., r\}$ $(r \in \mathbb{N}^*)$ then the notation $\sum_{n\in\mathbb{I}} x_n$ will represent, of course, the sum $\sum\limits_{n=1}^{r} x_n$. If $\mathbb{I} = \mathbb{N}^*$ and the series $\sum x_n$ is convergent in X, then the notation $\sum_{n\in\mathbb{I}} x_n$ will represent, of course, the sum $\sum\limits_{n=1}^{+\infty} x_n$ of the series $\sum x_n$.

(ix) We denote by $\mathbb{B}_X$ the closed unit ball of X :

$$\mathbb{B}_X := \{x \in X : \|x\|_X \leq 1\}$$

(x) We denote by $L(X)$ the set of all bounded linear operators $f : X \to X$. It is well-known that $L(X)$ is a Banach space for the norm $\|\cdot\|_{L(X)}$ defined by the formula :

$$\|f\|_{L(X)} := \sup_{x\in\mathbb{B}_X} \|f(x)\|_X$$

(xi) Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence of elements of X. The series $\sum x_n$ is said to be weakly unconditionally convergent if for every functional $x^* \in X^*$ the scalar series $\sum |x^*(x_n)|$ is convergent.

(xii) The Banach space X is called weakly sequentially complete if for each sequence $(x_n)_{n\in\mathbb{N}^*}$ of X such that $\lim\limits_{n\to+\infty} y^*(x_n)$ exists for every $y^* \in X^*$, there exists $x \in X$ such that $\lim\limits_{n\to+\infty} y^*(x_n) = y^*(x)$ for every $y^* \in X^*$.

(xiii) Let $(\mathcal{H}, \langle .,. \rangle)$ be a Hilbert space on $\mathbb{K}$ and f an element of $\mathcal{H}$. We denote by T_f the continuous linear form $T_f : \mathcal{H} \longrightarrow \mathbb{K}$ defined by: $T_f(x) = \langle x, f \rangle$, $x \in \mathcal{H}$.

(xiv) Let $(\mathcal{H}, \langle .,. \rangle)$ be a Hilbert space and $(y_n)_{n\in\mathbb{N}^*}$ be a sequence of elements of $\mathcal{H}$. $(y_n)_{n\in\mathbb{N}^*}$ is called a Bessel sequence if there exists a constant $B > 0$ such that:

$$\sum_{n=1}^{+\infty} |\langle y, y_n \rangle|^2 \leq B \|y\|^2_{\mathcal{H}}, \quad y \in \mathcal{H}$$

(xv) Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence of elements of X and $q \in]1, +\infty[$. We say that the sequence $(x_n)_{n\in\mathbb{N}^*}$ is weakly q-summable in X if the series $\sum |f^*(x_n)|^q$ is convergent for each $f^* \in X^*$.

(xvi) Let $(f_n^*)_{n\in\mathbb{N}^*}$ be a sequence of elements of X^*. We say that the sequence $(f_n^*)_{n\in\mathbb{N}^*}$ is *-weakly q-summable in X^* if the series $\sum |f_n^*(x)|^q$ is convergent for each $x \in X$.

(xvii) The sequence $((x_n, y_n^*))_{n\in\mathbb{I}}$ is called a paire of X.

(xviii) A paire of X of the form $((x_n, y_n^*))_{n\in\{1,...,r\}}$ is called a Schauder frame of X if for all $x \in X$, the following formula holds :

$$\sum_{n=1}^{r} y_n^*(x)\, x_n = x$$

(xix) A paire of X of the form $((x_n, y_n^*))_{n\in\mathbb{N}^*}$ is called a Schauder frame (resp. unconditional Schauder frame) of X if for all $x \in X$, the serie $\sum y_n^*(x)\, x_n$ is convergent (resp. unconditionally convergent) in X to x.

(xx) The paire $((x_n, y_n^*))_{n\in\mathbb{I}}$ of X is said to be a besselian paire of X if there exists a constant $A > 0$ such that

$$\sum_{n\in\mathbb{I}} |y_n^*(x)|\, |y^*(x_n)| \leq A\, \|x\|_X\, \|y^*\|_{X^*}$$

for each $(x, y^*) \in X \times X^*$.

(xxi) The paire $((x_n, y_n^*))_{n\in\mathbb{I}}$ of X is said to be a besselian Schauder frame of X if it is both a Schauder frame and a besselian paire.

Remark 2.1. For a besselian paire $\mathcal{F} := ((x_n, y_n^*))_{n\in\mathbb{I}}$ of X, the quantity

$$\mathcal{L}_{\mathcal{F}} := \sup_{(u,v^*)\in\mathbb{B}_X\times\mathbb{B}_{X^*}} \left(\sum_{n\in\mathbb{I}} |y_n^*(u)|\, |v^*(x_n)|\right)$$

is finite and for each $(x, y) \in X \times X^*$, the following inequality holds

$$\sum_{n\in\mathbb{I}} |y_n^*(x)|\, |y^*(x_n)| \leq \mathcal{L}_{\mathcal{F}}\, \|x\|_X\, \|y^*\|_{X^*}$$

The quantity $\mathcal{L}_{\mathcal{F}}$ is then called the constant of the besselian paire $\mathcal{F}$.

For all the material on Banach spaces or Hilbertian frames, one can refer to [24], [21], [22], [32] and [9]. In the sequel $(E, \|\|_E)$ will represent a given separable Banach space, $((a_n, b_n^*))_{n\in\mathbb{N}^*}$ a paire of E and $p \in]1, +\infty[$ a constant. When $((a_n, b_n^*))_{n\in\mathbb{N}^*}$is a Schauder frame of E, we will always denote, for each $n \in \mathbb{N}^*$, by S_n the operator

$$\begin{array}{rccc} S_n: & E & \to & E \\ & x & \mapsto & \sum\limits_{j=1}^{n} b_j^*(x)\, a_j \end{array}$$

3 Examples of besselian Schauder frames

Example 3.1. Let E be a vector space over $\mathbb{K}$, and of finite dimension N. Let $(f_j)_{1\leq j\leq N}$ be a basis of E and $(f_j^*)_{1\leq j\leq N}$ its dual basis. Let $\|.\|$ be a norm on E. We denote by $\|.\|_1$ (resp. $\|.\|_1^*$) the norm defined on E (resp E^*), for each $(x_1, ..., x_N) \in \mathbb{K}^N$, by the formulas :

$$\left\|\sum_{j=1}^{N} x_j f_j\right\|_1 = \sum_{j=1}^{N} |x_j|\,, \left\|\sum_{j=1}^{N} x_j f_j^*\right\|_1^* = \sum_{j=1}^{N} |x_j|_N$$

For each $x = \sum\limits_{j=1}^{N} x_j f_j$ and $y^* = \sum\limits_{j=1}^{N} y_j f_j^*$, we have

$$x = \sum_{j=1}^{N} f_j^*(x) f_j$$

and

$$\begin{aligned}\sum_{j=1}^{N}\left|f_j^*(x)\right|\left|y^*(f_j)\right| &= \sum_{j=1}^{N}|x_j|\,|y_j| \\ &\leq \left(\sum_{j=1}^{N}|x_j|\right)\left(\sum_{j=1}^{N}|y_j|\right) \\ &\leq \|x\|_1\|y^*\|_1^*\end{aligned}$$

Since E and E^* are finite dimensional, there exists a constant $M > 0$ independent of x and y such that: $\|x\|_1 \leq M\|x\|_E$ and $\|y^*\|_1^* \leq M\|y^*\|_{E^*}$. It follows that:

$$\sum_{j=1}^{N}\left|f_j^*(x)\right|\left|y^*(f_j)\right| \leq M^2\|x\|_E\|y^*\|_{E^*}$$

Consequently, $\left((f_j, f_j^*)\right)_{1\leq j\leq N}$ is a besselian Schauder frame of E.

Example 3.2. Let us consider the well-known Banach space $l^1(\mathbb{K})$. It is well-known that the dual of $l^1(\mathbb{K})$ is isometrically isomorphic to the Banach space $l^\infty(\mathbb{K})$. More precisely, the following mapping :

$$\begin{array}{rccc}\varphi: & l^\infty(\mathbb{K}) & \to & \left(l^1(\mathbb{K})\right)^* \\ & (\alpha_n)_{n\in\mathbb{N}^*} & \mapsto & \varphi\left((\alpha_n)_{n\in\mathbb{N}^*}\right)\end{array}$$

defined for each $(x_n)_{n\in\mathbb{N}^*} \in l^1(\mathbb{K})$ by the formula

$$\varphi\left((\alpha_n)_{n\in\mathbb{N}^*}\right)\left((x_n)_{n\in\mathbb{N}^*}\right) = \sum_{n=1}^{+\infty}\alpha_n x_n$$

is an isometric isomorphism from $\left(l^\infty(\mathbb{K}), \|\cdot\|_{l^\infty(\mathbb{K})}\right)$ onto $\left(\left(l^1(\mathbb{K})\right)^*, \|\cdot\|_{(l^1(\mathbb{K}))^*}\right)$. Let

$$(e_n)_{n\in\mathbb{N}^*} := \left((\delta_{nm})_{m\in\mathbb{N}^*}\right)_{n\in\mathbb{N}^*}$$

be the canonical Schauder basis of $l^1(\mathbb{K})$. It is clear that $(e_n)_{n\in\mathbb{N}^*}$ is a also sequence of vectors of $l^\infty(\mathbb{K})$. We set, for each $n \in \mathbb{N}^*$:

$$e_n^* := \varphi(e_n)$$

For each $y^* \in \left(l^1(\mathbb{K})\right)^*$ and $x = (x_n)_{n\in\mathbb{N}^*} \in l^1(\mathbb{K})$, we have :

$$x = \sum_{n=1}^{+\infty} e_n^*(x)\, e_n$$

Furthermore we have :

$$\begin{aligned}\sum_{n=1}^{+\infty}|e_n^*(x)|\,|y^*(e_n)| &\leq \sum_{n=1}^{+\infty}|x_n|\,\|y^*\|_{(l^1(\mathbb{K}))^*}\,\|e_n\|_{l^1(\mathbb{K})} \\ &\leq \|x\|_{l^1(\mathbb{K})}\,\|y^*\|_{(l^1(\mathbb{K}))^*}\end{aligned}$$

It follows that $((e_n, e_n^*))_{n\in\mathbb{N}}$ is a besselian Schauder frame of $l^1(\mathbb{C})$.

Example 3.3. We consider now the Banach space $l^p(\mathbb{K})$. It is well-known that the dual of $l^p(\mathbb{K})$ is isometrically isomorphic to the Banach space $l^{p^*}(\mathbb{K})$. More precisely, the following mapping :

$$\begin{array}{rccc}\psi: & l^{p^*}(\mathbb{K}) & \to & \left(l^p(\mathbb{K})\right)^* \\ & (\alpha_n)_{n\in\mathbb{N}^*} & \mapsto & \psi\left((\alpha_n)_{n\in\mathbb{N}^*}\right)\end{array}$$

defined for each $(x_n)_{n\in\mathbb{N}^*} \in l^p(\mathbb{K})$ by the formula

$$\psi\left((\alpha_n)_{n\in\mathbb{N}^*}\right)\left((x_n)_{n\in\mathbb{N}^*}\right) = \sum_{n=1}^{+\infty}\alpha_n x_n$$

is a well-defined isometric isomorphism from $\left(l^{p^*}(\mathbb{K}), \|\cdot\|_{l^{p^*}(\mathbb{K})}\right)$ onto $\left((l^p(\mathbb{K}))^*, \|\cdot\|_{(l^p(\mathbb{K}))^*}\right)$. The sequence

$$(e_n)_{n\in\mathbb{N}^*} := \left((\delta_{nm})_{m\in\mathbb{N}^*}\right)_{n\in\mathbb{N}^*}$$

is the canonical Schauder basis of $l^p(\mathbb{K})$. It is clear that $(e_n)_{n\in\mathbb{N}^*}$ is a also the canonical Schauder basis of $l^{p^*}(\mathbb{K})$. We set, for each $n \in \mathbb{N}^*$:

$$u_n^* := \psi(e_n)$$

Then it is clear that we have for each $x \in l^p(\mathbb{K})$, $y^* \in (l^p(\mathbb{K}))^*$:

$$\begin{cases} \|x\|_{l^p(\mathbb{K})} = \left(\sum\limits_{n=1}^{+\infty} |u_n^*(x)|^p\right)^{\frac{1}{p}} \\ \|y^*\|_{(l^p(\mathbb{K}))^*} = \left(\sum\limits_{n=1}^{+\infty} |y^*(e_n)|^{p^*}\right)^{\frac{1}{p^*}} \end{cases}$$

It follows then that :

$$\begin{aligned} \sum_{n=1}^{+\infty} |u_n^*(x)|\,|y^*(e_n)| &\leq \left(\sum_{n=1}^{+\infty} |u_n^*(x)|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{+\infty} |y^*(e_n)|^{p^*}\right)^{\frac{1}{p^*}} \\ &\leq \|x\|_{l^p(\mathbb{K})}\,\|y^*\|_{l^{p^*}(\mathbb{K})} \end{aligned}$$

Consequently, $((e_n, u_n^*))_{n\in\mathbb{N}}$ is a besselian Schauder frame of $l^p(\mathbb{K})$.

Example 3.4. We consider now the Banach space $c_0(\mathbb{K})$. It is well-known that the dual of $c_0(\mathbb{K})$ is isometrically isomorphic to the Banach space $l^1(\mathbb{K})$. More precisely, the following mapping :

$$\begin{array}{rccc} \chi : & l^1(\mathbb{K}) & \to & (c_0(\mathbb{K}))^* \\ & (\alpha_n)_{n\in\mathbb{N}^*} & \mapsto & \chi\left((\alpha_n)_{n\in\mathbb{N}^*}\right) \end{array}$$

defined for each $(x_n)_{n\in\mathbb{N}^*} \in l^p(\mathbb{K})$ by the formula

$$\chi\left((\alpha_n)_{n\in\mathbb{N}^*}\right)\left((x_n)_{n\in\mathbb{N}^*}\right) = \sum_{n=1}^{+\infty}\alpha_n x_n$$

is a well-defined isometric isomorphism from $\left(l^1(\mathbb{K}), \|\cdot\|_{l^1(\mathbb{K})}\right)$ onto $\left((c_0(\mathbb{K}))^*, \|\cdot\|_{(c_0(\mathbb{K}))^*}\right)$. The sequence

$$(e_n)_{n\in\mathbb{N}^*} := \left((\delta_{nm})_{m\in\mathbb{N}^*}\right)_{n\in\mathbb{N}^*}$$

is the canonical Schauder basis of $c_0(\mathbb{K})$ and also the canonical Schauder basis of $l^1(\mathbb{K})$. We set, for each $n \in \mathbb{N}^*$:

$$v_n^* := \chi(e_n)$$

Then it is clear that we have for each $x \in c_0(\mathbb{K})$, $y^* \in (c_0(\mathbb{K}))^*$:

$$\begin{cases} \|x\|_{c_0(\mathbb{K})} = \sup\limits_{n\in\mathbb{N}^*} |v_n^*(x)| \\ \|y^*\|_{(c_0(\mathbb{K}))^*} = \sum\limits_{n=1}^{+\infty} |y^*(e_n)| \end{cases}$$

It follows then that :

$$\sum_{n=1}^{+\infty} |v_n^*(x)|\,|y^*(e_n)| \leq \|x\|_{c_0(\mathbb{K})}\,\|y^*\|_{(c_0(\mathbb{K}))^*}$$

Consequently $((e_n, v_n^*))_{n\in\mathbb{N}}$ is a besselian Schauder frame of $c_0(\mathbb{K})$.

Example 3.5. Let $(\mathcal{H}, \langle .,. \rangle)$ be a separable real or complex Hilbert space and $\mathcal{G} := (x_n)_{n\in\mathbb{N}^*}$ a Hilbert frame of $\mathcal{H}$. That is $(x_n)_{n\in\mathbb{N}^*}$ is a sequence of vectors of $\mathcal{H}$ for which there exist two constants $\alpha, \beta > 0$ such that the following condition holds for every $x \in \mathcal{H}$:

$$\alpha \|x\|_{\mathcal{H}}^2 \leq \sum_{n=1}^{+\infty} |\langle x_n, x\rangle|^2 \leq \beta \|x\|_{\mathcal{H}}^2$$

Then the operator

$$\begin{array}{rccc} S_{\mathcal{G}}: & \mathcal{H} & \to & \mathcal{H} \\ & x & \mapsto & \sum\limits_{n=1}^{+\infty} \langle x_n, x\rangle x_n \end{array}$$

is a well defined isomorphism of Banach spaces invertible which is an autoadjoint operator of $\mathcal{H}$ [9, pages 122-124)]. It follows that we have for each $x \in \mathcal{H}$:

$$x = \sum_{n=1}^{+\infty} \left\langle S_{\mathcal{G}}^{-1}(x_n), x \right\rangle x_n$$

On the other hand, by virtue of the Cauchy-Schwarz inequality, we have for each $(x, y) \in \mathcal{H} \times \mathcal{H}$:

$$\begin{aligned} \sum_{n=1}^{+\infty} \left| T_{S_{\mathcal{G}}^{-1}(a_n)}(x) \right| |T_y(a_n)| &= \sum_{n=1}^{+\infty} \left| \left\langle S_{\mathcal{G}}^{-1}(a_n), x \right\rangle \right| |\langle a_n, y\rangle| \\ &\leq \left(\sum_{n=1}^{+\infty} \left| \left\langle S_{\mathcal{G}}^{-1}(a_n), x \right\rangle \right|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{+\infty} |\langle a_n, y\rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=1}^{+\infty} \left| \left\langle a_n, S_{\mathcal{G}}^{-1}(x) \right\rangle \right|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{+\infty} |\langle a_n, y\rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \alpha \left\| S_{\mathcal{G}}^{-1}(x) \right\|_{\mathcal{H}} \|y\|_{\mathcal{H}} \\ &\leq \alpha \left\| S_{\mathcal{G}}^{-1} \right\|_{L(\mathcal{H})} \|x\|_{\mathcal{H}} \|T_y\|_{\mathcal{H}^*} \end{aligned}$$

Consequently $\left(\left(x_n, T_{S_{\mathcal{G}}^{-1}(x_n)}\right)\right)_{n\in\mathbb{N}^*}$ is a besselian Schauder frame of $\mathcal{H}$.

4 Fundamental results

Proposition 4.1. *Let $(f_n)_{n\in\mathbb{N}^*}$ and $(g_n)_{n\in\mathbb{N}^*}$ be a Bessel sequences of a Hilbert space $(\mathcal{H}, \langle .,. \rangle)$. Then the paire $((f_n, T_{g_n}))_{n\in\mathbb{N}^*}$ is a besselian paire of $\mathcal{H}$.*

Proof. We have for each $x \in \mathcal{H}$ and $y \in \mathcal{H}$:

$$\begin{aligned} \sum_{n=1}^{+\infty} |T_{g_n}(x)| |T_y(f_n)| &= \sum_{n=1}^{+\infty} |\langle x, g_n\rangle \langle f_n, y\rangle| \\ &\leq \sqrt{\sum_{n=1}^{+\infty} |\langle x, g_n\rangle|^2} \sqrt{\sum_{n=1}^{+\infty} |\langle f_n, y\rangle|^2} \end{aligned}$$

But there exists a constant $B > 0$ such that:

$$\begin{cases} \sum\limits_{n=1}^{+\infty} |\langle x, g_n\rangle|^2 \leq B \|x\|_{\mathcal{H}}^2, & x \in \mathcal{H} \\ \sum\limits_{n=1}^{+\infty} |\langle f_n, y\rangle|^2 \leq B \|y\|_{\mathcal{H}}^2, & y \in \mathcal{H} \end{cases}$$

Consequently, we have for each $x \in \mathcal{H}$ and $y \in \mathcal{H}$:

$$\sum_{n=1}^{+\infty} |T_{g_n}(x)|\,|T_y(f_n)| \leq B\,\|x\|_{\mathcal{H}}\,\|y\|_{\mathcal{H}}$$

Then $((f_n, T_{g_n}))_{n\in\mathbb{N}^*}$ is a besselian paire of $\mathcal{H}$.
We can generalize the last proposition to the Banach space E by means of the notions of weakly p-summable sequences of E and *-weakly p-summable sequences of E^*. We need first to prove a useful lemma.

Lemma 4.2. *Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence of elements of E and $(f_n^*)_{n\in\mathbb{N}^*}$ be a sequence of elements of E^*.*

1. The sequence $(x_n)_{n\in\mathbb{N}^*}$ is weakly p-summable in E if and only if there exists a constant $C > 0$ such that :

$$\left(\sum_{n=1}^{+\infty} |f^*(x_n)|^p\right)^{\frac{1}{p}} \leq C\,\|f^*\|_{E^*},\ f^* \in E^*$$

*2. The sequence $(f_n^*)_{n\in\mathbb{N}^*}$ is *-weakly p-summable in E^* if and only if there exists a constant $C > 0$ such that :*

$$\left(\sum_{n=1}^{+\infty} |f_n^*(x)|^p\right)^{\frac{1}{p}} \leq C\,\|x\|_E,\ x \in E$$

Proof. (1) The sufficiency, in the first equivalence is obvious. We prove the necessity by means of the closed graph theorem applied to the operator

$$\begin{array}{rccc} U: & E^* & \to & l^p(\mathbb{K}) \\ & f^* & \mapsto & (f^*(x_n))_{n\in\mathbb{N}^*} \end{array}$$

Indeed U is well-defined on E^* and is linear. Then let $(f_n^*)_{n\in\mathbb{N}^*}$ be a sequence of elements of E^* such that $(f_n^*)_{n\in\mathbb{N}^*}$ is convergent in E^* to y^* and $(U(f_n^*))_{n\in\mathbb{N}^*}$ is convergent in $l^p(\mathbb{K})$ to $(\lambda_j)_{j\in\mathbb{N}^*}$. It follows that:

$$\lim_{n\to+\infty} |f_n^*(x_j) - \lambda_j| = 0,\ j \in \mathbb{N}^*$$

So $\lim_{n\to+\infty} f_n^*(x_j) = \lambda_j$ for each $j \in \mathbb{N}^*$, but we have also:

$$\lim_{n\to+\infty} \|f_n^* - y^*\|_{E^*} = 0$$

It follows that

$$y^*(x_j) = \lambda_j,\ j \in \mathbb{N}^*$$

Consequently, $(\lambda_n)_{n\in\mathbb{N}^*} \in l^p(\mathbb{K})$ and $(U(f_n^*))_{n\in\mathbb{N}^*}$ is convergent in E^* to $(y^*(x_j))_{j\in\mathbb{N}^*} = U(y^*)$. Hence the graph of U is closed. It follows that the operator U is continuous. That is, there exists a constant $C > 0$ such that:

$$\|U(f^*)\|_{l^p} \leq C\,\|f^*\|_{E^*},\quad f^* \in E^*$$

Consequently,

$$\left(\sum_{n=1}^{+\infty} |f^*(x_n)|^p\right)^{1/p} \leq C\,\|f^*\|_{E^*},\quad f^* \in E^*$$

Hence we achieve the proof of the first equivalence of the proposition.
(2). The sufficiency in the second equivalence is also obvious. In a similar way to the proof of the first equivalence we prove the necessity by means of the closed graph theorem applied to the operator

$$\begin{array}{rccc} V: & E & \to & l^p(\mathbb{K}) \\ & x & \mapsto & (f_n^*(x))_{n\in\mathbb{N}^*} \end{array}$$

Proposition 4.3. *Let $(x_n)_{n\in\mathbb{N}^*}$ be sequence of elements of E which is weakly p-summable in E and $(y_n^*)_{n\in\mathbb{N}^*}$ be sequence of elements of E^* which is *-weakly p^*-summable in E^*. Then $((x_n, y_n^*))_{n\in\mathbb{N}^*}$ is a besselian paire of E.*

Proof. Let $x \in E$ and $y^* \in E^*$. We have then by virtue of Hölder's inequality:

$$\sum_{n=1}^{+\infty} |y_n^*(x)|\,|y^*(x_n)| \leq \left(\sum_{n=1}^{+\infty} |y_n^*(x)|^{p^*}\right)^{1/p^*} \left(\sum_{n=1}^{+\infty} |y^*(x_n)|^{p}\right)^{1/p}$$

Since $(x_n)_{n\in\mathbb{N}^*}$ is weakly p-summable and $(y_n^*)_{n\in\mathbb{N}^*}$ is $*$-weakly p^*-summable, there exists a constant $C > 0$ such that the following inequalities hold for each $x \in E$, $y^* \in E^*$:

$$\begin{cases} \left(\sum_{n=1}^{+\infty} |y^*(x_n)|^p\right)^{1/p} \leq C\,\|y^*\|_{E^*} \\ \left(\sum_{n=1}^{+\infty} |y_n^*(x)|^{p^*}\right)^{1/p^*} \leq C\,\|x\|_E \end{cases}$$

It follows that:

$$\sum_{n=1}^{+\infty} |y_n^*(x)|\,|y^*(x_n)| \leq C^2\,\|x\|_E\,\|y^*\|_{E^*}$$

for each $x \in E$ and $y^* \in E^*$.
Consequently, $((x_n, y_n^*))_{n\in\mathbb{N}^*}$ is a besselian paire of E.

Proposition 4.4. *Assume that E has a Schauder frame.Then E has the approximation property.*

Proof. For each $n \in \mathbb{N}^*$ the operator S_n is a finite rank operator on E and we have for every $x \in E$

$$\|S_n(x) - x\|_E = \left\|\sum_{j=n+1}^{+\infty} b_j^*(x)\,a_j\right\|_E$$

So the following relation holds for each $x \in E$:

$$\lim_{n\to+\infty} \|S_n(x) - x\|_E = 0$$

It follows from [29, Proposition 4.3, page 73] that E has the approximation property. □

Proposition 4.5. *Let E be a Banach space and $((a_n, b_n^*))_{n\in\mathbb{N}^*}$ be a Schauder frame of E. If the following condition holds*

$$\sum_{n=1}^{+\infty} \|a_n\|_E\,\|b_n\|_{E^*} < +\infty \tag{4.1}$$

then E will be finite dimensional.

Proof. For each $n \in \mathbb{N}^*$, S_n is a finite rank linear operator. On the other hand we have for every $n \in \mathbb{N}^*$:

$$\|S_n - Id_E\|_{L(E)} \leq \sum_{j=n+1}^{+\infty} \|a_j\|_E\,\|b_j\|_{E^*}$$

It follows, according to the assumption (4.1) of the proposition, that the identity mapping Id_E belongs to the topological closure in $L(E)$ of the set of finite rank operators. Hence Id_E is a compact operator. Consequently the Banach space E is finite dimensional. □

Remark 4.6. Using the concept of nuclear operators between Banach spaces, we can give another version of the proof of the proposition 4.5

Proof. Under the assumption (4.1) of the proposition 4.5, the tensor product $\sum_{n=1}^{+\infty} b_n^* \otimes_\pi a_n$ is well-defined, belongs to $E^* \widehat{\otimes}_\pi E$ and fullfiles the relation :

$$Id_E = J\left(\sum_{n=1}^{+\infty} b_n^* \otimes_\pi a_n\right)$$

where J is the operator $J : E^* \widehat{\otimes}_\pi E \to L(E)$ that associates with the tensor $u := \sum_{n=1}^{+\infty} y_n^* \otimes_\pi x_n \in E^* \widehat{\otimes}_\pi E$ the linear operator F_u defined by

$$F_u(x) := \sum_{n=1}^{+\infty} y_n^*(x)\, x_n,\ x \in E$$

It follows that Id_E is a nuclear operator of $L(E)$. But, we know that all the nuclear operator of of $L(E)$ are compact [29, Chapter 2, page 42]. It follows that Id_E is compact. Consequently E is finite dimensional. □

Proposition 4.7. *We assume that E is a weakly sequentially complete Banach space and that $\mathcal{F} := ((a_n, b_n^*))_{n\in\mathbb{N}^*}$ is a besselian paire of E. Then for each $x \in E$, the series $\sum b_n^*(x)\, a_n$ is unconditionally convergent in E.*

Proof. For each $x \in E$, $y^* \in E^*$ we have:

$$\begin{aligned}\sum_{n=1}^{+\infty} |y^*(b_n^*(x)\, a_n)| &= \sum_{k=1}^{+\infty} |b_n^*(x)\, y^*(a_n)| \\ &\leq \mathcal{L}_{\mathfrak{F}} \|x\|_E \|y^*\|_{E^*} \\ &< +\infty\end{aligned}$$

Hence the series $\sum b_n^*(x)\, a_n$ is weakly unconditionally convergent. Then, since E is weakly sequentially complete, the well-known Orlicz's theorem (1929) [32, Proposition.4 ,page 59 and page 66], entails that the series $\sum b_n^*(x)\, a_n$ is unconditionally convergent. □

Proposition 4.8. *Let $E_1, ..., E_N$ be a closed subspaces of E such that*

$$E = E_1 \oplus ... \oplus E_N \tag{4.2}$$

1. Assume that E has a Schauder frame (resp. a besselian Schauder frame) then the Banach space E_j (as a closed subspace of E) has, for each $j \in \{1, ..., N\}$, a Schauder frame (resp. a besselian Schauder frame).

2. Assume that the Banach space E_j (as a closed subspace of E) has, for each $j \in \{1, ..., N\}$, a Schauder frame (resp. a besselian Schauder frame). Then the space E has a Schauder frame (resp. a besselian Schauder frame).

Proof. 1. a. It follows from (4.2), that there exists a continuous projections $\varrho_j : E \to E$, $j \in \{1, ..., N\}$ such that ϱ_j is a projection onto E_j for each $j \in \{1, ..., N\}$ and $\sum_{j=1}^{N} \varrho_j = Id_E$. Let $\mathcal{F} := ((a_n, b_n^*))_{n\in\mathbb{I}}$ be a Schauder frame of E. Then we have for each $x \in E$:

$$x = \sum_{n\in\mathbb{I}} b_n^*(x) a_n$$

If $x \in E_j$ for some $j \in \{1, ..., N\}$, then since ϱ_j is continuous, we have :

$$\begin{aligned}x &= \varrho_j(x) \\ &= \sum_{n\in\mathbb{I}} b_n^*(x) \varrho_j(a_n)\end{aligned}$$

But we have for each $j \in \{1, ..., N\}$ and $n \in \mathbb{N}^*$:

$$\varrho_j(a_n) \in E_j,\ b^*_{n|E_j} \in E^*_j$$

It follows that the paire $\left(\left(\varrho_j(a_n), b^*_{n|E_j}\right)\right)_{n\in\mathbb{N}^*}$ is a Schauder frame of E_j, for each $j \in \{1, ..., N\}$.

1. b. We assume now that $((a_n, b^*_n))_{n\in\mathbb{I}}$ is a besselian Schauder frame of E. Then we have for each $x \in E$ and $y^* \in E^*$:

$$\sum_{n\in\mathbb{I}} |b^*_n(x)| |y^*(a_n)| \le \mathcal{L}_{\mathcal{F}} \|x\|_E \|y^*\|_{E^*}$$

When $x \in E_j$ and $y^* \in E^*_j$ for some $j \in \{1, ..., N\}$, then there exists, thanks to Hahn-Banach theorem a linear form $Y^* \in E^*$ such that :

$$\begin{cases} Y^*_{|E_j} = y^* \\ \|Y^*\|_{E^*} = \|y^*\|_{E^*_j} \end{cases}$$

It follows that we have for each $x \in E_j$ and $y^* \in E^*_j$:

$$\begin{aligned}\sum_{n\in\mathbb{I}} \left|\left(b^*_{n|E_j}\right)(x)\right| |y^*(\varrho_j(a_n))| &= \sum_{n\in\mathbb{I}} |(b^*_n)(x)| |Y^*(\varrho_j(a_n))| \\ &= \sum_{n\in\mathbb{I}} |(b^*_n)(x)| |(Y^* \circ \varrho_j)(a_n)| \\ &\le \mathcal{L}_{\mathcal{F}} \|x\|_E \|Y^* \circ \varrho_j\|_{E^*} \\ &\le \mathcal{L}_{\mathcal{F}} \|\varrho_j\|_{L(E)} \|x\|_E \|Y^*\|_{E^*} \\ &\le \mathcal{L}_{\mathcal{F}} \|\varrho_j\|_{L(E)} \|x\|_{E_j} \|y^*\|_{E^*_j}\end{aligned}$$

But the paire $\mathcal{F}_j := \left(\left(\varrho_j(a_n), b^*_{n|E_j}\right)\right)_{n\in\mathbb{N}^*}$ is already Schauder frame of E_j, for each $j \in \{1, ..., N\}$. Consequently, $\mathcal{F}_j$ is a besselian Schauder frame of E_j, for each $j \in \{1, ..., N\}$.

2. a. Assume that, for each $j \in \{1, ..., N\}$, the subspace E_j has a Schauder frame $\mathcal{F}_j := \left((a_{n,j}, b^*_{n,j})\right)_{n\in\mathbb{I}_j}$ where $\mathbb{I}_j$ is a set of the form $\mathbb{I}_j = \{1, ..., r_j\}$ $(r_j \in \mathbb{N}^*)$ or $\mathbb{I}_j = \mathbb{N}^*$. Let us set then for each $j \in \{1, ..., N\}$

$$\mathcal{R}_j := \left((x_{n,j}, y^*_{n,j})\right)_{n\in\mathbb{N}^*}$$

where :

$$\left(x_{n,j}, y^*_{n,j}\right) = \begin{cases} \left(a_{n,j}, b^*_{n,j}\right) \text{ if } n \in \mathbb{I}_j \\ (0_E, 0_{E^*}) \text{ if } n \in \mathbb{N}^* \backslash \mathbb{I}_j \end{cases}$$

Then $\mathcal{R}_j$ is, for each $j \in \{1, ..., N\}$, is a Schauder frame of E_j.

On the other hand we have for each $x \in E$ and $j \in \{1, ..., N\}$:

$$\varrho_j(x) = \sum_{n=1}^{+\infty} b^*_{n,j}(\varrho_j(x))\, a_{n,j}$$

It follows that we have for each $x \in E$:

$$\begin{aligned} x &= \sum_{j=1}^{N} \varrho_j(x) \\ &= \sum_{j=1}^{N}\sum_{n=1}^{+\infty} y^*_{n,j}(\varrho_j(x))\, x_{n,j}\end{aligned}$$

Let us now consider the paire $((x_n, y^*_n))_{n\in\mathbb{N}^*}$ defined by :

$$\begin{cases} x_{kN+l} = x_{k+1,l} \\ y^*_{kN+l} = y^*_{k+1,l} \circ \varrho_l \end{cases}$$

for $k \in \mathbb{N}$ and $l \in \{1, ..., N\}$. It is clear that the series $\sum y_n^* (x) x_n$ is convergent for each $x \in E$ to the vector x. Hence the paire $((x_n, y_n^*))_{n\in\mathbb{N}^*}$ is a Schauder frame of E.
2. b. Assume now that, for each $j \in \{1, ..., N\}$, the paire $\mathcal{F}_j$ is a besselian Schauder frame of the subspace E_j. Then $\mathcal{R}_j$ is, for each $j \in \{1, ..., N\}$, a besselian Schauder frame of the subspace E_j. Hence we have for each $x \in E$, $y^* \in E^*$ and $j \in \{1, ..., N\}$:

$$\begin{aligned}\sum_{n=1}^{+\infty} \left|b_{n,j}^* \left(\varrho_j \left(x\right)\right)\right| \left|y^* \left(a_{n,j}\right)\right| \leq & \mathcal{L}_{\mathcal{F}_j} \left\|\varrho_j \left(x\right)\right\|_{E_j} \left\|y_{|E_j}^*\right\|_{E_j^*} \\ \leq & \mathcal{L}_{\mathcal{F}_j} \left\|\varrho_j\right\|_{L(E)} \left\|x\right\|_E \left\|y^*\right\|_{E^*}\end{aligned}$$

It follows that we have for each $x \in E$, $y^* \in E^*$:

$$\begin{aligned}\sum_{n=1}^{+\infty} \left|y_n^* \left(x\right)\right| \left|y^* \left(x_n\right)\right| =& \sum_{j=1}^{N} \sum_{n=1}^{+\infty} \left|b_{n,j}^* \left(\varrho_j \left(x\right)\right)\right| \left|y^* \left(a_{n,j}\right)\right| \\ \leq & \left(\sum_{j=1}^{N} \mathcal{L}_{\mathcal{F}_j} \left\|\varrho_j\right\|_{L(E)}\right) \left\|x\right\|_E \left\|y^*\right\|_{E^*}\end{aligned}$$

It follow that $((x_n, y_n^*))_{n\in\mathbb{N}^*}$ is a besselian Schauder frame of E.

Corollary 4.9. *Let E be a Banach space with a Schauder frame (resp. a besselian Schauder frame) and a subspace F of E which is complemented in E. Then F has a Schauder frame (resp. a besselian Schauder frame) .*

Proof. The corollary is a direct consequence of the previous proposition (with $N = 2$). □

Corollary 4.10. *Let $E_1, ..., E_N$ be a Banach spaces. Then the Banach space $E_1 \times ... \times E_N$ (endowed with the norm $\|(x_1, ..., x_N)\|_{E_1\times...\times E_N} := \sum_{j=1}^{N} \|x_j\|_{E_j}$) has a Schauder frame (resp. a besselian Schauder frame) if and only if E_j has, for each $j \in \{1, ..., N\}$ a Schauder frame (resp. a besselian Schauder frame).*

Proof. We set for each $j \in \{1, ..., N\}$.

$$\tilde{E}_j = \left\{ \begin{array}{l} (x_1, ..., x_N) \in E_1 \times ... \times E_N : \\ x_k = 0_E \text{ if } k \neq j \text{ and } x_j \in E_j \end{array} \right\}$$

1. For each $j \in \{1, ..., N\}$, $\tilde{E}_j$ has a Schauder frame (resp. a besselian Schauder frame) if and only if E_j has a Schauder frame (resp. a besselian Schauder frame);
2. Since $E_1 \times ... \times E_N = \tilde{E}_1 \oplus ... \oplus \tilde{E}_N$, it follows from the proposition above that $E_1 \times ... \times E_N$ has a Schauder frame (resp. a besselian Schauder frame) if and only if $\tilde{E}_j$ has for every $j \in \{1, ..., N\}$ a Schauder frame (resp. a besselian Schauder frame) if and only if $\tilde{E}_j$ has for every $j \in \{1, ..., N\}$ a Schauder frame (resp. a besselian Schauder frame). □

5 Conditions for the existence of Schauder frames and besselian Schauder frames.

Theorem 5.1. *1. If E has a Schauder frame then there exists a Banach space Z such that Z has a Schauder basis and E is isometrically isomorphic to a complemented subspace of Z.*

2. If E is isomorphic to a complemented subspace of a Banach space Z which has a Schauder basis then E has a Schauder frame.

Proof. 1. Let $\mathcal{F} := ((a_n, b_n^*))_{n\in\mathbb{I}}$ be a Schauder frame of E.

If the set $\{n \in \mathbb{I} : a_n \neq 0_E \text{ and } b_n^* \neq 0_{E^*}\}$ is finite then E is finite dimensional. In this case E has an algebraic basis which is also a Schauder basis. Hence, in this case, the assertion of the direct part of the theorem is true for $Z = E$.

Now we can assume without loss of the generality that $\mathbb{I} = \mathbb{N}^*$and

$$a_n \neq 0_E,\ b_n^* \neq 0_{E^*},\ n \in \mathbb{N}^*$$

Let Z_E be the set of the sequences $z := (b_n^*(x_n)a_n)_{n\in\mathbb{N}^*}$ such that $x_n \in E$ for each $n \in \mathbb{N}^*$ and the series $\sum b_n^*(x_n)a_n$ is convergent. It is clear that Z_E is a vector space over $\mathbb{K}$, when endowed with the usual operations of addition of sequences and multiplication of sequences by scalars. We can easily show that the mapping $\|\cdot\|_{Z_E} : Z_E \to \mathbb{R}$ defined for each $z := (b_n^*(x_n)a_n)_{n\in\mathbb{N}^*} \in Z_E$ by :

$$\|z\|_{Z_E} := \sup_{n\in\mathbb{N}^*} \left\| \sum_{j=1}^{n} b_j^*(x_j)\, a_j \right\|_E$$

is well-defined and is a norm on the vector space Z_E.

Claim 1. $\left(Z_E, \|\cdot\|_{Z_E}\right)$ *is a Banach space.*

Proof of the claim :

Let $(z_m)_{m\in\mathbb{N}^*} := \left((b_n^*(x_{n,m})a_n)_{n\in\mathbb{N}^*}\right)_{m\in\mathbb{N}^*}$ be a Cauchy sequence in $\left(Z_E, \|\cdot\|_{Z_E}\right)$. Hence there exists, for each $\varepsilon > 0$, an integer $N_\varepsilon \in \mathbb{N}^*$ such that :

$$\|z_m - z_s\|_{Z_E} \leq \frac{\varepsilon}{4},\ \ s \geq m \geq N_\epsilon$$

which means that the following inequality holds for each $j, n, s, m \in \mathbb{N}^*$ such that $s \geq m \geq N_\varepsilon$:

$$\left\| \sum_{j=1}^{n} b_j^*(x_{j,m} - x_{j,s})\, a_j \right\|_{Z_E} \leq \frac{\varepsilon}{4} \tag{5.1}$$

Hence the sequence $\left(b_j^*(x_{j,m})\right)_{m\in\mathbb{N}^*}$ are, for each $j \in \mathbb{N}^*$, a Cauchy sequences in E. Hence the sequences $\left(b_j^*(x_{j,m})\right)_{m\in\mathbb{N}^*}$ are convergent in E. We set then :

$$l_j := \lim_{m\to+\infty} b_j^*(x_{j,m}),\ j \in \mathbb{N}^*$$

Since $b_j^* \neq 0_{E^*}$ for each $j \in \mathbb{N}^*$, it follows that there exists for each $j \in \mathbb{N}^*$, $x_j \in E$ such that

$$b_j^*(x_j) = l_j,\ \ j \in \mathbb{N}^*$$

Tending s to infinity in (5.1), we obtain for each $j, n, m \in \mathbb{N}^*$ such that $m \geq N_\varepsilon$:

$$\sup_{n\in\mathbb{N}^*} \left\| \sum_{j=1}^{n} b_j^*(x_{j,m})\, a_j - \sum_{j=1}^{n} b_j^*(x_j)\, a_j \right\|_E \leq \frac{\varepsilon}{4}$$

It follows that we have for each integers $n_2 \geq n_1$:

$$\left\| \sum_{j=n_1}^{n_2} b_j^*(x_{j,N_\varepsilon})\, a_j - \sum_{j=n_1}^{n_2} b_j^*(x_j)\, a_j \right\|_E \leq \frac{\varepsilon}{2}$$

But since the series $\sum b_j^*(x_{n,N_\varepsilon})\, a_j$ is convergent, it follows that there exists $M_\varepsilon \in \mathbb{N}^*$ such that the following inequality holds for each integers n_1, n_2 such that $n_2 \geq n_1 \geq M_\varepsilon$:

$$\left\| \sum_{j=n_1}^{n_2} b_j^*(x_{j,N_\varepsilon})\, a_j \right\|_E \leq \frac{\varepsilon}{2}$$

It follows that

$$\left\| \sum_{j=n_1}^{n_2} b_j^*(x_j)\, a_j \right\|_E \leq \varepsilon,\ n_2 \geq n_1 \geq M_\varepsilon$$

Consequently, the series $\sum b_n^*(x_n)\,a_n$ is convergent in E. Let us set $z_\infty := \sum_{n=1}^{+\infty} b_n^*(x_n)\,a_n$ then $u \in Z_E$ and

$$\|z_m - z_\infty\|_{Z_E} \leq \frac{\varepsilon}{4}, \quad m \geq N_\varepsilon$$

Hence $\left(Z_E, \|\cdot\|_{Z_E}\right)$ is a Banach space. □

Claim 2. *Let us set for each $n \in N^*$:*

$$A_n := (\delta_{nm} a_n)_{m \in \mathbb{N}^*}$$

Then $A_n \in Z_E$ for each $n \in \mathbb{N}^*$ and the sequence $(A_n)_{n \in \mathbb{N}^*}$ is a monotone Schauder basis of Z_E.

Proof of the claim.

Since $b_n^* \neq 0_{E^*}$ for each $n \in \mathbb{N}^*$, there exists $y_n \in E$ such that $b_n^*(y_n) = 1$. We set $m \in \mathbb{N}^*$, $c_m = \delta_{m,n} y_n$. It follows that

$$\begin{cases} A_n = (b_m^*(c_m) a_m)_{m \in \mathbb{N}^*} \\ \sum\limits_{j=1}^{m} b_j^*(c_j) a_j = a_n,\ m \geq n \end{cases}$$

Consequently :

$$A_n \in Z_E,\ n \in \mathbb{N}^*$$

Let $z := (b_n^*(x_n) a_n)_{n \in \mathbb{N}^*}$ be an element of Z_E. Then we have for each $n \in \mathbb{N}^*$:

$$z - \sum_{j=1}^{n} b_j^*(x_j)\,A_j = \begin{pmatrix} \underbrace{0_E, ..., 0_E}_{n \text{ times}}, b_{n+1}^*(x_{n+1}) a_{n+1}, \\ b_{n+2}^*(x_{n+2}) a_{n+2}, ... \end{pmatrix}$$

It follows that:

$$\left\| z - \sum_{j=1}^{n} b_j^*(x_j)\,A_j \right\|_{Z_E} = \sup_{r \geq n+1} \left\| \sum_{j=n+1}^{r} b_j^*(x_j)\,a_j \right\|_E$$

It follows that we have in Z_E

$$z = \sum_{n=1}^{+\infty} b_n^*(x_n)\,A_n$$

On the other hand, we have for each $(\alpha_1, \alpha_2, ..., \alpha_{n+1}) \in \mathbb{K}^{n+1}$ $(n \in \mathbb{N}^*)$:

$$\begin{aligned} \left\| \sum_{s=1}^{n} \alpha_s A_s \right\|_{Z_E} &= \|(b_1^*(\alpha_1 y_1) a_1, ..., b_n^*(\alpha_n y_n) a_n, 0_E, 0_E, ...)\|_{Z_E} \\ &= \sup_{1 \leq r \leq n} \left\| \sum_{j=1}^{r} b_j^*(\alpha_j y_j)\,a_j \right\|_E \\ &\leq \sup_{1 \leq r \leq n+1} \left\| \sum_{j=1}^{r} b_j^*(\alpha_j y_j)\,a_j \right\|_E \\ &\leq \left\| \sum_{j=1}^{n+1} \alpha_j a_j \right\|_{Z_E} \end{aligned}$$

Consequently $(A_n)_{n \in \mathbb{N}^*}$ is a monotone Schauder basis of Z_E. □.

Claim 3.

1. *The mapping* :

$$\begin{array}{rccc} T_0 : & E & \to & Z_E \\ & x & \mapsto & \left(b_n^*(x)a_n\right)_{n\in\mathbb{N}^*} \end{array}$$

is well-defined with a closed image $T_0(E)$*. Furthermore* T_0 *is an isomorphism from* E *onto* $T_0(E)$.

2. *The subspace* $T_0(E)$ *is complemented in* Z_E.

Proof of the claim.

1. We have for each $x \in E$:

$$\lim_{n\to+\infty} \|x - S_n(x)\|_E = 0 \tag{5.2}$$

It follows, thanks to [27, 239], page, that the quantity

$$\mathcal{K}_{\mathcal{F}} := \sup_{n\in\mathbb{N}^*} \|S_n\|_{L(E)}$$

is finite, that is :

$$\mathcal{K}_{\mathcal{F}} := \sup_{n\in\mathbb{N}^*,\, y\in\mathbb{B}_E} \left\|\sum_{j=1}^{n} b_j^*(y)\, a_j\right\|_E < +\infty$$

It follows that we have for each $x \in E$:

$$\sup_{n\in\mathbb{N}^*} \left\|\sum_{j=1}^{n} b_j^*(x)\, a_j\right\|_E \leq \mathcal{K}_{\mathcal{F}} \|x\|_E$$

Hence, for each $x \in E$, we have $\left(b_n^*(x)a_n\right)_{n\in\mathbb{N}^*} \in Z_E$. It follows that the mapping T_0 is a well-defined linear operator of E and fulfilles for each $x \in E$ the inequality :

$$\begin{aligned} \|T_0(x)\|_{Z_E} &= \sup_{n\in\mathbb{N}^*} \left\|\sum_{j=1}^{n} b_j^*(x)\, a_j\right\|_E \\ &\leq \mathcal{K}_{\mathcal{F}} \|x\|_E \end{aligned}$$

On the other hand, we have for each $x \in E$:

$$\left\|\sum_{j=1}^{n} b_j^*(x)\, a_j\right\|_E \leq \|T_0(x)\|_{Z_E},\ n \in \mathbb{N}^*$$

It follows, in view of (5.2), that :

$$\|x\|_E \leq \|T_0(x)\|_{Z_E},\ x \in E$$

Consequently we have :

$$\|x\|_E \leq \|T_0(x)\|_{Z_E} \leq \mathcal{K}_{\mathcal{F}} \|x\|_E,\ x \in E \tag{5.3}$$

It follows, from (5.3), that $T_0(E)$ is a closed subspace of E and that T_0 is an isomorphism from E onto $T_0(E)$.

2. Let us consider the mapping $\varrho : Z_E \longrightarrow Z_E$ defined for each $z := \left(b_n^*(x_n)a_n\right)_{n\in\mathbb{N}^*} \in Z_E$ by the formula :

$$\varrho(z) := \left(b_n^*\left(\sum_{m=1}^{+\infty} b_m^*(x_m)a_m\right) a_n\right)_{n\in\mathbb{N}^*}$$

It is easy to check that ϱ is a well-defined linear mapping and that :

$$\begin{cases} \varrho(z) = T_0\left(\sum\limits_{m=1}^{+\infty} b_m^*(x_m)a_m\right) \in T_0(E),\ z \in Z_E \\ \varrho(z) = z,\ z \in T_0(E) \end{cases}$$

It follows that $\varrho(Z_E) = T_0(E)$. For each $z \in Z_E$, we have also :

$$\begin{aligned}\|\varrho(z)\|_{Z_E} =& \left\| T_0 \left(\sum_{m=1}^{+\infty} b_m^*(x_m) a_m \right) \right\|_{Z_E} \\ \leq& \mathcal{K}_{\mathcal{F}} \left\| \sum_{m=1}^{+\infty} b_m^*(x_m) a_m \right\|_{Z_E} \\ \leq& \mathcal{K}_{\mathcal{F}} \sup_{n \in \mathbb{N}^*} \left\| \sum_{j=1}^{n} b_j^*(x_j) a_j \right\|_E \\ \leq& \mathcal{K}_{\mathcal{F}} \|z\|_{Z_E}\end{aligned}$$

It follows that $\varrho : Z_E \to Z_E$ is a bounded linear operator. On the other hand we have for each $z \in Z_E$:

$$\begin{aligned}(\varrho \circ \varrho)(z) =& \varrho(\varrho(z)) \\ =& \varrho \left(\left(b_n^* \left(\sum_{m=1}^{+\infty} b_m^*(x_m) a_m \right) a_n \right)_{n \in \mathbb{N}^*} \right) \\ =& \left(b_n^* \left(\sum_{k=1}^{+\infty} b_k^* \left(\sum_{l=1}^{+\infty} b_l^*(x_l) a_l \right) a_k \right) a_n \right)_{n \in \mathbb{N}^*} \\ =& \left(b_n^* \left(\sum_{l=1}^{+\infty} b_l^*(x_l) a_l \right) a_n \right)_{n \in \mathbb{N}^*} \\ =& \varrho(z)\end{aligned}$$

It follows that $\varrho \circ \varrho = \varrho$.

Consequently ϱ is a bounded projection of Z_E onto $T_0(E)$. It follows that $T_0(E)$ is a complemented subspace of Z_E. □

End of the proof of the direct part of the theorem :

In the claims above, it is proved that the Banach space E is isomorphic to the complemented subspace $T_0(E)$ of the Banach space Z_E which has a monotone Schauder basis $(A_n)_{n \in \mathbb{N}^*}$. But there exists, according to [25, proposition 1,page 211], a Banach space Z and an isomorphism $T_1 : Z_E \to Z$ such that E is isometrically isomorphic to the subspace $T_1(T_0(E))$. It follows that $(T_1(A_n))_{n \in \mathbb{N}^*}$ is a Schauder basis of the Banach space Z and that $T_1(T_0(E))$ is a complemented subspace of Z. Hence we achieve the proof of the direct part of the theorem.

2. We assume now that E is isomorphic to the complemented closed subspace Y of a Banach space Z which has a Schauder basis $\mathcal{F}$:=$((u_n, v_n^*))_{n \in \mathbb{I}}$. Then $\mathcal{F}$ is a Schauder frame of Z. It follows, thanks to the corollary 4.1., that the Banach space E has a Schauder frame.

Theorem 5.2. *Assume that E is a weakly sequentially complete Banach space such that E has a besselian Schauder frame, then E is isometrically isomorphic to a complemented subspace of a Banach space W wich has an unconditional Schauder basis.*

Proof. Let $\mathcal{F} := ((a_n, b_n^*))_{n \in \mathbb{I}}$ be a besselian Schauder frame of E.

If the set $\{n \in \mathbb{I} : a_n \neq 0_E \text{ and } b_n^* \neq 0_{E^*}\}$ is finite then E is finite dimensional. In this case E has an unconditional Schauder frame. Hence, in this case, the assertion of the theorem is true for $W = E$.

Now we can assume without loss of the generality that $\mathbb{I} = \mathbb{N}^*$and

$$a_n \neq 0_E,\ b_n^* \neq 0_{E^*},\ n \in \mathbb{N}^*$$

We denote by W_E the sequences $(b^*(x_n) a_n)_{n \in \mathbb{N}^*}$ such that $x_n \in E$ for each $n \in \mathbb{N}^*$ and the series $\sum b^*(x_n) a_n$ is unconditionally convergent in E. It is clear that W_E is a vector space over $\mathbb{K}$, when endowed with the usual operations of addition of sequences and multiplication of sequences by scalars.

Claim 1: *For each* $z=\left(b_n^*(x_n)a_n\right)_{n\in\mathbb{N}^*}$ *in* W_E, *the quantity*

$$\|z\|_{W_E}:=\sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\,n\in\mathbb{N}^*}\left\|\sum_{j=1}^{n}b_{\sigma(j)}^*\left(x_{\sigma(j)}\right)a_{\sigma(j)}\right\|_E$$

is finite.

Proof of the claim :

Since the series $\sum b_n^*(x_n)a_n$ is unconditionally convergent in E, it follows that there exists for each $\varepsilon>0$ an integer $k(\varepsilon)\in\mathbb{N}^*$ such that the inegality

$$\left\|\sum_{j\in L}b_j^*(x_j)a_j\right\|_E\leq\varepsilon$$

holds for each $L\in\mathcal{D}_{\mathbb{N}^*}$ which fulfilles the condition $\min(L)\geq k(\varepsilon)+1$. Let $A\in\mathcal{D}_{\mathbb{N}^*}$. Hence for $\varepsilon=1$, there exists $k(1)\in\mathbb{N}^*$ such that

$$\begin{cases}\left\|\sum\limits_{j\in A\cap\{1,\dots k(1)\}}b_j^*(x_j)a_j\right\|_E\leq\sum\limits_{j=1}^{k(1)}\left\|b_j^*(x_j)a_j\right\|_E\\ \left\|\sum\limits_{j\in A\setminus\{1,\dots k(1)\}}b_j^*(x_j)a_j\right\|_E\leq 1\end{cases}$$

It follows that:

$$\left\|\sum_{j\in A}b_j^*(x_j)a_j\right\|_E\leq 1+\sum_{j=1}^{k(1)}\left\|b_j^*(x_j)a_j\right\|_E$$

Consequently:

$$\sup_{A\in\mathcal{D}_{\mathbb{N}^*}}\left\|\sum_{j\in A}b_j^*\left(x_j\right)a_j\right\|_E<+\infty$$

It follows that:

$$\sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\,n\in\mathbb{N}^*}\left\|\sum_{j=1}^{n}b_{\sigma(j)}^*\left(x_{\sigma(j)}\right)a_{\sigma(j)}\right\|_E<+\infty$$

Hence the quantity $\|z\|_{W_E}$ is finite. □

Claim 2 : *The mapping*

$$\begin{array}{rccc}\|.\|_{W_E}: & W_E & \to & \mathbb{R}\\ & z & \mapsto & \|z\|_{W_E}\end{array}$$

is a norm on W_E *and* $(W_E,\|.\|_{W_E})$ *is a Banach space.*

Proof of the claim :

Direct computations show that $\|.\|_{W_E}$ is a norm on W_E. Let us show that $(W_E,\|.\|_{W_E})$ is a Banach space. Indeed, let $(z_m)_{m\in\mathbb{N}^*}:=\left((b_n^*(x_{n,m})a_n)_{n\in\mathbb{N}^*}\right)_{m\in\mathbb{N}^*}$ be a Cauchy sequence in $(W_E,\|.\|_{W_E})$. Then for each $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}^*$ such that the following inequality holds for each $\sigma\in\mathfrak{S}_{\mathbb{N}^*}$, $s\geq r\geq N_\varepsilon$ and $n\in\mathbb{N}^*$:

$$\left\|\sum_{j=1}^{n}\left(b_{\sigma(j)}^*\left(x_{\sigma(j),s}\right)a_{\sigma(j)}-b_{\sigma(j)}^*\left(x_{\sigma(j),r}\right)a_{\sigma(j)}\right)\right\|_E\leq\frac{\varepsilon}{4}$$

It follows that:

$$\left\|b_{\sigma(n)}^*\left(x_{\sigma(n),s}\right)a_{\sigma(n)}-b_{\sigma(n)}^*\left(x_{\sigma(n),r}\right)a_{\sigma(n)}\right\|_E\leq\frac{\varepsilon}{2}$$

for $s \geq r \geq N_\varepsilon$, $n \in \mathbb{N}^*$ and $\sigma \in \mathfrak{S}_{\mathbb{N}^*}$. Consequently, for each $n \in \mathbb{N}^*$ the sequence $\left(b_n^*\left(x_{n,r}\right) a_n\right)_{n\in\mathbb{N}^*}$ is convergent in E to a vector of the form $b_n^*\left(x_n\right) a_n$. Tending s to infinity, we obtain for each $n, r \in \mathbb{N}^*$, $\sigma \in \mathfrak{S}_{\mathbb{N}^*}$ with $r \geq N_\varepsilon$, the following inequality :

$$\left\|\sum_{j=1}^{n}\left(b_{\sigma(j)}^*\left(x_{\sigma(j)}\right) a_{\sigma(j)} - b_{\sigma(j)}^*\left(x_{\sigma(j),r}\right) a_{\sigma(j)}\right)\right\|_E \leq \frac{\varepsilon}{4}$$

It follows, for each $n_1, n_2 \in \mathbb{N}^*$, $n_2 \geq n_1$, that :

$$\left\|\sum_{j=n_1}^{n_2} b_{\sigma(j)}^*\left(x_{\sigma(j)}\right) a_{\sigma(j)} - \sum_{j=n_1}^{n_2} b_{\sigma(j)}^*\left(x_{\sigma(j),N_\varepsilon}\right) a_{\sigma(j)}\right\|_E \leq \frac{\varepsilon}{2}$$

But the series $\sum_j b_{\sigma(j)}^*\left(x_{\sigma(j),N_\varepsilon}\right) a_{\sigma(j)}$ is convergent for each $\sigma \in \mathfrak{S}_{\mathbb{N}^*}$, hence there exists for each $\sigma \in \mathfrak{S}_{\mathbb{N}^*}$, an integer $M_{\varepsilon,\sigma} \in \mathbb{N}^*$ such that:

$$\left\|\sum_{j=n_1}^{n_2} b_{\sigma(j)}^*\left(x_{\sigma(j),N_\varepsilon}\right) a_{\sigma(j)}\right\|_E \leq \frac{\varepsilon}{2},\ n_2 \geq n_1 \geq M_{\varepsilon,\sigma}$$

It follows that:

$$\left\|\sum_{j=n_1}^{n_2} b_{\sigma(j)}^*\left(x_{\sigma(j)}\right) a_{\sigma(j)}\right\|_E \leq \varepsilon,\ n_2 \geq n_1 \geq M_{\varepsilon,\sigma}$$

for each $\sigma \in \mathfrak{S}_{\mathbb{N}^*}$. Consequently, the series $\sum_n b_n^*\left(x_n\right) a_n$ is unconditionally convergent in E. Hence $z_\infty := \left(b_n^*\left(x_n\right) a_n\right)_{n\in\mathbb{N}^*}$ belongs to Z_E. Then

$$\|z_r - z_\infty\|_{W_E} \leq \frac{\varepsilon}{4},\ \ r \geq N_\varepsilon$$

Consequently, $(W_E, \|.\|_{W_E})$ is a Banach space.□

Claim 3 : *The sequence* $(A_n)_{n\in\mathbb{N}^*} := \left((\delta_{nr} a_n)_{r\in\mathbb{N}^*}\right)_{n\in\mathbb{N}^*}$ *is a sequence of elements of* W_E *and is an unconditional Schauder basis of* $(W_E, \|.\|_{W_E})$.

Proof of the claim

Since $b_n^* \neq 0_{E^*}$ for each $n \in \mathbb{N}^*$, there exists, for each $n \in \mathbb{N}^*$, $y_n \in E$ such that $b_n^*(y_n) = 1$. It is clear that $A_n := \left(b_r^*(\delta_{n,r} y_n) a_r\right)_{r\in\mathbb{N}^*} \in W_E$ for each $n \in \mathbb{N}^*$. Let $z := \left(b_n^*(x_n) a_n\right)_{n\in\mathbb{N}^*} \in W_E$. We have for every $n \in \mathbb{N}^*$ and $\sigma_0 \in \mathfrak{S}_{\mathbb{N}^*}$:

$$\left\|z - \sum_{r=1}^{n} b_{\sigma_0(r)}^*(x_{\sigma_0(r)}) A_{\sigma_0(r)}\right\|_{Z_E} = \sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\, r\in\mathbb{N}^*}\left\|\sum_{1\leq j\leq r\ ,\ \sigma(j)\in\mathbb{N}^*\setminus\{\sigma_0(1),\ldots,\sigma_0(n)\}} b_{\sigma(j)}^*(x_{\sigma(j)}) a_{\sigma(j)}\right\|_E$$

Let us recall that there exists for each $\varepsilon > 0$, an integer $k(\varepsilon) \in \mathbb{N}^*$ such that:

$$\left\|\sum_{j\in L} b_j^*(x_j) a_j\right\|_E \leq \varepsilon$$

for each $L \in \mathcal{D}_{\mathbb{N}^*}$ such that $\min(L) \geq k(\varepsilon) + 1$. It follows that we have for each integer $n \in \mathbb{N}^*$ such that $n \geq \max_{1\leq j\leq k(\varepsilon)}\left(\sigma_0^{-1}(j)\right) + 1$:

$$\sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\, r\in\mathbb{N}^*}\left\|\sum_{1\leq j\leq r,\ \sigma(j)\in\mathbb{N}^*\setminus\{\sigma_0(1),\ldots,\sigma_0(n)\}} b_{\sigma(j)}^*(x_{\sigma(j)}) A_{\sigma(j)}\right\|_E \leq \varepsilon$$

Consequently:

$$\lim_{n\to+\infty}\left\|z - \sum_{r=1}^{n} b_{\sigma_0(r)}^*(x_{\sigma_0(r)}) A_{\sigma_0(r)}\right\|_{W_E} = 0$$

for every $\sigma_0 \in \mathfrak{S}_{\mathbb{N}^*}$. Hence the series $\sum b_n^*(x_n)A_n$ is unconditionally convergent in W_E to z. On the other hand, we have for each $n \in \mathbb{N}^*$ and $(\alpha_1, ...\alpha_{n+1}) \in \mathbb{K}^{n+1}$:

$$\alpha_1 A_1 + ... + \alpha_n A_n = (b_1^*(z_r)a_r)_{r\in\mathbb{N}^*}$$

where :

$$z_r := \begin{cases} \alpha_r y_r \text{ if } 1 \leq r \leq n \\ 0 \text{ if } r \geq n+1 \end{cases}$$

It follows that:

$$\begin{aligned}
\|\alpha_1 A_1 + ... + \alpha_n A_n\|_{W_E} &= \sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\, r\in\mathbb{N}^*} \left\| \sum_{j=1}^{r} b^*_{\sigma(j)}(z_{\sigma(j)})a_{\sigma(j)} \right\|_E \\
&= \sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\, r\in\mathbb{N}^*} \left\| \sum_{1\leq j\leq r,\, 1\leq\sigma(j)\leq n} b^*_{\sigma(j)}(z_{\sigma(j)})a_{\sigma(j)} \right\|_E \\
&= \sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\, r\in\mathbb{N}^*} \left\| \sum_{1\leq\sigma^{-1}(j)\leq r, 1\leq j\leq n} b^*_j(z_j)a_j \right\|_E \\
&= \sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\, r\in\mathbb{N}^*} \left\| \sum_{j\in\{\sigma(1),...,\sigma(r)\}\cap\{1,...,n\}} b^*_j(z_j)a_j \right\|_E \\
&= \sup_{L\in\mathcal{P}_{\{1,...,n\}}} \left\| \sum_{j\in L} b^*_j(z_j)a_j \right\|_E \\
&\leq \sup_{L\in\mathcal{P}_{\{1,...,n+1\}}} \left\| \sum_{j\in L} b^*_j(z_j)a_j \right\|_E \\
&\leq \|\alpha_1 A_1 + ... + \alpha_{n+1} A_{n+1}\|_{W_E}
\end{aligned}$$

It follows that:

$$\|\alpha_1 A_1 + ... + \alpha_n A_n\|_{W_E} \leq \|\alpha_1 A_1 + ... + \alpha_{n+1} A_{n+1}\|_{W_E}$$

Consequently $(A_n)_{n\in\mathbb{N}^*}$ is an unconconditionally monotone Schauder basis of $(W_E, \|.\|_{W_E})$.□

Claim 4 :

1. *The mapping*

$$\begin{array}{rccc} T_2 : & E & \to & W_E \\ & x & \mapsto & (b_n^*(x)\, a_n)_{n\in\mathbb{N}^*} \end{array}$$

is well-defined with a closed image $T_2(E)$. Furthermore T_2 is an isomorphism from E onto $T_2(E)$.

2. *$T_2(E)$ is a complemented subspace of W_E.*

Proof of the claim :

1. Since $((a_n, b_n^*))_{n\in\mathbb{N}^*}$ is a besselian Schauder frame of E and that E is weakly sequentially complete, it follows thanks to the proposition 4.7, that the series $\sum b_n^*(x)a_n$ is unconditionally convergent in E, for each $x \in E$. It follows that the sequence $((b_n^*(x)a_n))_{n\in\mathbb{N}^*}$ belongs to W_E for each $x \in E$. Consequently the mapping T_2 is well defined. It is clear that T_2 is a linear operator. Furthermore, since $((a_n, b_n^*))_{n\in\mathbb{N}^*}$ is a besselian Schauder frame of E, we have for all

$x \in E$:

$$\begin{aligned}
\|T_2(x)\|_{Z_E} &= \sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\, n\in\mathbb{N}^*} \left\| \sum_{j=1}^{n} b^*_{\sigma(j)}(x) a_{\sigma(j)} \right\|_E \\
&= \sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*},\, n\in\mathbb{N}^*} \left(\sup_{y^*\in\mathbb{B}_{E^*}} \left| \sum_{j=1}^{n} b^*_{\sigma(j)}(x) y^*(a_{\sigma(j)}) \right| \right) \\
&\leq \sup_{\sigma\in\mathfrak{S}_{\mathbb{N}^*}} \left(\sup_{y^*\in\mathbb{B}_{E^*}} \sum_{n=1}^{+\infty} \left| b^*_{\sigma(n)}(x) y^*(a_{\sigma(n)}) \right| \right) \\
&\leq \sup_{y^*\in\mathbb{B}_{E^*}} \sum_{n=1}^{+\infty} |b^*_n(x) y^*(a_n)| \\
&\leq \mathcal{L}_{\mathcal{F}} \|x\|_E
\end{aligned}$$

Consequently, T_2 is a bounded linear operator. We have also for each $x \in E$:

$$\begin{aligned}
\|T_2(x)\|_{W_E} &\geq \sup_{n\in\mathbb{N}^*} \left(\left\| \sum_{j=1}^{n} b^*_j(x) a_j \right\|_E \right) \\
&\geq \|x\|_E
\end{aligned}$$

Consequently we have :

$$\|x\|_E \leq \|T_2(x)\|_{Z_E} \leq \mathcal{L}_{\mathcal{F}} \|x\|_E\, ,\ x \in E \tag{5.4}$$

It follows, from (5.4), that $T_2(E)$ is a closed subspace of E and that T_2 is an isomorphism from E onto $T_2(E)$.

2. Let us consider the mapping $\widetilde{\varrho} : W_E \longrightarrow W_E$ defined for each $z := (b^*_n(x_n)a_n)_{n\in\mathbb{N}^*} \in W_E$ by the formula :

$$\widetilde{\varrho}(z) := \left(b^*_n \left(\sum_{m=1}^{+\infty} b^*_m(x_m) a_m \right) a_n \right)_{n\in\mathbb{N}^*}$$

It is easy to check that ϱ is a well-defined linear mapping and that :

$$\begin{cases} \widetilde{\varrho}(z) = T_2\left(\sum\limits_{m=1}^{+\infty} b^*_m(x_m) a_m \right) \in T_2(E)\, ,\ z \in W_E \\ \qquad\qquad \varrho(z) = z,\ z \in T_2(E) \end{cases}$$

It follows that $\widetilde{\varrho}(W_E) = T_2(E)$. We have also :

$$\begin{aligned}
\|\widetilde{\varrho}(z)\|_{W_E} &= \left\| T_2 \left(\sum_{m=1}^{+\infty} b^*_m(x_m) a_m \right) \right\|_{Z_E} \\
&\leq \mathcal{L}_{\mathcal{F}} \left\| \sum_{m=1}^{+\infty} b^*_m(x_m) a_m \right\|_E \\
&\leq \mathcal{L}_{\mathcal{F}} \sup_{n\in\mathbb{N}^*} \left\| \sum_{j=1}^{n} b^*_j(x_j) a_j \right\|_E \\
&\leq \mathcal{L}_{\mathcal{F}} \|z\|_{Z_E}
\end{aligned}$$

It follows that $\widetilde{\varrho} : W_E \to W_E$ is a bounded linear operator.

By a similar way as for the proof of the direct part of the theorem 5.1, we show that $\widetilde{\varrho} \circ \widetilde{\varrho} = \widetilde{\varrho}$. Consequently $\widetilde{\varrho}$ is a bounded projection from W_E onto $T_2(E)$. It follows that the closed subspace

$T_2(E)$ of W_E, is a complemented subspace of W_E. □

End of the proof of the theorem :

E is isomorphic to the closed complemented subspace $T_2(E)$ of W_E which has a monotone unconditional Schauder basis $(A_n)_{n\in\mathbb{N}^*}$. But there exists, according to [25, proposition 1, page 211]), a Banach space W and an isomorphism $T_3 : W_E \to W$ such that E is isometrically isomorphic to the subspace $T_3\left(T_2\left(E\right)\right)$. It follows that $\left(T_3\left(A_n\right)\right)_{n\in\mathbb{N}^*}$ is an unconditional Schauder basis of the Banach space W and that $T_3\left(T_2\left(E\right)\right)$ is a complemented subspace of W. Hence we achieve the proof of the theorem. □

Corollary 5.3. *The space $L_1\left([0,1]\right)$ has no besselian Schauder frame.*

Proof. For the sake of contradiction, we assume that $L_1([0,1])$ has a besselian Schauder frame. It is well known that the Banach space $L_1([0,1])$ is weakly sequentially complete [22, page 31], [13, theorem 6, IV.8.6, pages 290, 291]. It follows, thanks to the theorem 5.2, that $L_1([0,1])$ is isomorphic to a subspace of a Banach space with an unconditional basis. But this consequence contradict a well-known result [21, proposition 1.d.1, page 24] on the space $L_1([0,1])$. Consequently the Banach space $L_1([0,1])$ has no besselian Schauder frame. □

Corollary 5.4. *1. There exists a separable Banach space $\mathcal{B}$ with a Schauder basis such that a Banach space E has a Schauder frame if and only if E is isomorphic to a complemented subspace of $\mathcal{B}$.*
2. There exists a separable Banach space $\mathcal{U}$ with an unconditional basis such that each weakly sequentially complete Banach space E which has a besselian Schauder frame is isomorphic to a complemented subspace of $\mathcal{U}$.

Proof. 1. a. Assume that E has a Schauder frame. Then there exist thanks to the theorem 5.1, a Banach space Z which has a Schauder basis and a bounded linear operator $\varphi : E \longrightarrow Z$ such that φ is an isomorphism from E onto $\varphi(E)$ and $\varphi(E)$ is a closed complemented subspace in Z . But we know, thanks to [25, Corollary 1, page 248], that there exists a universal separable Banach space $\mathcal{B}$ with a Schauder basis such that every Banach space with a Schauder basis is isomorphic to a closed subspace of $\mathcal{B}$ which is complemented in $\mathcal{B}$. Hence there exists a linear operator $\Phi : Z \to \mathcal{B}$ such that Φ is an isomorphism from Z onto $\Phi(Z)$ and $\Phi(Z)$ is both a closed subspace of $\mathcal{B}$ and a complemented subspace in $\mathcal{B}$. So there exists a closed subspace V_0 of Z and a closed subspace V_1 of $\mathcal{B}$ such that :

$$\begin{cases} \varphi(E) \oplus V_0 = Z \\ \Phi(Z) \oplus V_1 = \mathcal{B} \end{cases}$$

It follows, from the assumptions on φ and Φ, that $\left(\Phi \circ \varphi\right)(E)$ is closed in $\mathcal{B}$, $\Phi(V_0)$ is closed in $\mathcal{B}$ and

$$\Phi(Z) = \left(\Phi \circ \varphi\right)(E) \oplus \Phi(V_0)$$

Hence we have :

$$\mathcal{B} = \left(\Phi \circ \varphi\right)(E) \oplus \Phi(V_0) \oplus V_1$$

It follows that E is isomorphic to $\left(\Phi \circ \varphi\right)(E)$ which is a closed subspace of $\mathcal{B}$ and complemented in $\mathcal{B}$.

b. It is clear that if E is isomorphic to a complemented subspace of $\mathcal{B}$ then, thanks to the theorem 5.2, E will have a Schauder frame.

2. Assume that E is weakly sequentially complete and that E has a besselian Schauder frame. Then there exist a Banach space W wich has an unconditional Schauder basis E, a bounded linear operator $\psi : E \longrightarrow W$ such that ψ is an isomorphism from E onto $\psi(E)$ and $\psi(E)$ is a closed complemented subspace in W. But we know, thanks to [26, Corollary 1, page 248], that there exists a universal separable Banach space $\mathcal{U}$ with an unconditional Schauder basis such that every Banach space with an unconditional Schauder basis is isomorphic to a closed subspace of $\mathcal{U}$ which is complemented in $\mathcal{U}$. Hence there exists a linear operator $\Psi : W \to \mathcal{U}$ such that Ψ is an isomorphism from W onto $\Psi(W)$ and $\Psi(W)$ is both a closed subspace of $\mathcal{U}$ and a complemented

subspace in $\mathcal{U}$. So there exist a closed subspace V_2 of W and a closed subspace V_3 of $\mathcal{U}$ such that :

$$\begin{cases} \psi(E) \oplus V_2 = W \\ \Psi(Z) \oplus V_3 = \mathcal{U} \end{cases}$$

It follows, from the assumptions on ψ and Ψ, that $(\Psi \circ \psi)(E)$ is closed in $\mathcal{U}$, $\Psi(V_2)$ is closed in $\mathcal{U}$ and

$$\Psi(Z) = (\Psi \circ \psi)(E) \oplus \Psi(V_2)$$

Hence we have :

$$\mathcal{U} = (\Psi \circ \psi)(E) \oplus \Psi(V_2) \oplus V_3$$

It follows that E is isomorphic to $(\Psi \circ \psi)(E)$ which is a closed subspace of $\mathcal{U}$ and complemented in $\mathcal{U}$.□

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