

THE CHOQUARD LOGARITHMIC EQUATION INVOLVING FRACTIONAL LAPLACIAN OPERATOR AND A NONLINEARITY WITH EXPONENTIAL CRITICAL GROWTH

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 35J60, 35R11; Secondary 35Q55, 35B25.

Keywords and phrases: Choquard logarithmic equations, fractional operator, exponential growth, ground state solution.

The first author was supported by Coordination of Superior Level Staff Improvement-(CAPES), Finance Code 001 and São Paulo Research Foundation-(FAPESP), grant number 2019/22531-4, while the second author was supported by National Council for Scientific and Technological Development-(CNPq), grant number 307061/2018-3 and FAPESP grant number 2019/24901-3.

Abstract In the present work we investigate the existence and multiplicity of nontrivial solutions for the Choquard Logarithmic equation $(-\Delta)^{\frac{1}{2}}u + a(x)u + \lambda(\ln|\cdot| * |u|^2)u = f(u)$ in \mathbb{R} , for $a : \mathbb{R} \rightarrow \mathbb{R}$ a continuous potential, $\lambda > 0$ and a nonlinearity f with exponential critical growth. We prove the existence of a nontrivial solution at the mountain pass level and a nontrivial ground state solution under exponential critical and subcritical growth. Moreover, when f has subcritical growth we guarantee the existence of infinitely many solutions, via genus theory.

1 Introduction

In the present paper we are concerned with existence and multiplicity results for the Choquard logarithmic equation

$$(-\Delta)^{\frac{1}{2}}u + u + (\ln|\cdot| * |u|^2)u = f(u) \text{ in } \mathbb{R}, \tag{1.1}$$

where $(-\Delta)^{\frac{1}{2}}$ is the fractional Laplacian, \ln is the neperian logarithm and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous with exponential growth and primitive $F(s) = \int_0^s f(t)dt$. The study was motivated by recent works dealing with logarithmic kernel, the possible applications to this kind of equations and the development of an approach that can be applied to general potentials with not definite sign.

We recall that a function h has *subcritical* exponential growth at $+\infty$, if

$$\lim_{s \rightarrow +\infty} \frac{h(s)}{e^{\alpha s^2} - 1} = 0, \text{ for all } \alpha > 0,$$

and we say that h has *critical* exponential growth at $+\infty$, if there exists $\omega \in (0, \pi]$ and $\alpha_0 \in (0, \omega]$ such that

$$\lim_{s \rightarrow +\infty} \frac{h(s)}{e^{\alpha s^2} - 1} = \begin{cases} 0, & \forall \alpha > \alpha_0. \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

So, in the following we present some necessary conditions to obtain our main results. This kind of hypothesis are usual in works with Moser-Trudinger inequality, such as [18, 19]. We assume that f satisfies

(f₁) $f \in C(\mathbb{R}, \mathbb{R}), f(0) = 0$ and has critical exponential growth.

$$(f_2) \quad \lim_{|t| \rightarrow 0} \frac{f(t)}{|t|} = 0.$$

From (f_1) and (f_2) , for $q > 2$, $\varepsilon > 0$ and $\alpha > \alpha_0$, there exists a constant $b_2 > 0$ such that

$$|F(u)| \leq \frac{\varepsilon}{p} |u|^2 + b_2 |u|^q (e^{\alpha |u|^2} - 1), \quad \forall u \in X. \quad (1.2)$$

and there exists a constant $b_3 > 0$ satisfying

$$|f(u)| \leq \varepsilon |u| + b_3 |u|^{q-1} (e^{\alpha |u|^2} - 1), \quad \forall u \in X. \quad (1.3)$$

We also can derive from (1.2) a very useful inequality. Consider $r_1, r_2 > 1$, $r_1 \sim 1$ and $r_2 > 2$, such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then, for all $u \in H^{\frac{1}{2}}(\mathbb{R})$, $\alpha > \alpha_0$ and $\varepsilon > 0$, we get that

$$\int_{\mathbb{R}} |F(u)| dx \leq \frac{\varepsilon}{2} \|u\|_2^2 + b_2 \|u\|_{r_2 q}^q \left(\int_{\mathbb{R}} (e^{r_1 \alpha u^2} - 1) dx \right)^{\frac{1}{r_1}}. \quad (1.4)$$

Our strategy to prove Theorem 1.1 will consist in finding a Cerami sequence for the mountain pass level. In order to verify that such sequence is bounded in $H^{\frac{1}{2}}(\mathbb{R})$, we will need the following condition

$$(f_3) \quad \text{there exists } \theta > 4 \text{ such that } f(t)t \geq \theta F(t) > 0, \text{ for all } t \in \mathbb{R} \setminus \{0\}.$$

Moreover, as we will be working with an exponential term, to guarantee that the mentioned Cerami sequence and the minimizing sequence for the ground state satisfy the exponential estimate, we rely in the next condition.

$$(f_4) \quad \text{there exist } q > 4 \text{ and } C_q > \frac{[2(q-2)]^{\frac{q-2}{2}} S_q^q}{q^{\frac{q}{2}} \rho_0^{q-2}} \text{ such that } F(t) \geq C_q |t|^q, \text{ for all } t \in \mathbb{R},$$

for $S_q, \rho_0 > 0$ to be defined in Lemma 2.13.

Hence, we are able to enunciate our first main result.

Theorem 1.1. *Assume $(f_1) - (f_4)$, $q > 4$ and $C_q > 0$ sufficiently large. Then,*

(i) *Problem (1.1) has a solution $u \in X \setminus \{0\}$ such that*

$$I(u) = c_{mp} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) ; \gamma(0), I(\gamma(1)) < 0\}$.

(ii) *Problem (1.1) has a ground state solution $u \in X \setminus \{0\}$ in the sense that*

$$I(u) = c_g = \inf\{I(v) ; v \in X \text{ is a nontrivial solution of (1.1)}\}.$$

For the second main result, we are concerned with multiplicity of solutions. However, to obtain this we need to exchange the condition (f_1) by the condition below.

$$(f'_1) \quad f \in C(\mathbb{R}, \mathbb{R}), f \text{ is odd, has subcritical exponential growth and } F(t) \geq 0, \text{ for all } t \in \mathbb{R}.$$

Also, we need to add a condition that gives us the desired geometry for the associated functional. That is,

$$(f_5) \quad \text{the function } t \mapsto \frac{f(t)}{t^3} \text{ is increasing in } (0, +\infty).$$

From this condition, since f is odd, it follows that $\frac{f(t)}{t^3}$ is decreasing in $(-\infty, 0)$.

Moreover, in this case we can also weaken the condition (f_4) , as follows.

$$(f'_4) \quad \text{there exists } q > 4 \text{ and } M_1 > 0 \text{ such that } F(t) \geq M_1 |t|^q, \quad \forall t \in \mathbb{R}.$$

Our strategy to prove the second main result consists in applying the genus theory and it is inspired by the construction presented in [12].

Theorem 1.2. *Suppose $(f'_1), (f_2), (f_3), (f'_4), (f_5)$. Then, problem (1.1) admits a sequence of solution pairs $\pm u_n \in X$ such that $I(u_n) \rightarrow +\infty$ as $n \rightarrow +\infty$.*

The reader should pay attention that many difficulties arise while dealing with Choquard logarithmic equations with exponential nonlinearities. Concerning the logarithmic term, one will see that it is very difficult to guarantee when a Cerami sequence has a subsequence that converges strongly in X (defined in (2.1)). Inspired by [12], we obtain such convergence module translations. But, since the norm of X is not invariant under translations, new difficulties arise. Also, we can mention that the key convergence result that we have so far, does not work for functions satisfying $u_n(x) \rightarrow 0$ a.e. in \mathbb{R} . Besides that, we have the difficulties concerning the controlling of exponential term, as we will mention later.

Next we make a quick overview of literature. In the first half of this overview, we recall that problems with nonlocal operators arise in many areas, such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws and water waves. See e.g. [15, 9]. For fractional problems involving $(-\Delta)^s$, with $N > 2s$ and $s \in (0, 1)$, without logarithmic kernel, we refer to [13, 21], where the authors have obtained ground state solution under subcritical polynomial growth and, in addition, in [21], they studied the regularity and derived some properties for those solutions. Moreover, in [35], authors dealt with coercive potentials, while in [17], they study the equation with potentials that vanish at infinity. In [18], the authors guarantee the existence of ground state solution when the nonlinearity f has maximal exponential growth. We also refer the reader for the works [19, 26, 3, 27], in which the authors dealt with fractional Laplacian operator, and the works [10, 28, 32], for general problems with Moser-Trudinger type behaviour.

For the second half, we take a look into works that deal with Choquard equations which are well-known for their great range of applications in science, specially in physics and mechanics. We cite, as example, the seminal papers due to Fröhlich and Pekar, [24, 23, 39], where the authors describe the quantum mechanics of a polaron at rest, in the particular case, when $V(x) \equiv a > 0$ and $\gamma > 0$. It is curious that, although such equations are called “Choquard equations”, they were introduced in the above mentioned papers.

We focus our overview in the recent works [12, 11, 20, 38, 2], where the authors study problems involving the logarithmic kernel, considering equation (1.1) in the local situation.

In [12], the authors have proved the existence of infinitely many geometrically distinct solutions and a ground state solution, considering $V : \mathbb{R}^2 \rightarrow (0, \infty)$ continuous and \mathbb{Z}^2 -periodic, $\gamma > 0$, $b \geq 0$, $p \geq 4$ and $f(u) = b|u|^{p-2}u$. Here because of the periodic setting, the global Palais–Smale condition can fail, since the corresponding functional become invariant under \mathbb{Z}^2 -translations. Then, intending to fill the gap, in [20] it is proved the existence of a mountain pass solution and a ground state solution for the local problem (1.1) in the case $V(x) \equiv a > 0$, $\lambda > 0$, $2 < p < 4$ and $f(u) = |u|^{p-2}u$. Also, they verified that, if $p \geq 3$, both levels are equal and provided a characterization for them. Moreover, in [11], the authors dealt with the existence of stationary waves with prescribed norm considering $\lambda \in \mathbb{R}$ in a similar setting. Then, following the ideas of the predecessors, in [38], they discuss a local version of problem (1.1) in \mathbb{R}^3 with a polynomial nonlinearity. Finally, in [2], the authors proved the existence of a ground state solution for local problem (1.1), with a nonlinearity of Moser-Trudinger type.

The present work aims to extend or complement those results already found in the literature, combining the fractional Laplacian operator with Choquard logarithmic equations and exponential nonlinearities.

Throughout the paper, we will use the following notations: $L^t(\mathbb{R})$ denotes the usual Lebesgue space with norm $\|\cdot\|_t$; X' denotes the dual space of X ; $B_r(x) = (x - r, x + r)$ is the ball in \mathbb{R} centred in x with radius $r > 0$ and simply B_r when $x = 0$; A^c stands for $\mathbb{R} \setminus A$, for any subset $A \subset \mathbb{R}$; C, C_1, C_2, \dots will denote different positive constants whose exact values are not essential to the exposition of arguments.

The paper is organized as follows: in section 2 we present the framework’s problem and some technical and essential results, some of them already derived in previous works and whose application to our problem is immediate. Section 3 consists in the proof of a key proposition and our first main result. Finally, in section 4, we prove our second main result, considering the subcritical case.

2 Preliminary Results

In this section we present the reader for the framework necessary to study problem (1.1) and provide some technical results.

We start remembering that the operator $(-\Delta)^{\frac{1}{2}} : \mathcal{S}(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is given by

$$(-\Delta)^{\frac{1}{2}}u(x) = C \left(1, \frac{1}{2}\right) \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R} \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^2} dy, \quad \forall x \in \mathbb{R},$$

where the normalizing constant $C(1, \frac{1}{2})$ is defined in [15] and $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space. Equivalently, for $u \in \mathcal{S}(\mathbb{R})$, by [15, Proposition 3.3], if \mathcal{F} denotes the Fourier Transform,

$$(-\Delta)^{\frac{1}{2}}u = \mathcal{F}^{-1}(|\xi|(\mathcal{F}u)), \quad \forall x \in \mathbb{R}.$$

Moreover, in light of [15, Proposition 3.6], we have

$$\|(-\Delta)^{\frac{1}{4}}u\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy, \quad \forall u \in H^{\frac{1}{2}}(\mathbb{R}),$$

even though we usually consider this equality omitting the normalizing constant $\frac{1}{2\pi}$.

Now, we turn our attention to the Hilbert space

$$W^{\frac{1}{2},2}(\mathbb{R}) = H^{\frac{1}{2}}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) ; \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy < +\infty \right\},$$

endowed with the norm $\|\cdot\|^2 = [\cdot]_{\frac{1}{2},2}^2 + \|\cdot\|_2^2$, where

$$[u]_{\frac{1}{2},2}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy.$$

For further considerations about $H^{\frac{1}{2}}(\mathbb{R})$ and $(-\Delta)^{\frac{1}{2}}$ and some useful results, we refer to [15, 14, 1, 26, 3, 27].

Next, inspired in [37], in order to guarantee that the associated functional with the problem is well-defined, we consider the slightly smaller space

$$X = \left\{ u \in H^{\frac{1}{2}}(\mathbb{R}) ; \int_{\mathbb{R}} \ln(1 + |x|)u^2(x)dx < +\infty \right\}. \quad (2.1)$$

The space X endowed with the norm $\|\cdot\|_X^2 = \|\cdot\|^2 + \|\cdot\|_*^2$, where

$$\|u\|_*^2 = \int_{\mathbb{R}} \ln(1 + |x|)u^2(x)dx,$$

is a Hilbert space.

Inspired by [12], we define three auxiliary symmetric bilinear forms

$$(u, v) \mapsto B_1(u, v) = \int_{\mathbb{R}} \int_{\mathbb{R}} \ln(1 + |x - y|)u(x)v(y)dx dy,$$

$$(u, v) \mapsto B_2(u, v) = \int_{\mathbb{R}} \int_{\mathbb{R}} \ln\left(1 + \frac{1}{|x - y|}\right)u(x)v(y)dx dy,$$

$$(u, v) \mapsto B_0(u, v) = B_1(u, v) - B_2(u, v) = \int_{\mathbb{R}} \int_{\mathbb{R}} \ln(|x - y|)u(x)v(y)dx dy.$$

These definitions are understood to being over measurable functions $u, v : \mathbb{R} \rightarrow \mathbb{R}$, such that the integrals are defined in the Lebesgue sense. We also define the functionals $V_1 : H^{\frac{1}{2}}(\mathbb{R}) \rightarrow [0, \infty]$, $V_2 : L^4(\mathbb{R}) \rightarrow [0, \infty)$ and $V_0 : H^{\frac{1}{2}}(\mathbb{R}) \rightarrow \mathbb{R} \cup \{\infty\}$, given by $V_1(u) = B_1(u^2, u^2)$, $V_2(u) = B_2(u^2, u^2)$ and $V_0(u) = B_0(u^2, u^2)$, respectively.

By Hardy-Littlewood-Sobolev inequality (HLS) [31] and using that $0 \leq \ln(1 + r) \leq r$, for $r > 0$, one can easily obtain

$$|B_2(u, v)| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{|x - y|} u(x)v(y) dx dy \leq K_0 \|u\|_2 \|v\|_2, \quad \forall u, v \in L^2(\mathbb{R}), \tag{2.2}$$

where $K_0 > 0$ is the HLS constant.

As a consequence of (2.2),

$$|V_2(u)| \leq K_0 \|u\|_4^4, \quad \forall u \in L^4(\mathbb{R}), \tag{2.3}$$

so V_2 takes finite values over $L^4(\mathbb{R})$. Also, observing that

$$\ln(1 + |x \pm y|) \leq \ln(1 + |x| + |y|) \leq \ln(1 + |x|) + \ln(1 + |y|), \quad \text{for } x, y \in \mathbb{R}, \tag{2.4}$$

and applying Hölder inequality, we estimate

$$B_1(uv, wz) \leq \|u\|_* \|v\|_* \|w\|_2 \|z\|_2 + \|u\|_2 \|v\|_2 \|w\|_* \|z\|_*, \tag{2.5}$$

for all $u, v, w, z \in L^2(\mathbb{R})$.

In our first lemma we determine in which spaces X can be embedded either continuously or compactly.

Lemma 2.1. *X is continuously embedded in $H^{\frac{1}{2}}(\mathbb{R})$ and compactly embedded in $L^t(\mathbb{R})$, for all $t \in [2, +\infty)$.*

Proof. The continuous embedding is immediate, since $\|u\| \leq \|u\|_X$, for all $u \in X$. In order to prove the compact embeddings, let $(u_n) \subset X$ such that $u_n \rightharpoonup 0$ in X . Then, the result follows from the coerciveness of $\ln(1 + |x|)$, [1, Theorem 7.41], [15, Theorem 7.1], the interpolation inequality for $L^t(\mathbb{R})$ spaces and a diagonal argument. \square

Next we make some considerations concerning the exponential behaviour of the nonlinearity. First of all, we mention the celebrated Moser-Trudinger Lemma [10, 33, 18]

Lemma 2.2 (Moser-Trudinger, [33]). *There exists $0 < \omega \leq \pi$ such that, for all $\alpha \in (0, \omega)$, there exists a constant $H_\alpha > 0$ satisfying*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx \leq H_\alpha \|u\|_2^2,$$

for all $u \in H^{\frac{1}{2}}(\mathbb{R})$ with $\|(-\Delta)^{\frac{1}{4}} u\|_2^2 \leq 1$.

Based on the above lemma, [18] proved the following result.

Lemma 2.3. [18, Proposition 2.1] *For any $\alpha > 0$ and $u \in H^{\frac{1}{2}}(\mathbb{R})$,*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) dx < +\infty.$$

Hence, from Lemma 2.3 and equation (1.2), for any $u \in X$ we have

$$\int_{\mathbb{R}} F(u) dx \leq \frac{1}{2} \|u\|_2^2 + b_2 \|u\|_{q_2}^q \left(\int_{\mathbb{R}} (e^{r_1 \alpha u^2} - 1) dx \right)^{\frac{1}{r_1}} < +\infty. \tag{2.6}$$

The following lemma plays a key role on the continuity of the associated functional and while guaranteeing that the functional is lower semicontinuous for $H^{\frac{1}{2}}(\mathbb{R})$. In order to do that, we need a result that allow us to control the exponential term.

Lemma 2.4. *Let $(u_n) \subset X$ such that (u_n) is strongly convergent in $H^{\frac{1}{2}}(\mathbb{R})$. Then, there exists a subsequence $(u_{n_k}) \subset (u_n)$ and a function $g \in H^{\frac{1}{2}}(\mathbb{R})$ satisfying $|u_{n_k}(x)| \leq g(x)$ a.e. in \mathbb{R} , for all $k \in \mathbb{N}$.*

Proof. The proof can be done similarly as in [16]. We only highlight here why the function w obtained in [16] belongs to $H^{\frac{1}{2}}(\mathbb{R})$. Since (w_n) is bounded in $H^{\frac{1}{2}}(\mathbb{R})$, passing to a subsequence, if necessary, $w_n \rightharpoonup v \in H^{\frac{1}{2}}(\mathbb{R})$. Then, by [15, Theorem 7.1], $w_n \rightarrow v$ in $L^2(B_R)$, for all $R > 0$. Hence, up to a subsequence, $w_n(x) \rightarrow v(x)$ a.e. in \mathbb{R} . Once $w_n(x) \rightarrow w(x)$ a.e. in \mathbb{R} , we conclude that $w = v$ a.e. in \mathbb{R} . Therefore, $w \in H^{\frac{1}{2}}(\mathbb{R})$. \square

Lemma 2.5. *Let $(u_n) \subset X$ and $u \in X$ such that $u_n \rightarrow u$ on $H^{\frac{1}{2}}(\mathbb{R})$. Then, we have*

$$\int_{\mathbb{R}} F(u_n) \rightarrow \int_{\mathbb{R}} F(u), \quad \int_{\mathbb{R}} f(u_n)u_n \rightarrow \int_{\mathbb{R}} f(u)u \quad \text{and} \quad \int_{\mathbb{R}} f(u_n)v \rightarrow \int_{\mathbb{R}} f(u)v, \quad \forall v \in X.$$

Proof. Since $u_n \rightarrow u$ in $H^{\frac{1}{2}}(\mathbb{R})$, $u_n(x) \rightarrow u(x)$ a.e. in \mathbb{R} and, from [15, Theorem 6.9], $u_n \rightarrow u$ in $L^t(\mathbb{R})$ for all $t \geq 2$. By Lemma 2.4 and using the Dominated Convergence Theorem, the result follows. \square

We need some technical lemmas in order to obtain boundedness and convergence when dealing with logarithmic parts. Once the proofs found in [12] remains essentially the same, by only exchanging \mathbb{R}^2 for \mathbb{R} , we omit it here to make the paper concise.

Lemma 2.6. ([12, Lemma 2.1]) *Let (u_n) be a sequence in $L^2(\mathbb{R})$ and $u \in L^2(\mathbb{R}) \setminus \{0\}$ such that $u_n \rightarrow u$ pointwise a.e. on \mathbb{R} . Moreover, let (v_n) be a bounded sequence in $L^2(\mathbb{R})$ such that*

$$\sup_{n \in \mathbb{N}} B_1(u_n^2, v_n^2) < \infty.$$

Then, there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that $\|u_n\|_ < C$, for $n \geq n_0$. If, moreover,*

$$B_1(u_n^2, v_n^2) \rightarrow 0 \quad \text{and} \quad \|v_n\|_2 \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then

$$\|v_n\|_* \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Lemma 2.7. ([12, Lemma 2.6]) *Let (u_n) , (v_n) and (w_n) be bounded sequences in X such that $u_n \rightarrow u$ in X . Then, for every $z \in X$, we have $B_1(v_n w_n, z(u_n - u)) \rightarrow 0$, as $n \rightarrow +\infty$.*

Lemma 2.8. ([12, Lemma 2.2]) (i) *The functionals V_0, V_1, V_2 are of class C^1 on X . Moreover, $V_i'(u)(v) = 4B_i(u^2, uv)$, for $u, v \in X$ and $i = 0, 1, 2$.*

(ii) *V_2 is continuous (in fact continuously differentiable) on $L^4(\mathbb{R})$.*

(iii) *V_1 is weakly lower semicontinuous on $H^{\frac{1}{2}}(\mathbb{R})$. (iv) I is lower semicontinuous on $H^{\frac{1}{2}}(\mathbb{R})$.*

Therefore, from Lemmas 2.5 and 2.8, (2.6) and [27, Lemma 2], we conclude that $I : X \rightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|^2 + V_0(u) - \int_{\mathbb{R}} F(u)dx$$

is well-defined in X and $I \in C^1(X, \mathbb{R})$.

Next, let us verify that I has the mountain pass geometry, in order to obtain a Cerami sequence for the mountain pass level c_{mp} .

Lemma 2.9. *There exists $\rho > 0$ such that*

$$m_\beta = \inf\{I(u) ; u \in X, \|u\| = \beta\} > 0, \quad \forall \beta \in (0, \rho) \tag{2.7}$$

and

$$n_\beta = \inf\{I'(u)(u) ; u \in X, \|u\| = \beta\} > 0, \quad \forall \beta \in (0, \rho]. \tag{2.8}$$

Proof. First of all, we choose $r_1, r_2 > 1$ such that $r_1 \sim 1$ and $r_2 > 2$. Then, consider $u \in X \setminus \{0\}$ such that $r_1 \alpha \|u\|^2 < \omega$, in order to apply the exponential inequalities. Thus, from (1.4), HLS, Lemma 2.2 and Sobolev embeddings, we have

$$I(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} V_0(u) - \int_{\mathbb{R}} F(u) dx \geq \frac{\|u\|^2}{2} [1 - \varepsilon - C_2 \|u\|^2 - C_3 \|u\|^{q-2}].$$

Therefore, since $q - 2 > 0$, we can choose $\rho > 0$ sufficiently small, such that (2.7) is valid. Similarly, as

$$I'(u)(u) = \|u\|^2 + V_0(u) - \int_{\mathbb{R}} f(u) u dx \geq \|u\|^2 [1 - \varepsilon - C_4 \|u\|^2 - C_5 \|u\|^{q-2}],$$

once again one can pick $\rho > 0$ sufficiently small, such that (2.8) holds. \square

Lemma 2.10. *Let $u \in X \setminus \{0\}$ and $q > 4$. Then,*

$$\lim_{t \rightarrow 0} I(tu) = 0, \quad \sup_{t > 0} I(tu) < +\infty \quad \text{and} \quad I(tu) \rightarrow -\infty, \quad \text{as } t \rightarrow +\infty.$$

Proof. Let $u \in X \setminus \{0\}$. First, from (f_4) and $q > 4$,

$$I(tu) = \frac{t^2}{2} \|u\|^2 + \frac{t^4}{4} V_0(u) - \int_{\mathbb{R}} F(tu) \leq \frac{t^2}{2} \|u\|^2 + \frac{t^4}{4} V_0(u) - C_q t^q \|u\|_q^q \rightarrow -\infty,$$

as $t \rightarrow +\infty$. Now, from (1.2) and Lemma 2.2, taking $t > 0$ sufficiently small such that $r_1 \alpha t \|u\|^2 < \omega$, we have

$$\int_{\mathbb{R}} F(tu) dx \leq \frac{t^2}{2} \|u\|_2^2 + H_\alpha t^q \|u\|_{qr_2}^q \rightarrow 0,$$

as $t \rightarrow 0$. Hence, $I(tu) \rightarrow 0$ as $t \rightarrow 0$. Finally, this behaviour combined with the fact that I is C^1 , tell us that $\sup_{t > 0} I(tu) < +\infty$. \square

For the next results, consider a sequence $(u_n) \subset X$ satisfying

$$\exists d > 0 \text{ s.t. } I(u_n) < d, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad \|I'(u_n)\|_{X'} (1 + \|u_n\|_X) \rightarrow 0, \quad \text{as } n \rightarrow +\infty \quad (2.9)$$

Lemma 2.11. *Let $(u_n) \subset X$ satisfying (2.9). Then, (u_n) is bounded in $H^{\frac{1}{2}}(\mathbb{R})$.*

Proof. From (2.9) and (f_3) , we have

$$d + o(1) \geq I(u_n) - \frac{1}{4} I'(u_n)(u_n) \geq \frac{1}{4} \|u_n\|^2 + \left(\frac{\theta}{4} - 1\right) \int_{\mathbb{R}} F(u_n) dx \geq \frac{1}{4} \|u_n\|^2.$$

Hence, we conclude that (u_n) is bounded in $H^{\frac{1}{2}}(\mathbb{R})$. \square

Remark 2.12. One can easily verify, from Lemma 2.10 and the Intermediate Value Theorem, that $0 < m_\rho \leq c_{mp} < +\infty$. Then, since I has the mountain pass geometry, similarly as in [20, Lemma 3.2], there exists a sequence $(u_n) \subset X$ such that

$$I(u_n) \rightarrow c_{mp} \quad \text{and} \quad \|I'(u_n)\|_{X'} (1 + \|u_n\|_X) \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \quad (2.10)$$

Moreover, this sequence satisfies (2.9).

Subsequently, we will guarantee that the norms of the sequence obtained in Remark 2.12 can be taken sufficiently small such that the exponential inequalities are valid for it. Moreover, this will be valid for any sequence $(u_n) \subset X$ satisfying $I(u_n) \leq c_{mp}$, for all $n \geq n_0$, $n_0 \in \mathbb{N}$.

Lemma 2.13. *Let $(u_n) \subset X$ satisfying (2.10) and $q > 4$. Then, for some $\rho_0 > 0$ sufficiently small,*

$$\limsup_n \|u_n\|^2 < \rho_0^2.$$

Proof. From Lemma 2.11, we have that (u_n) is bounded in $H^{\frac{1}{2}}(\mathbb{R})$ and $4c_{mp} + o(1) \geq \|u_n\|^2$, for all $n \in \mathbb{N}$. Then, $\limsup_n \|u_n\|^2 \leq 4c_{mp}$. Hence, we need to find an estimative for the value c_{mp} . To do that, consider the set $\mathcal{A} = \{u \in X ; u \neq 0, V_0(u) \leq 0\}$. Defining $u_t(x) = t^2 u(tx)$, for all $t > 0$, $u \in X \setminus \{0\}$ and $x \in \mathbb{R}$, we verify that

$$V_0(u_t) = t^6 V_0(u) - t^6 \ln t \|u\|_2^4 \rightarrow -\infty,$$

as $t \rightarrow +\infty$, and hence $\mathcal{A} \neq \emptyset$. Moreover, from immersions [15, Theorem 6.9], there exists $C > 0$ such that $\|u\| \geq C\|u\|_q$, for all $u \in H^{\frac{1}{2}}(\mathbb{R}) \setminus \{0\}$. Thus, it makes sense to define

$$S_q(v) = \frac{\|v\|}{\|v\|_q} \quad \text{and} \quad S_q = \inf_{v \in \mathcal{A}} S_q(v) \geq \inf_{v \neq 0} S_q(v) > 0.$$

Now we are ready to estimate c_{mp} . From Lemma 2.10, for $v \in \mathcal{A}$ and $T > 0$ sufficiently large, $I(Tv) < 0$. So, we can define $\gamma \in \Gamma$ by $\gamma(t) = tTv$, such that

$$c_{mp} \leq \max_{0 \leq t \leq 1} I(\gamma(t)) = \max_{0 \leq t \leq 1} I(tTv) \leq \max_{t \geq 0} I(tv).$$

Consequently, for $\psi \in \mathcal{A}$, we have

$$c_{mp} \leq \max_{t \geq 0} I(t\psi) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \|\psi\|^2 - C_q t^q \|\psi\|_q^q \right\} \leq \left(\frac{1}{2} - \frac{1}{q} \right) \frac{S_q(\psi)^{\frac{2q}{q-2}}}{(qC_q)^{\frac{2}{q-2}}}.$$

Thus, taking the infimum over $\psi \in \mathcal{A}$, we obtain

$$\limsup_n \|u_n\|^2 \leq \frac{2(q-2)}{q} \frac{S_q^{\frac{2q}{q-2}}}{(qC_q)^{\frac{2}{q-2}}} \leq \rho_0^2,$$

for $C_q > 0$ sufficiently large. □

Lemma 2.14. *Let $(u_n) \subset X$ be bounded in $H^{\frac{1}{2}}(\mathbb{R})$ such that*

$$\liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{Z}} \int_{B_2(y)} |u_n(y)|^2 dx > 0. \quad (2.11)$$

*Then, there exists $u \in H^{\frac{1}{2}}(\mathbb{R}) \setminus \{0\}$ and $(y_n) \subset \mathbb{Z}$ such that, up to a subsequence, $y_n * u_n = \tilde{u}_n \rightarrow u \in H^{\frac{1}{2}}(\mathbb{R})$.*

Proof. From (2.11), lim inf properties, the boundedness of (u_n) in $H^{\frac{1}{2}}(\mathbb{R})$, [1, Theorem 7.41] and [15, Theorem 7.1], one can construct the desired subsequence and obtain the result. □

3 Proof of Theorem 1.1

In the present section, we finish the proof of Theorem 1.1. As a first step, we prove a key proposition which provide us with nontrivial critical points.

Proposition 3.1. *Let $q > 4$ and $(u_n) \subset X$ satisfying either (2.10) or (2.9) with $d = c_{mp}$. Then, passing to a subsequence if necessary, only one between the following items is true:*

- (a) $\|u_n\| \rightarrow 0$ and $I(u_n) \rightarrow 0$, as $n \rightarrow +\infty$.
- (b) *There exists points $y_n \in \mathbb{Z}$ such that $\tilde{u}_n = y_n * u_n \rightarrow u$ in X , for a nontrivial critical point $u \in X$ for I .*

Proof. First of all, from Lemma 2.11, (u_n) is bounded in $H^{\frac{1}{2}}(\mathbb{R})$ and, passing to a subsequence if necessary, by Lemma 2.13,

$$\int_{\mathbb{R}} (e^{r_1 \alpha u_n^2} - 1) dx \leq H_\alpha, \forall n \in \mathbb{N}.$$

Suppose that (a) does not happen.

Claim 1: $\liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{Z}} \int_{B_2(y)} |u_n(x)|^2 dx > 0.$

Suppose the contrary. Then, from an easy adaptation of Lion’s Lemma found in [40, Lemma 2.4], one has $u_n \rightarrow 0$ in $L^t(\mathbb{R})$, for all $t > 2$. Thus, from (2.3), $V_2(u_n) \rightarrow 0$. Moreover, since $q > 2$, from (2.9),

$$\left| \int_{\mathbb{R}} f(u_n) u_n dx \right| \leq \varepsilon \|u_n\|_2^2 + C_1 \|u_n\|_{q_{r_2}}^q \rightarrow 0,$$

as $\varepsilon \rightarrow 0$ and $n \rightarrow +\infty$. Consequently,

$$\|u_n\|_2^2 + V_1(u_n) = I'(u_n)(u_n) + V_2(u_n) + \int_{\mathbb{R}} f(u_n) u_n dx \rightarrow 0,$$

as $n \rightarrow +\infty$. Thus, $\|u_n\| \rightarrow 0$ and $V_1(u_n) \rightarrow 0$. From the continuous immersions [15, Theorem 6.9], $\|u_n\|_2 \rightarrow 0$. Then, from (1.4), $\int_{\mathbb{R}} F(u_n) dx \rightarrow 0$. As a consequence, $I(u_n) \rightarrow 0$, which is a contradiction. Therefore, the claim is valid.

From Claim 1 and Lemma 2.14, up to a subsequence, there exists $(y_n) \subset \mathbb{Z}$ and $u \in H^{\frac{1}{2}}(\mathbb{R}) \setminus \{0\}$ such that $\tilde{u}_n = y_n * u_n \rightharpoonup u$ in $H^{\frac{1}{2}}(\mathbb{R})$. We can conclude that (\tilde{u}_n) is bounded in $L^t(\mathbb{R})$, for all $t \geq 2$, and $\tilde{u}_n(x) \rightarrow u(x)$ a.e. in \mathbb{R} .

Observe that, as $q > 2$,

$$\begin{aligned} V_1(\tilde{u}_n) &= V_1(u_n) = I'(u_n)(u_n) + V_2(u_n) + \int_{\mathbb{R}} f(u_n) u_n dx - \|u_n\|_2^2 \\ &\leq K_0 \|u_n\|_4^2 + \varepsilon C_2 + C_1 \|u_n\|_{q_{r_2}}^q + o(1) \leq C_3 + o(1). \end{aligned}$$

That is, $\sup_n V_1(\tilde{u}_n) < +\infty$. So, from (\tilde{u}_n) being bounded in $L^2(\mathbb{R})$ and Lemma 2.6, $(\|\tilde{u}_n\|_*)$ is bounded. Therefore, (\tilde{u}_n) is bounded in X and, once X is reflexive, passing to a subsequence, if necessary, $\tilde{u}_n \rightharpoonup u$ in X . From Lemma 2.1, $\tilde{u}_n \rightarrow u$ in $L^t(\mathbb{R})$, for all $t \geq 2$.

Claim 2: $I'(\tilde{u}_n)(\tilde{u}_n - u) \rightarrow 0$, as $n \rightarrow +\infty$.

One can easily see, by a change of variable, that $I'(\tilde{u}_n)(\tilde{u}_n - u) = I'(u_n)(u_n - (-y_n) * u)$. Thus,

$$|I'(\tilde{u}_n)(\tilde{u}_n - u)| = |I'(u_n)(u_n - (-y_n) * u)| \leq \|I'(u_n)\|_{X'} (\|u_n\|_X + \|(-y_n) * u\|_X), \forall n \in \mathbb{N}. \tag{3.1}$$

Then, we first seek for an useful inequality for $\|(-y_n) * u\|_X$. Once $\|u_n\|_X$ already appears in (3.1), a natural way is to look for a constant $C > 0$ such that $\|(-y_n) * u\|_X \leq C \|u_n\|_X$.

Observe that, since $\tilde{u}_n \rightarrow u$ in $L^2(\mathbb{R})$ and $u \in L^2(\mathbb{R}) \setminus \{0\}$, there are $R_1 > 0$, $n_1 \in \mathbb{N}$ and $C_1 > 0$ such that $\|\tilde{u}_n\|_{2, B_{R_1}}^2 \geq C_1 > 0$, for all $n \geq n_1$. Let us prove that (y_n) is bounded in \mathbb{Z}^2 . In fact, suppose, by contradiction, that any subsequence of (y_n) satisfies $|y_n| \rightarrow +\infty$. Thus, there exists $n_2 \in \mathbb{N}$ such that $|y_n| \geq 2R_1$, for all $n \geq n_2$. Moreover, we recall that $1 + |x + y_n| \geq \sqrt{1 + |y_n|}$, for all $x \in B_{R_1}$ and $n \geq n_2$. Therefore, passing to a subsequence if necessary, we have

$$\begin{aligned} \|\tilde{u}_n\|_*^2 &= \int_{\mathbb{R}} \ln(1 + |x + y_n|) |u_n(x)|^2 dx \\ &\geq C_2 \|u_n\|_{2, B_{R_1}}^2 \ln(1 + |y_n|) = \tilde{C}_3 \ln(1 + |y_n|), \forall n \in \mathbb{N}, \end{aligned}$$

which contradicts the fact that (\tilde{u}_n) is bounded in X .

Now, observe that

$$\|u_n\|_*^2 = \int_{\mathbb{R}} \ln(1 + |x - y_n|) |\tilde{u}_n(x)|^2 dx \leq \|\tilde{u}_n\|_*^2 + \ln(1 + |y_n|) \|\tilde{u}_n\|_2^2, \forall n \in \mathbb{N}. \quad (3.2)$$

Since (y_n) is bounded in \mathbb{Z}^2 , there are $y_0 \in \mathbb{Z}^2$ and $\bar{n} \in \mathbb{N}$ such that $y_n = y_0$, for all $n \geq \bar{n}$. Then, from (3.2) and the fact that (\tilde{u}_n) is bounded in X , we have

$$\|u_n\|_*^2 \leq C_4 \quad \text{where} \quad C_4 = \max\{\ln(1 + |y_{n_1}|), \dots, \ln(1 + |y_{\bar{n}}|), \ln(1 + |y_0|)\} > 0.$$

Therefore, once (u_n) is already bounded in $H^{\frac{1}{2}}(\mathbb{R})$, we conclude that (u_n) is bounded in X .

Thus, from (2.10),

$$|I'(\tilde{u}_n)(\tilde{u}_n - u)| \leq (1 + C_8^{\frac{1}{2}}) \|I'(u_n)\|_{X'} \|u_n\|_X \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Claim 3: $\int_{\mathbb{R}} f(\tilde{u}_n)(\tilde{u}_n - u) dx \rightarrow 0$, as $n \rightarrow +\infty$.

Since $\|\cdot\|$ is \mathbb{Z} -invariant, Moser-Trudinger inequality and the exponential results above hold for the sequence (\tilde{u}_n) as well. It follows that

$$\left| \int_{\mathbb{R}} f(\tilde{u}_n)(\tilde{u}_n - u) dx \right| \leq \|\tilde{u}_n\|_2 \|\tilde{u}_n - u\|_2 + C_9 \|\tilde{u}_n - u\|_{q_{r_2}}^q \rightarrow 0,$$

as $n \rightarrow +\infty$, concluding the claim.

Moreover, observe that

$$|V_2'(\tilde{u}_n)(\tilde{u}_n - u)| \leq K_0 \|\tilde{u}_n\|_4^3 \|\tilde{u}_n - u\|_4 \rightarrow 0$$

and

$$V_1'(\tilde{u}_n)(\tilde{u}_n - u) = B_1(\tilde{u}_n^2, \tilde{u}_n(\tilde{u}_n - u)) = B_1(\tilde{u}_n^2, (\tilde{u}_n - u)^2) + B_1(\tilde{u}_n^2, u(\tilde{u}_n - u)).$$

Also, since (\tilde{u}_n) is bounded in X , from Lemma 2.7, $B_1(\tilde{u}_n^2, u(\tilde{u}_n - u)) \rightarrow 0$. So, from the above estimations

$$o(1) = \|\tilde{u}_n\|^2 - \|u\|^2 + B_1(\tilde{u}_n^2, (\tilde{u}_n - u)^2) + o(1) \geq \|\tilde{u}_n\|^2 - \|u\|^2 + o(1).$$

Hence, $\|\tilde{u}_n\| \rightarrow \|u\|$ and $B_1(\tilde{u}_n^2, (\tilde{u}_n - u)^2) \rightarrow 0$. Therefore, we conclude that $\|\tilde{u}_n - u\| \rightarrow 0$ and, combined with Lemma 2.6, we have $\|\tilde{u}_n - u\|_* \rightarrow 0$. Consequently, $\tilde{u}_n \rightarrow u$ in X .

Finally, we have to show that u is a critical point of I . Let $v \in X$. So, as we deduced above, it is possible to find $C_{10} > 0$ such that

$$\|(-y_n) * v\|_X \leq C_{10} \|u_n\|_X, \forall n \in \mathbb{N}.$$

Thus,

$$|I'(u)(v)| = \lim_{n \rightarrow +\infty} |I'(\tilde{u}_n)(v)| = \lim_{n \rightarrow +\infty} |I'(u_n)((-y_n) * v)| \leq C_{10} \lim_{n \rightarrow +\infty} \|I'(u_n)\|_{X'} \|u_n\|_X = 0.$$

□

Proof of Theorem 1.1. (i) From Lemma 2.9 and Proposition 3.1 there exists a nontrivial critical point of I , $u_0 \in X$, such that $I(u_0) = c_{mp}$.

(ii) Define $K = \{v \in X \setminus \{0\} ; I'(v) = 0\}$. Since $u_0 \in K$, $K \neq \emptyset$. Thus, we can consider $(u_n) \subset K$ satisfying $I(u_n) \rightarrow c_g = \inf_{v \in K} I'(v)$.

Observe that $c_g \in [-\infty, c_{mp}]$. If $c_g = c_{mp}$, nothing remains to be proved. Otherwise, if $c_g < c_{mp}$, passing to a subsequence if necessary, we can assume that $I(u_n) \leq c_{mp}$, for all $n \in \mathbb{N}$. So, Lemma 2.13 holds for this sequence and we can apply all the results derived earlier. Therefore, from Lemma 2.9, $(u_n) \subset K$ and Proposition 3.1 there exists $(y_n) \subset \mathbb{Z}$ such that $\tilde{u}_n \rightarrow u$ in X , for a nontrivial critical point u of I in X . Consequently, $I'(u) = \lim I'(\tilde{u}_n) = \lim I'(u_n) = 0$ and we conclude that $u \in K$ and $I(u) = \lim I(\tilde{u}_n) = \lim I(u_n) = c_g$. Particularly, we see that $c_g > -\infty$. □

4 Proof of Theorem 1.2

To finish this section, we will verify some properties of an important auxiliary function, namely, $\varphi_u : \mathbb{R} \rightarrow \mathbb{R}$, given by $\varphi_u(t) = I(tu)$, for all $u \in X \setminus \{0\}$ and $t \in \mathbb{R}$.

Lemma 4.1. *Let $u \in X \setminus \{0\}$. Then, φ_u is even and there exists a unique $t_u \in (0, +\infty)$ such that $\varphi'_u(t) > 0$, for all $t \in (0, t_u)$, and $\varphi'_u(t) < 0$, for all $t \in (t_u, \infty)$. Moreover, $\varphi_u(t) \rightarrow -\infty$, as $t \rightarrow +\infty$.*

Proof. Since f is odd, I is even and, consequently, φ_u is even as well.

(i) For $t > 0$ sufficiently small and $\alpha > 0$, for (1.3), we have

$$\varphi'_u(t) \geq t\|u\|^2[1 - C_2t^2\|u\|^2 - \varepsilon - C_4t^{q-2}\|u\|^{q-2}].$$

Thus, $\varphi'_u(t) > 0$ for $t > 0$ sufficiently small.

(ii) From (f_3) and (f'_4) , once $q > 4$,

$$\varphi'_u(t) \leq t\|u\|^2 + t^3V_1(u) - C_3t^{q-1}\|u\|_q^q \rightarrow -\infty, \text{ as } t \rightarrow +\infty.$$

Hence, from (i)-(ii), since I is C^1 , there exists $t_u \in (0, +\infty)$ such that $\varphi'_u(t_u) = 0$, which is unique by (f_5) . \square

Corollary 4.2. *Let $u \in X \setminus \{0\}$. Then, there exists a unique $t'_u \in (0, +\infty)$ such that $\varphi_u(t) > 0$, for $t \in (0, t'_u)$, and $\varphi_u(t) < 0$, for $t \in (t'_u, +\infty)$. Moreover, t_u given by Lemma 4.1 is a global maximum for φ_u .*

Lemma 4.3. *For each $u \in X \setminus \{0\}$, the map $u \mapsto t'_u$ is continuous.*

Proof. First of all, observe that, from the uniqueness of t'_u , the map is well-defined for all $u \in X \setminus \{0\}$. Consider now $(u_n) \subset X \setminus \{0\}$ such that $u_n \rightarrow u$ in $X \setminus \{0\}$.

Suppose, by contradiction, that (t'_{u_n}) is not bounded. So, from Lemma 4.1, the definition of φ_u , the continuity of I and Corollary 4.2, there exists $t_0 > 0$ such that $0 \leq \varphi_u(t_0) < 0$, which is a contradiction. Therefore, (t'_{u_n}) is bounded.

Hence, there exists $(t'_{u_{n_k}}) \subset (t'_{u_n})$ such that $t'_{u_{n_k}} \rightarrow \bar{t} \in (0, +\infty)$. Thus, we infer that

$$0 \leq \|t'_{u_{n_k}} u_{n_k} - \bar{t}u\|_X \leq |t'_{u_{n_k}} - \bar{t}|\|u_{n_k}\|_X + \bar{t}\|u_{n_k} - u\|_X \rightarrow 0.$$

Consequently, $t'_{u_{n_k}} u_{n_k} \rightarrow \bar{t}u$ in X . From continuity of I , $I(t'_{u_{n_k}} u_{n_k}) \rightarrow I(\bar{t}u)$.

On the other hand, from $t'_{u_{n_k}}$ definition, $I(t'_{u_{n_k}} u_{n_k}) = 0$. Hence, $I(\bar{t}u) = 0$. From uniqueness given by Corollary 4.2, $\bar{t} = t'_u$. So, we conclude that $t'_{u_{n_k}} \rightarrow t'_u$. Since this argument can be done to any subsequence of (t_{u_n}) , by standard arguments one can see that $t'_{u_n} \rightarrow t'_u$, proving the continuity. \square

In this section we will provide the proof of the multiplicity result stated. In order to do so, we will need to verify some results concerned with the genus theory, denoted by γ and which definition, given over $\mathcal{A} = \{A \subset X ; A \text{ is symmetric and closed}\}$ (with respect to the continuity in X), and basic properties can be found in [36, Chapter II.5].

We start defining the sets

$$K_c = \{u \in X ; I'(u) = 0, I(u) = c\}, \quad c \in (0, +\infty)$$

and

$$A_{c,\rho} = \{u \in X ; \|u - v\| \leq \rho, \text{ for some } v \in K_c\}, \quad c \in (0, +\infty).$$

It is easy to verify that the sets K_c and $A_{c,\rho}$ are symmetric, closed (with respect to $\|\cdot\|_X$) and invariant under \mathbb{Z} translations, i.e., if $u \in K_c, A_{c,\rho}$, then $z * u \in K_c, A_{c,\rho}$, for all $z \in \mathbb{Z}$.

Next, we fix a continuous map $\beta : L^2(\mathbb{R}) \setminus \{0\} \rightarrow \mathbb{R}$ that is equivariant under \mathbb{Z} translations, i.e., $\beta(x * u) = x + \beta(u)$, for $x \in \mathbb{Z}$ and $u \in L^2(\mathbb{R}) \setminus \{0\}$. We also require that $\beta(-u) = \beta(u)$. Such map is called a *generalized barycenter map* and an example can be constructed as that one in [4]. Hence, we can define the sets

$$\tilde{K}_c = \{u \in K_c ; \beta(u) \in [-4, 4]\},$$

which are clearly symmetric. Moreover, before we enunciate our first result, we need to recall the Gauss bracket $[\cdot] : \mathbb{R} \rightarrow \mathbb{Z}$, given by $[s] = \max\{n \in \mathbb{Z} ; n \leq s\}$ (see [25, Chapter 3]). We recall some properties of the Gauss bracket, that are needed inside of the proof of our result.

Lemma 4.4. *Let $[\cdot] : \mathbb{R} \rightarrow \mathbb{Z}$, given by $[s] = \max\{n \in \mathbb{Z} ; n \leq s\}$. Then,*

(i) $0 \leq s - [s] < 1$, for all $s \in \mathbb{R}$.

(ii) if $z \in \mathbb{Z}$ and $s \in \mathbb{R}$, then $[z + s] = z + [s]$.

(iii) Let $s \in \mathbb{R}$ such that $s - [s] \geq \frac{1}{2}$, then $s - \frac{1}{2} - \left[s - \frac{1}{2} \right] < \frac{1}{2}$.

(iv) $0 \leq s - [s] < 1$, for all $s \in \mathbb{R}$.

(v) Let $s \in \mathbb{R}$. Then, $s - \left[s - \frac{1}{2} \right] < 1$.

Proposition 4.5. *Let $c > 0$. Then, there exists $\rho_0 = \rho_0(c) > 0$ such that $\gamma(A_{c,\rho}) < \infty$, for all $\rho \in (0, \rho_0)$.*

Proof. Fix a barycenter map $\beta : L^2(\mathbb{R}) \setminus \{0\} \rightarrow \mathbb{R}$.

Claim 1: The sets \tilde{K}_c are compact in X .

Let $(u_n) \subset \tilde{K}_c$. From Proposition 3.1, there exists $(y_n) \subset \mathbb{Z}$ such that, passing to a subsequence if necessary, $w_n = y_n * u_n \rightarrow u \in K_c$ in X . Consequently, as β is continuous,

$$y_n + \beta(u_n) = \beta(y_n * u_n) = \beta(w_n) \rightarrow \beta(u).$$

Since $(u_n) \subset \tilde{K}_c$, $\beta(u_n) \subset [-4, 4]$. From compactness, passing to a subsequence if necessary, there exists $r \in [-4, 4]$ such that $\beta(u_n) \rightarrow r$ in \mathbb{R} . Also, since $(\beta(u_n))$ and $(y_n + \beta(u_n))$ are bounded, $(y_n) \subset \mathbb{Z}$ is bounded. Thus, we can assume that (y_n) is convergent. Therefore,

$$\beta(u) = \lim y_n + \lim \beta(u_n) \Rightarrow \lim y_n = \beta(u) - r.$$

Moreover, since $(y_n) \subset \mathbb{Z}$, $\beta(u) - r = \bar{y} \in \mathbb{Z}$.

Lets prove that $u_n \rightarrow u_0 = (-\bar{y}) * u$, with respect to $\|\cdot\|_X$. First of all, observe that, since $(y_n) \subset \mathbb{Z}$ and $y_n \rightarrow \bar{y} \in \mathbb{Z}$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, $y_n = \bar{y}$. Moreover,

$$\begin{aligned} \|u_n - (-y_n) * u\|_* &\leq \int_{\mathbb{R}} \ln(1 + |y|) |y_n * u_n - u|^2 dy + \int_{\mathbb{R}} \ln(1 + |y_n|) |y_n * u_n - u|^2 dy \\ &\leq (1 + \ln(1 + |\bar{y}|)) \|y_n * u_n - u\|_X^2. \end{aligned}$$

Since $y_n = \bar{y}$, for all $n \geq n_0$, and $y_n * u_n \rightarrow u$ in X , we have

$$\begin{aligned} \lim \|u_n - (-y_n) * u\|_*^2 &\leq \lim [(1 + \ln(1 + |\bar{y}|)) \|y_n * u_n - u\|_X^2] \\ &= (1 + \ln(1 + |\bar{y}|)) \lim \|y_n * u_n - u\|_X^2 = 0. \end{aligned}$$

Also, as $\|y_n * u_n - u\| = \|u_n - (-y_n) * u\|$ and $y_n * u_n \rightarrow u$ in $H^{\frac{1}{2}}(\mathbb{R})$, $\|u_n - (-y_n) * u\| \rightarrow 0$, as $n \rightarrow +\infty$. Hence, $\|u_n - (-y_n) * u\|_X \rightarrow 0$. Finally, since $y_n = \bar{y}$, for all $n \geq n_0$, we conclude that $u_n \rightarrow u_0$, as desired.

Therefore, from the \mathbb{Z} -invariance of K_c , $(-\bar{y}) * u \in K_c$ and, by continuity of β , $\beta(u_n) \rightarrow \beta(u_0) \in [-4, 4]$. Consequently, $u_0 \in \tilde{K}_c$, and the claim is valid.

For $c > 0$, from the facts that β is even and $X \hookrightarrow H^{\frac{1}{2}}(\mathbb{R})$, one can see that \tilde{K}_c is compact in $H^{\frac{1}{2}}(\mathbb{R})$ and, also, $\tilde{K}_c \in \mathcal{A}$. Thus, since $0 \notin \tilde{K}_c$, $\gamma(\tilde{K}_c) < +\infty$.

Hence, from the genus definition, there exists a continuous and odd function $g_0 : \tilde{K}_c \rightarrow \mathbb{R}^k \setminus \{0\}$, for some $k \geq 0$. Then, from Claim 1, one can easily verify that g_0 is continuous with respect to $\|\cdot\|$.

Therefore, since $H^{\frac{1}{2}}(\mathbb{R})$ is a normal topological space and \tilde{K}_c is closed in $H^{\frac{1}{2}}(\mathbb{R})$, we can apply a corollary of Tietze's Theorem in order to extend g_0 to a continuous and odd function (with respect to $\|\cdot\|$), $g : H^{\frac{1}{2}}(\mathbb{R}) \rightarrow \mathbb{R}$.

Claim 2: There exists $\bar{\rho} > 0$ such that $g(u) \neq 0$, for all $u \in U_{\bar{\rho}} = \{u \in H^{\frac{1}{2}}(\mathbb{R}) ; \|u - v\| \leq \bar{\rho}, \text{ for some } v \in \tilde{K}_c\}$.

First of all, observe that $g(\tilde{K}_c) = g_0(\tilde{K}_c) \subset \mathbb{R}^k \setminus \{0\}$, that is, $\{0\} \cap g(\tilde{K}_c) = \emptyset$ or, equivalently, $g^{-1}(\{0\}) \cap \tilde{K}_c = \emptyset$. From continuity of g , we have that $g^{-1}(\{0\})$ is closed in $H^{\frac{1}{2}}(\mathbb{R})$. Consequently, $d = \text{dist}(g^{-1}(\{0\}), \tilde{K}_c) > 0$.

Set $\bar{\rho} = \frac{d}{2}$. Suppose, by contradiction, that there exists $u \in U_{\bar{\rho}}$ such that $g(u) = 0$, i.e., $u \in g^{-1}(\{0\})$. Thus, there exists $v \in \tilde{K}_c$ satisfying

$$\text{dist}(u, v) = \|u - v\| \leq \bar{\rho} = \frac{d}{2}.$$

Hence, $d \leq \|u - v\| \leq \frac{d}{2}$, which is a contradiction.

Therefore, $g(u) \neq 0$ for all $u \in U_{\bar{\rho}}$.

Also, one can verify that $U_{\bar{\rho}} \in \mathcal{A}$. In the following, we need to define the set

$$L_1 = \left\{ u \in L^p(\mathbb{R}) \setminus \{0\}; |\beta(u) - [\beta(u)]| \leq \frac{1}{2} \right\}.$$

One can easily see that $z * u \in L_1$, for all $z \in \mathbb{Z}$ and $u \in L_1$. Now, define a map $h_1 : L_1 \rightarrow L_1$ by $h_1(u) = (-[\beta(u)]) * u$. It is easy to verify that h_1 is well-defined, odd, invariant under \mathbb{Z} translations and an isometry. Also, setting $a = \frac{1}{2}$ and $L_2 = a * L_1 \subset L^2(\mathbb{R}) \setminus \{0\}$, it is easy to see, from item (iii) of Lemma 4.4, that $L^2(\mathbb{R}) \setminus \{0\} \subset L_1 \cup L_2$.

Now, define the map $h_2 : L_2 \rightarrow L_2$ by $h_2(u) = a * [h_1((-a) * u)]$, for $u \in L_2$. Once again, it is easy to verify that h_2 is odd, \mathbb{Z} -invariant and an isometry. Moreover, it follows immediately from the properties of β , definition of h_2 and h_1 and Lemma 4.4-(v), that $\beta(h_2(u)) \in [0, 1]$ for $u \in L_2$.

Next, define the sets $A_i = h_i^{-1}(U_{\bar{\rho}}) \subset L_i$, for $i = 1, 2$. One can see that $A_i \in \mathcal{A}$ and that they are \mathbb{Z} -invariant, for $i = 1, 2$. Set $A = A_1 \cup A_2$.

Claim 3: There exists $\rho_c \in (0, +\infty)$, such that $A_{c, \rho} \subset A$, for all $\rho \in (0, \rho_c)$.

Suppose, by contradiction, that for every $n \in \mathbb{N}$, there exists $\rho_n \in (0, \frac{1}{n})$ such that $A_{c, \rho_n} \not\subset A$. Then, there exists $u_n \in A_{c, \rho_n} \setminus A$, for every $n \in \mathbb{N}$.

We can suppose that $u_n \in L_1$, for all $n \in \mathbb{N}$. By definition, there exists $v_n \in K_c$ such that $\|u_n - v_n\| \leq \frac{1}{n}$, for all $n \in \mathbb{N}$.

Now, observe that, setting $\tilde{u}_n = (-[\beta(u_n)]) * u_n$ and $\tilde{v}_n = (-[\beta(v_n)]) * v_n$, we have $\tilde{u}_n \in L_1$ and $\tilde{v}_n \in K_c$, for all $n \in \mathbb{N}$. From \mathbb{Z} -invariance, $\|\tilde{u}_n - \tilde{v}_n\| = \|u_n - v_n\| \leq \frac{1}{n}$.

Moreover,

$$\beta(\tilde{v}_n) = \beta(-[\beta(v_n)] * v_n) = -[\beta(\tilde{v}_n)] + \beta(\tilde{v}_n)$$

and, from item (iv) of Lemma 4.4, $\beta(\tilde{v}_n) \in [0, 1]$. Hence, $(\tilde{v}_n) \subset \tilde{K}_c$. Since \tilde{K}_c is compact in $H^{\frac{1}{2}}(\mathbb{R})$, considering a subsequence if necessary, $\tilde{v}_n \rightarrow v \in \tilde{K}_c$. From the continuity of β , $\beta(\tilde{v}_n) \rightarrow \beta(v) \in [0, 1]$.

On the other hand,

$$\|\tilde{u}_n - v\| \leq \|\tilde{u}_n - \tilde{v}_n\| + \|\tilde{v}_n - v\| \leq \frac{1}{n} + o(1) \rightarrow 0,$$

as $n \rightarrow +\infty$. Therefore, $\tilde{u}_n \rightarrow v$ in $H^{\frac{1}{2}}(\mathbb{R})$. Once more, from β continuity, there exists $n_0 \in \mathbb{N}$ such that $\beta(\tilde{u}_n) \in [-2, 2]$, for all $n \geq n_0$.

Define $w_n = (-[\beta(\tilde{u}_n)]) * \tilde{v}_n$. Since $\tilde{v}_n \in K_c$, $w_n \in K_c$ and $\beta(w_n) = \beta(\tilde{v}_n) - [\beta(\tilde{u}_n)]$. Consequently,

$$|\beta(\tilde{v}_n) - \beta(\tilde{u}_n) + (\beta(\tilde{u}_n) - [\beta(\tilde{u}_n)])| \leq 1 + \frac{1}{2} + 2 < 4.$$

Hence, $\beta(w_n) \in [-4, 4]$, for all $n \geq n_0$. So, we conclude that $w_n \in \tilde{K}_c$, for n sufficiently large.

Now, as $\tilde{u}_n \in L_1$, makes sense to apply h_1 . Thus,

$$\begin{aligned} \|h_1(\tilde{u}_n) - w_n\| &= \|(-[\beta(\tilde{u}_n)]) * \tilde{u}_n - (-[\beta(\tilde{u}_n)]) * \tilde{v}_n\| \\ &= \|(-[\beta(\tilde{u}_n)]) * (\tilde{u}_n - \tilde{v}_n)\| = \|\tilde{u}_n - \tilde{v}_n\| \leq \frac{1}{n}. \end{aligned}$$

Therefore, taking $n \geq \max\{n_0, \frac{1}{\bar{\rho}}\}$, we have that $w_n \in \tilde{K}_c$ and $\|h_1(\tilde{u}_n) - w_n\| \leq \bar{\rho}$, which implies that $h_1(\tilde{u}_n) \in U_{\bar{\rho}}$.

Hence, $\tilde{u}_n \in A_1$ and, since A_1 is \mathbb{Z} -invariant, $u_n \in A_1 \subset A$, which is a contradiction. So, the claim is valid.

Claim 4: $\gamma(U_{\bar{\rho}}) \leq \gamma(\tilde{K}_c)$.

If $\gamma(\tilde{K}_c) = 0$, then $\tilde{K}_c = \emptyset$, which implies $U_{\bar{\rho}} = \emptyset$ and, by definition, $\gamma(U_{\bar{\rho}}) = 0$. Suppose then, $\gamma(\tilde{K}_c) > 0$. So, we can consider a function $\tilde{g} : U_{\bar{\rho}} \rightarrow \mathbb{R}^k \setminus \{0\}$, given by $\tilde{g}(u) = g(u)$. Observe that \tilde{g} is well-defined, since $U_{\bar{\rho}} \subset H^{\frac{1}{2}}(\mathbb{R})$ and $g \neq 0$ over $U_{\bar{\rho}}$. Moreover, as g is odd and continuous, so is \tilde{g} . Consequently, from genus definition, $\gamma(U_{\bar{\rho}}) \leq k = \gamma(\tilde{K}_c)$, proving the claim.

Finally, since $\tilde{K}_c, A_{c,\rho}, U_{\bar{\rho}}, A_i, A \in \mathcal{A}$, for $i = 1, 2$, make sense to apply γ in all of this sets. Therefore, from the upper considerations, Claim 3, and standard properties of genus, for $\rho \in (0, \rho_c)$, we have

$$\begin{aligned} \gamma(A_{c,\rho}) &\leq \gamma(A) = \gamma(A_1 \cup A_2) \\ &\leq \gamma(A_1) + \gamma(A_2) = \gamma(h_1^{-1}(U_{\bar{\rho}})) + \gamma(h_2^{-1}(U_{\bar{\rho}})) \\ &\leq \sum_{i=1}^2 \gamma(U_{\bar{\rho}}) \leq \sum_{i=1}^2 \gamma(\tilde{K}_c) = 2\gamma(\tilde{K}_c) = 2k < \infty. \end{aligned}$$

Hence, taking $\rho_0 = \rho_c$, we have the proposition. \square

For the next results we will need the definition and some basic properties of relative genus. So, for convenience of the reader, we will include it here.

Definition 4.6. Let $D, Y \in \mathcal{A}$ with $D \subset Y$. We say that $U, V \in \mathcal{A}$ is a covering of Y relative to D if it satisfies

- (i) $Y \subset U \cup V$ and $D \subset U$;
- (ii) there exists an even continuous (in X) function $\chi : U \rightarrow D$, such that $\chi(u) = u$, for all $u \in D$.

If $U, V \in \mathcal{A}$ is a covering of Y relative to D , then the genus of this covering is $\gamma(V) = k$.

Definition 4.7. Let $D, Y \in \mathcal{A}$ with $D \subset Y$. We define the Krasnoselskii's genus of Y relative to D , denoted by $\gamma_D(Y)$, as

- (i) There exists a covering for Y relative to D and, in this case, $\gamma_D(Y) = k$, where k is the lowest genus of this coverings.
- (ii) If we cannot find any such covering of Y relative to D , we set $\gamma_D(Y) = +\infty$.

In the following, we list some useful properties of relative genus that are needed to guarantee that the results are valid.

Lemma 4.8. (i) Let $D \subset \mathcal{A}$. Then, $\gamma_D(D) = 0$.

(ii) Let $D, Y, Z \in \mathcal{A}$ satisfying $D \subset Y$ and $D \subset Z$. If there exists a function $\varphi : Y \rightarrow Z$, even and continuous (in X), such that $\varphi(u) = u$, for all $u \in D$, then $\gamma_D(Y) \leq \gamma_D(Z)$.

(iii) Let $D \subset Y \subset Z \in \mathcal{A}$. Then, $\gamma_D(Y) \leq \gamma_D(Z)$.

(iv) Let $D, Y, Z \in \mathcal{A}$ satisfying $D \subset Y$. Then, $\gamma_D(Y \cup Z) \leq \gamma_D(Y) + \gamma_D(Z)$.

Proof. For item (i), take $U = D, V = \emptyset$ and $\chi = id$, in the definition of relative genus. Proofs for items (ii) and (iv) can be found, for example, in [12]. Finally, item (iii) is an immediate consequence of item (ii). \square

For the next results, we define the sets

$$I^c = \{u \in X ; I(u) \leq c\}, \text{ for } c \in \mathbb{R} \text{ and } D = I^0,$$

and the values

$$c_k = \inf\{c \geq 0 ; \gamma_D(I^c) \geq k\}, \forall k \in \mathbb{N}.$$

Remark 4.9. (1) Since I is unbounded from below, $D \neq \emptyset$.

(2) Let $c_1, c_2 \in \mathbb{R}$ with $c_1 > c_2$. Then, if $u \in I^{c_2}$, $I(u) \leq c_2 < c_1$, so $u \in I^{c_1}$. That is, if $c_1 > c_2$, then $I^{c_2} \subset I^{c_1}$.

(3) If $c_1 > c_2 \geq 0$, then $D \subset I^{c_2} \subset I^{c_1}$. Consequently, $\gamma_D(I^{c_2}) \leq \gamma_D(I^{c_1})$.

(4) For $\varepsilon > 0$, $\gamma_D(I^{c_k+\varepsilon}) \geq k$ and $\gamma_D(I^{c_k-\varepsilon}) < k$, for every $k \in \mathbb{N}$.

Lemma 4.10. *We have that*

$$\inf_{u \in X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) = \inf_{u \in X \setminus \{0\}} \sup_{t > 0} I(tu) < +\infty. \tag{4.1}$$

Proof. First of all, observe that, since I is even, $\sup_{t \in \mathbb{R}} I(tu) = \sup_{t > 0} I(tu)$. Moreover, from Lemma

2.10, $\sup_{t > 0} I(tu) < +\infty$ for all $u \in X \setminus \{0\}$. □

Consider the Nehari’s manifold for I , defined by

$$\mathcal{N} = \{u \in X \setminus \{0\} ; I'(u)(u) = 0\}. \tag{4.2}$$

Lemma 4.11. *Let \mathcal{N} as in (4.2). Then,*

$$\inf_{\mathcal{N}} I = \inf_{u \in X \setminus \{0\}} \sup_{t > 0} I(tu).$$

Proof. We start proving the following statement.

Claim 1: Let $u \in X \setminus \{0\}$. Then, $t_u u \in \mathcal{N}$ and $\sup_{t > 0} I(tu) = I(t_u u)$.

Indeed, from Lemma 4.1, there exists a unique $t_u \in (0, +\infty)$ such that $\varphi'_u(t_u) = 0$, which, from Corollary 4.2, is a global maximum for φ_u , concluding the proof of claim 1.

Claim 2: Let $u \in \mathcal{N}$. Then, $\sup_{t > 0} I(tu) = I(u)$.

Since, $u \in \mathcal{N}$ and $\varphi'_u(t) = I'(tu)(u)$, we have $0 = I'(u)(u) = \varphi'_u(1)$. From Lemma 4.1, 1 is a global maximum for φ_u . Consequently, $\sup_{t > 0} I(tu) = \varphi_u(1) = I(u)$, and the claim follows.

From Lemma 4.1 and Claims 1 and 2, we obtain the result. □

Lemma 4.12. *We have $\inf_{X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) > 0$.*

Proof. Suppose that $\alpha_1 = \inf_{X \setminus \{0\}} \sup_{t \in \mathbb{R}} I(tu) = 0$. Then, there exists a sequence $(u_n) \subset X \setminus \{0\}$ such that $\sup_{t \in \mathbb{R}} I(tu_n) \rightarrow 0$. From Lemma 4.11, the definition of φ_\circ and Lemma 2.9 we can reach a contradiction. □

In order to prove the next results, we need to introduce the following sets

$$\mathcal{N}^+ = \{u \in X ; I'(u)(u) > 0\} \quad \text{and} \quad \mathcal{N}^- = \{u \in X ; I'(u)(u) < 0\}.$$

Note that $X = \{0\} \cup \mathcal{N}^+ \cup \mathcal{N} \cup \mathcal{N}^-$ and that these sets are 2 – 2 disjoint. Moreover, using the definition of a set’s boundary and Lemmas 2.9 and 4.1, it is possible to verify that $\partial \mathcal{N}^- = \mathcal{N}$ and $\partial \mathcal{N}^+ = \{0\} \cup \mathcal{N}$.

Proposition 4.13. *We have $c_1 = \inf_{\mathcal{N}} I = \inf_{u \in X \setminus \{0\}} \sup_{t > 0} I(tu) > 0$.*

Proof. Observe that, from Lemma 4.11, remains to prove that $c_1 = \inf_{\mathcal{N}} I > 0$.

Claim 1: $c_1 > 0$.

Suppose that $c_1 = 0$. Then, by c_1 definition, $\gamma_D(D) \geq 1$, which contradicts Lemma 4.8-(i). Therefore, $c_1 > 0$.

Claim 2: $c_1 \geq \inf_{\mathcal{N}} I$.

Suppose, by contradiction, that $c_1 < \inf_{\mathcal{N}} I$. Choose $c \in (c_1, \inf_{\mathcal{N}} I)$. Define the function $\mathcal{F} : I^c \rightarrow X$ by

$$\mathcal{F}(u) = \begin{cases} 0, & \text{if } u \in \{0\} \cup \mathcal{N}^+ \\ \max\{1, t'_u\}u, & \text{if } u \in \mathcal{N}^- \end{cases} .$$

Then, we mention some properties of \mathcal{F} that can be easily verify: (i) \mathcal{F} is well-defined, since $X = \{0\} \cup \mathcal{N}^+ \cup \mathcal{N} \cup \mathcal{N}^-$; (ii) by the definition of \mathcal{F} , Lemma 4.3, the continuity of max function and $\partial(\{0\} \cup \mathcal{N}^+) = \mathcal{N} = \partial\mathcal{N}^-$, \mathcal{F} is continuous; (iii) from Corollary 4.2 and direct computations, one concludes that \mathcal{F} odd; (iv) $\mathcal{F}|_D = id$, it clearly follows from Corollary 4.2 and Lemma 4.1.

Consider $U = I^c$ and $V = \emptyset$. Since I is C^1 and even, U is closed and symmetric. That is, $U \in \mathcal{A}$. Also, $V = \emptyset \in \mathcal{A}$. Moreover, $I^c = U \cup V$ and, since $c \in (c_1, \inf_{\mathcal{N}} I)$, $c > 0$ and $D \subset I^c \subset U$.

From properties (i)-(iv) and Definition 4.7, U and V is a covering for I^c relative to D . Then, once $\gamma(\emptyset) = 0$, from definition, the genus of this covering is zero. Hence, $\gamma_D(I^c) = 0$. But it gives a contradiction, since $1 \leq \gamma_D(I^{c_1}) \leq \gamma_D(I^c) = 0$. Consequently, $c_1 \geq \inf_{\mathcal{N}} I$.

Claim 3: $c_1 \leq \inf_{\mathcal{N}} I$.

For $u_0 \in X \setminus \{0\}$, without loss of generality, we can assume $\|u_0\| = 1$.

Set $d = \sup_{t>0} I(tu_0)$. Lets prove that $\gamma_D(I^d) \geq 1$. Note that, if $u \in B = \{tu_0 ; t > 0\}$, then there exists $t_0 > 0$ such that $u = t_0u_0$. Then,

$$I(u) = I(t_0u_0) \leq \sup_{t>0} I(tu_0)$$

and, consequently, $u \in I^d$. Moreover, $\inf_{u \in X \setminus \{0\}} \sup_{t>0} I(tu)$ implies $d > 0$. On the other hand, if $u \in D$, $I(u) \leq 0 < d$. So, $u \in I^d$. Hence, $D \subset B \cup D \subset I^d$.

Thus, from Lemma 4.8-(iii), $\gamma_D(B \cup D) \leq \gamma_D(I^d)$. In this point of view, we will work to prove that $\gamma_D(B \cup D) \geq 1$.

Suppose that $\gamma_D(B \cup D) = 0$. Then, by definition, $U = B \cup D$ and $V = \emptyset$, once only the empty set has null genus. Moreover, there exists a function, continuous and odd, $\chi : B \cup D \rightarrow D$ such that $\chi(u) = u$, for all $u \in D$.

Define $g : (0, +\infty) \rightarrow (0, +\infty)$ by $g(t) = \|\chi(tu_0)\|$. Note that g is continuous.

Now, consider $\rho > 0$ given by Lemma 2.9. From Corollary 4.2, there exists $\bar{t} \in (t'_{u_0}, +\infty)$, with $\bar{t} > \rho$, such that $\varphi_{u_0}(\bar{t}) < 0$. It is equivalent to $I(\bar{t}u_0) < 0$, which implies $\bar{t}u_0 \in D$.

Consequently, $g(\bar{t}u_0) = \|\chi(\bar{t}u_0)\| = \bar{t}$. On the other hand, once $0 \in D$, $g(0) = 0$. So, since g is continuous, from the Intermediate Value Theorem, there exists $t_\rho \in (0, \bar{t})$ such that $\rho = g(t_\rho) = \|\chi(t_\rho u_0)\|$. But, from Lemma 2.9, $I(\chi(t_\rho u_0)) \geq m_\rho > 0$ and we have that $\chi(t_\rho u_0) \notin D$, which is a contradiction.

Therefore, claim 3 is valid and, combined with claim 2, we have the proposition. \square

Remark 4.14. (1) We can provide an equivalent definition to the the function $\mathcal{F} : I^c \rightarrow X$ by

$$\mathcal{F}(u) = \begin{cases} 0, & \text{if } u \in \{0\} \cup \mathcal{N}^+ \\ \sigma(u)u, & \text{if } u \in \mathcal{N}^- \end{cases},$$

where $\sigma : \mathcal{N}^- \rightarrow [1, +\infty)$ is given by $\sigma(u) = \inf\{t \geq 1 ; \varphi_u(t) = I(tu) \leq 0\}$.

(2) Let $i, j \in \mathbb{N}$ with $i > j$. Then,

$$\{c \in (0, +\infty) ; \gamma_D(I^c) \geq i\} \subset \{c \in (0, +\infty) ; \gamma_D(I^c) \geq j\}.$$

Therefore, $c_j \leq c_i$.

In the following, we will consider W as a k -dimensional subspace of X . As a consequence of $(f'_1), (f'_4), V_2(u) \geq 0$ and $V_1(u) \leq 2\|u\|_X^4$, once can see that

$$I(u) \rightarrow -\infty, \text{ as } \|u\|_X \rightarrow +\infty, \text{ and } \sup_{u \in W} I(u) < +\infty. \quad (4.3)$$

Lemma 4.15. *There exists $R > 0$ such that $\{u \in W ; \|u\| \geq R\} \subset D$.*

Proof. Set $A = \{u \in W ; \|u\| \geq R\}$. Suppose, by contradiction, that for all $n \in \mathbb{N}$, there exists $u_n \in A$ such that $u_n \notin D$. Allied with the definition of D , there exists a sequence $(u_n) \subset W$ satisfying $\|u_n\| \geq n$ and $I(u_n) > 0$, for all $n \in \mathbb{N}$. But it contradicts (4.3), since for $n \in \mathbb{N}$ sufficiently large we should have $I(u_n) \leq 0$. \square

For the next results, consider $\rho > 0$ given by Lemma 2.9 and $\chi : X \rightarrow X$ a continuous and odd function, such that $\chi(u) = u$, for all $u \in D$. Define the sets

$$\mathcal{O}_\chi = \{u \in W ; \|\chi(u)\| < \rho\}.$$

For convenience, we set χ as a function having these properties, except we say otherwise.

Remark 4.16. One can easily verify that the sets \mathcal{O}_χ have the following properties:

- (1) \mathcal{O}_χ is a neighbourhood of zero in W .
- (2) \mathcal{O}_χ is bounded and symmetric.
- (3) If $u \in \partial_W \mathcal{O}_\chi$, then $\|\chi(u)\| = \rho$.

Proposition 4.17. *Let $k \in \mathbb{N}$. Then $c_k < +\infty$.*

Proof. Set $\alpha = \sup_{u \in W} I(u)$. From (4.3), $\alpha < +\infty$. Also, by definition of I^α , we see that $W \subset I^\alpha$.

Suppose that $\gamma_D(I^\alpha) < k$. Then, from Definition 4.7, there exists $U, V \in \mathcal{A}$ and $\chi : U \rightarrow D$ continuous and odd, such that $I^\alpha \subset U \cup V$, $D \subset U$, $\gamma(V) \leq k - 1$ and $\chi|_D = id$.

Since $U \subset X$ is closed, by a corollary of Tietze's Theorem, there exists a unique extension of χ , that we still denote by χ , to X , which is continuous and odd. So, the sets \mathcal{O}_χ are well-defined.

Now, from Remark 4.16 and a genus property, since $\dim W = k$, $\gamma(\partial_W \mathcal{O}_\chi) = k$.

Let $u \in \partial_W \mathcal{O}_\chi$. From Remark 4.16, $\|\chi(u)\| = \rho$. By Lemma 2.9, $I(\chi(u)) \geq m_\rho > 0$. Thus, $\chi(u) \notin D$. Equivalently, $u \notin \chi^{-1}(D)$. Once the initial function χ satisfies $\chi(U) \subset D$, we conclude that $u \notin U$. Therefore, $\partial_W \mathcal{O}_\chi \cap U = \emptyset$.

On the other hand,

$$\partial_W \mathcal{O}_\chi \subset W \subset I^\alpha \subset U \cup V \implies \partial_W \mathcal{O}_\chi \subset V.$$

Then, we have $k = \gamma(\partial_W \mathcal{O}_\chi) \leq \gamma(V) \leq k - 1$, which is a contradiction.

Therefore, from the definition of values c_k and (4.3), $c_k \leq \alpha < +\infty$. □

Before we prove that the values c_k are critical values of I , we will provide a deformation lemma. We start defining the sets $S = X \setminus A_{c,\rho}$,

$$S_\delta = \{u \in X ; \|u - v\|_X \leq \delta, \text{ for some } v \in S\}$$

and

$$\tilde{S}_\delta = \{u \in X ; \|u - v\| \leq \delta, \text{ for some } v \in S\},$$

for $c, \rho, \delta \in (0, +\infty)$. Since the proofs of Lemma 4.18 and Corollary 4.19 can be done as [12, Lemma 4.6], we omit it here.

Lemma 4.18. *Let $c, \rho > 0$. Then, there exists $\delta_0 = \delta(c, \rho) > 0$ such that, if $\delta \in (0, \delta_0)$, then $\|I'(u)\|_{X'}(1 + \|u\|_X) \geq 8\delta$, for all $u \in \tilde{S}_{2\delta}$ with $I(u) \in [c - 2\delta^2, c + 2\delta^2]$.*

Corollary 4.19. *Let $c, \rho > 0$. Then, there exists $\tilde{\delta}_0 = \tilde{\delta}_0(c, \rho) > 0$ such that, if $\delta \in (0, \tilde{\delta}_0)$, then $\|I'(u)\|_{X'}(1 + \|u\|_X) \geq 8\delta$, for all $u \in S_{2\delta}$ with $I(u) \in [c - 2\delta^2, c + 2\delta^2]$.*

Lemma 4.20. *Let $c > 0$. Then, there exists $\rho_1 = \rho_1(c) > 0$ such that, for all $\rho \in (0, \rho_1)$, we have*

(i) $A_{c,\rho} \cap D = \emptyset$.

(ii) *There exists $\varepsilon = \varepsilon(c, \rho) > 0$ and a function $\phi : I^{c+\varepsilon} \setminus A_{c,\rho} \rightarrow I^{c-\varepsilon}$, continuous and odd, such that $D \subset I^{c-\varepsilon}$ and $\phi|_D = id$.*

Proof. (i) Suppose that the statement is false. Then, for all $n \in \mathbb{N}$, there exists $\rho_n \in (0, \frac{1}{n})$ and $u_n \in A_{c,\rho_n} \cap D$. Note that $\rho_n \rightarrow 0$. Thus, by A_{c,ρ_n} definition, there exists $v_n \in K_c$ such that $\|u_n - v_n\| \leq \rho_n$, for all $n \in \mathbb{N}$.

As a consequence, $I'(v_n) = 0$ and $I(v_n) = c$, for all $n \in \mathbb{N}$. Hence, (v_n) is a Cerami sequence for I in level $c > 0$. Then, by Proposition 3.1, there exists $(y_n) \subset \mathbb{Z}$ satisfying $\tilde{v}_n = y_n * v_n \rightarrow v$ in X , where v is a non-zero critical point of I . Therefore,

$$\|\tilde{u}_n - v\| \leq \|\tilde{u}_n - \tilde{v}_n\| + \|\tilde{v}_n - v\| = \|u_n - v_n\| + \|\tilde{v}_n - v\| \rightarrow 0.$$

Finally, as K_c is closed, $v \in K_c$ and, from the \mathbb{Z} -invariance of I and A_{c,ρ_n} , $\tilde{u}_n = y_n * u_n \in A_{c,\rho_n} \cap D$. So, since $(\tilde{u}_n) \subset D$, $I(\tilde{u}_n) \leq 0$, for all $n \in \mathbb{N}$. Consequently, from Lemma 2.8-(iv),

$$c = I(v) \leq \liminf_{n \rightarrow +\infty} I(\tilde{u}_n) \leq 0,$$

which is a contradiction.

(ii) Let δ_0 as given by Corollary 4.19 and $\delta \in (0, \delta_0)$, such that $\delta^2 < \frac{c}{2}$. Take $\varepsilon = \delta^2$. Then, from [29, Deformation's Lemma 2.6], there exists $\eta : [0, 1] \times X \rightarrow X$, continuous, satisfying

- (a) $\eta(t, u) = u$, if $t = 0$ or $u \notin I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$;
- (b) $\eta(1, I^{c+\varepsilon} \cap (X \setminus A_{c,\rho})) \subset I^{c-\varepsilon}$;
- (c) $t \mapsto I(\eta(t, u))$ is non-increasing, for all $u \in X$.

Moreover, since I is even, it is possible to modify the proof of the refereed lemma, such as in [34], to obtain as well

- (d) $\eta(t, -u) = -\eta(t, u)$, for all $t \in [0, 1]$ and $u \in X$.

Define $\phi : I^{c+\varepsilon} \setminus A_{c,\rho} \rightarrow I^{c-\varepsilon}$ by $\phi(u) = \eta(1, u)$. Note that item (b) is equivalent to $\eta(1, I^{c+\varepsilon} \cap (X \setminus A_{c,\rho})) \subset I^{c-\varepsilon}$, which guarantee that ϕ is well-defined. Also, as η is continuous, ϕ is continuous as well, and from item (d), ϕ is odd.

Moreover, once $\varepsilon = \delta^2 < \frac{c}{2}$, $0 \notin [c - 2\varepsilon, c + 2\varepsilon]$. Thus, $D \cap I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) = \emptyset$. Hence, from item (a), if $u \in D$, $\phi(u) = \eta(1, u) = u$, which implies that $\phi|_D = id$.

Finally, if $u \in D$, $I(u) \leq 0 < c - \varepsilon$, then $u \in I^{c-\varepsilon}$. □

Proposition 4.21. *Let $k \in \mathbb{N}$. Then, c_k is a critical value of I .*

Proof. Suppose, by contradiction, that for all $u \in X$, with $I'(u) = 0$, $I(u) \neq c_k$, for all $k \in \mathbb{N}$. Let $\rho > 0$. From definition of K_{c_k} , we have $K_{c_k} = \emptyset$ and, consequently, $A_{c_k,\rho} = \emptyset$, for all $\rho > 0$.

From Lemma 4.20, there exists $\varepsilon = \varepsilon(c_k, \rho) > 0$ and $\phi : I^{c_k+\varepsilon} \rightarrow I^{c_k-\varepsilon}$, continuous and odd, such that $\phi|_D = id$. Then, from Lemma 4.8-(ii), $\gamma_D(I^{c_k+\varepsilon}) \leq \gamma_D(I^{c_k-\varepsilon})$.

On the other hand, as $c_k - \varepsilon < c_k + \varepsilon$, $D \subset I^{c_k-\varepsilon} \subset I^{c_k+\varepsilon}$ and then, by Lemma 4.8-(iii), $\gamma_D(I^{c_k+\varepsilon}) \geq \gamma_D(I^{c_k-\varepsilon})$. Consequently, $\gamma_D(I^{c_k+\varepsilon}) = \gamma_D(I^{c_k-\varepsilon})$, which is a contradiction, since $\gamma_D(I^{c_k+\varepsilon}) \geq k$ and $\gamma_D(I^{c_k-\varepsilon}) < k$.

Therefore, c_k is a critical value of I . □

Proposition 4.22. *We have $c_k \rightarrow +\infty$, as $k \rightarrow +\infty$.*

Proof. Suppose that there exists $M > 0$, such that $c_k < M$, for all $k \in \mathbb{N}$. From Remark 4.14, c_k is monotonically nondecreasing. Then, there exists $c > 0$ such that $c_k \rightarrow c$.

From Proposition 4.5, there exists $\rho_0 > 0$ such that $\gamma(A_{c,\rho}) < +\infty$, for all $\rho \in (0, \rho_0)$. Also, by Lemma 4.20-(i), there exists $\rho_1 > 0$ such that $A_{c,\rho} \cap D = \emptyset$, for all $\rho \in (0, \rho_1)$. Set $\rho_2 = \min\{\rho_0, \rho_1\}$ and let $\rho \in (0, \rho_2)$.

From Lemma 4.20-(ii), there exists $\varepsilon > 0$ and $\phi : I^{c+\varepsilon} \setminus A_{c,\rho} \rightarrow I^{c-\varepsilon}$, continuous and odd, such that $\phi|_D = id$ and $D \subset I^{c-\varepsilon}$. Moreover, once $c_k \rightarrow c$ monotonically nondecreasing, there exists $k_0 \in \mathbb{N}$, such that $c_{k_0} \geq c - \varepsilon$. Then, by Lemma 4.8-(ii),

$$\gamma_D(I^{c+\varepsilon} \setminus A_{c,\rho}) \leq \gamma_D(I^{c-\varepsilon}) < k_0 < +\infty.$$

Consequently, from Lemma 4.8-(iv),

$$\gamma_D(I^{c+\varepsilon}) = \gamma_D((I^{c+\varepsilon} \setminus A_{c,\rho}) \cup A_{c,\rho}) \leq \gamma_D(I^{c+\varepsilon} \setminus A_{c,\rho}) + \gamma(A_{c,\rho}) < +\infty.$$

Once more, as $c_k \rightarrow c$ monotonically nondecreasing, $c + \varepsilon > c \leq c_k$, for all $k \in \mathbb{N}$. Also, by definition of values c_k , $\gamma_D(I^{c+\varepsilon}) \geq k$, for all $k \in \mathbb{N}$, that is $\gamma_D(I^{c+\varepsilon}) \rightarrow +\infty$, as $k \rightarrow +\infty$, which leads to a contradiction.

Therefore, $c_k \rightarrow +\infty$, as $k \rightarrow +\infty$. □

Proof of Theorem 1.2. From Proposition 4.22, we can extract a subsequence of (c_k) such that $c_k \rightarrow +\infty$ monotonically increasing. Then, from Proposition 4.21, there exists $u_k \in X$ satisfying $I(u_k) = c_k$ and $I'(u_k) = 0$, for all $k \in \mathbb{N}$. Since (c_k) is monotone, $c_i \neq c_j$, when $i \neq j$, and $c_k > c_1 > 0$, for all $k \geq 2$. Then, we conclude that the functions u_k are distinct and that $u_k \neq 0$, for all $n \in \mathbb{N}$. Also, as I is even, the same holds for $-u_k$ and $I(\pm u_k) \rightarrow +\infty$. □

5 Final Considerations

In this final section we call attention the reader that it is possible to adapt the results and techniques derived in this work to solve versions of equation (1.1) considering a continuous potential $a : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions

$$(a_0) \quad a : \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous, } \mathbb{Z}\text{-periodic, } a \in L^\infty(\mathbb{R}) \text{ and } \inf_{x \in \mathbb{R}} a(x) = a_0 > 0.$$

One also could investigate the case where the potential a is not invariant under \mathbb{Z} translations but is an asymptotically \mathbb{Z} -periodic function, that is, there exists a \mathbb{Z} -periodic potential $a_p : \mathbb{R} \rightarrow \mathbb{R}$ such that a_p satisfies (a_0) ,

$$(a_1) \quad 0 < \inf_{x \in \mathbb{R}} a(x) \leq a(x) \leq a_p(x), \quad \forall x \in \mathbb{R}^N$$

and

$$(a_2) \quad \lim_{|x| \rightarrow +\infty} |a(x) - a_p(x)| = 0.$$

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Received: September 12, 2021.

Accepted: January 30, 2022.