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NEW CONSTRUCTIONS OF K-FRAMES IN HILBERT $\mathbb{C}^*$-MODULES

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Abstract Nowadays, frame theory plays an important role in many areas and fields, from applied mathematics to engineering applications such as image processing and sampling theory. In the present paper, we will study some algebraic structures of K-frames for divisible submodule in Hilbert $\mathbb{C}^*$-modules and we are often interested in constructing some new K-frames.

1 Introduction and Preliminaries

Frames were first introduced by Duffin and Schaeffer [8] and used them as a tool in the study of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [6]. Today, frame has been a useful tool in many areas such signal processing [11], sampling theory [9]. For special applications, various generalizations of frames were proposed, such as K-frames by L. Găvruta [15] to study the atomic systems with respect to a bounded linear operator. Khosravi and Asgari [21] introduced frames in tensor product of Hilbert spaces that is useful in the approximation of multi-variate functions of combinations of univariate ones. Further, we pay attention that Our methods are derived from abelian group theory, which must be translated to $\mathcal{A}[X]$-modules and then specialized to the case of $\mathcal{A}[X]$-modules before it can be applied to linear algebra [14, 20]. Frank and Larson [13] introduced the notion of frames in Hilbert $\mathbb{C}^*$-module as a generalization of frames in Hilbert spaces. For background material on frame theory and related topics, we refer the reader to [5, 3, 4].

Definition 1.1 ([28]). A left Hilbert $\mathbb{C}^*$-module over the unital $\mathbb{C}^*$-Algebra $\mathcal{A}$ is a left $\mathcal{A}$-module $\mathcal{H}$ equipped with an $\mathcal{A}$-valued inner product:

$$\langle x, y\rangle \quad : \quad \mathcal{H}\times\mathcal{H} \quad\longrightarrow\quad \mathcal{A}$$

satisfying the following conditions:

1. $\langle x, x\rangle \geq 0$, for all $x \in \mathcal{H}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;
2. $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$, for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$;
3. $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in \mathcal{H}$;
4. $\mathcal{H}$ is complete with respect to the norm $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We also reserve the notation $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is abbreviated to $End^*_{\mathcal{A}}(\mathcal{H})$.

For $T \in End^*_{\mathcal{A}}(\mathcal{H})$, we denote by $R(T)$ and $N(T)$ the range and the kernel subspaces of T respectively. Throughout this paper, we suppose that $\mathcal{H}$ is a Hilbert $\mathbb{C}^*$-module and $\mathbb{J}$ is a countable index set of $\mathbb{N}$.

Example 1.2. Let us consider

$$l^2(\mathbb{C}) = \Big\{\{a_n\}_n \subset \mathbb{C} : \sum_{n=1}^{\infty} | a_n |^2 < \infty\Big\},$$

and

$$\mathcal{A}=\Big\{\{a_n\}_n\subset\mathbb{C}:\sup_{n\geq 0}\{\mid a_n\mid\}<\infty\Big\}.$$

It is easy to see that $l^2(\mathbb{C})$ with the $\mathcal{A}$-valued inner product

$$\langle\{a_n\},\{b_n\}\rangle=\sum_{n\geq 0}a_nb_n^*$$

is a Hilbert $\mathbb{C}^*$-module which is called the standard Hilbert $\mathbb{C}^*$-module over $\mathcal{A}$.

Example 1.3. If $\{\mathcal{H}_k\}_{k\in\mathbb{N}}$ is a countable set of Hilbert $\mathcal{A}$-modules, then their sum direct $\bigoplus_{k\in\mathbb{N}}\mathcal{H}_k$ is Hilbert $\mathcal{A}$-module. We define the inner product by

$$\langle x,y\rangle=\sum_{k\in\mathbb{N}}\langle x_k,y_k\rangle,\ \text{for all } x_k,y_k\in\mathcal{H}_k.$$

An operator T is called positive if

$$\langle Tx,x\rangle\geq 0,\ for\,all\,x\in\mathcal{H}.$$

Recall first that the center of $\mathcal{A}$ is defined as follows

$$Z(\mathcal{A})=\{a\in\mathcal{A}:ab=ba,\ for\,all\,b\in\mathcal{A}\}.$$

Remark 1.4. Let $T\in End^*_{\mathcal{A}}(\mathcal{H})$ and $a\in Z(\mathcal{A})$, then aT is adjointable.

Definition 1.5. [13] Let $\mathcal{H}$ be a Hilbert $\mathbb{C}^*$-module. A sequence $\{x_j\}_{j\in\mathbb{J}}$ is said to be a frame if there exist constants $\lambda,\mu>0$ such that

$$\lambda\langle x,x\rangle\leq\sum_{j\in\mathbb{J}}\langle x,x_j\rangle\langle x_j,x\rangle\leq\mu\langle x,x\rangle,\ (x\in\mathcal{H}).$$

The constants λ,μ are called frame bounds.

If just the last inequality in the above definition holds, we say that $\{x_j\}_{j\in\mathbb{J}}$ is a Bessel sequence. The operator

$$\Phi:\mathcal{H}\to l^2(\mathcal{A})\quad defined\ by:\ \ \Phi(x)=\{\langle x,x_i\rangle\}_{i\in\mathbb{I}}.$$

is called the analysis operator and it's adjoint operator is given by:

$$\Phi^*:l^2(\mathcal{A})\to\mathcal{H},\quad defined\ by:\ \ \Phi^*(a)=\sum_{j\in\mathbb{J}}a_jx_j,\quad a=(a_j)_{j\in\mathbb{J}}\in l^2(\mathcal{A})$$

is called synthesis operator.
The frame operator for $\{x_j\}_{j\in\mathbb{J}}$ is defined by

$$S:\mathcal{H}\to\mathcal{H},\ \ S(x)=\Phi(\Phi^*(x))=\sum_{j\in\mathbb{J}}\langle x,x_j\rangle x_j$$

For each x $\in\mathcal{H}$

$$\langle Sx,x\rangle=\sum_{j\in\mathbb{J}}\langle x,x_j\rangle\langle x_j,x\rangle.$$

Then, S is bounded positive, self-adjoint. Moreover, S verifie

$$\lambda I\leq S\leq\mu I.$$

Thus, S is invertible.

Definition 1.6. [15] Let $K\in End^*_{\mathcal{A}}(\mathcal{H})$. The sequence $\{x_j\}_{j\in\mathbb{J}}$ is called a K-frame for $\mathcal{H}$, if there exist constants $\lambda,\mu>0$ such that

$$\lambda\langle K^*x,K^*x\rangle\leq\sum_{j\in\mathbb{J}}\langle x,x_j\rangle\langle x_j,x\rangle\leq\mu\langle x,x\rangle,\ \ for\,all\,x\in\mathcal{H}.$$

Later we will need the following lemmas to prove our mains results.

Lemma 1.7. *[27] Let $\{x_j\}_{j\in\mathbb{J}}$ be a Bessel sequence. Then $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$ if and only if there exists $\alpha > 0$ such that*

$$S \geq \alpha K K^*.$$

where S is the frame operator for $\{x_j\}_{j\in\mathbb{J}}$.

Lemma 1.8. *[28] Let $\mathcal{H}$ be a Hilbert $\mathbb{C}^*$-module and $T \in End^*_{\mathcal{A}}(\mathcal{H})$. Then*

$$\langle Tx, Tx\rangle \leq \| T \|^2 \langle x, x\rangle, \; for\, all\, x \in \mathcal{H}.$$

Lemma 1.9. *([30]) Let $\mathcal{H}$ be a Hilbert $\mathbb{C}^*$-module and T,G $\in End^*_{\mathcal{A}}(\mathcal{H})$. If $R(G)$ is closed, then the following statements are equivalent:*

1. $R(T) \subseteq R(G)$.
2. $TT^* \leq \alpha GG^*$, *for some* $\alpha > 0$.

It is well known that $\mathcal{H}$ has a structure of $\mathcal{A}[X]$-module defined by setting $px = p(T)x$, for every $x \in \mathcal{H}$ and $p \in \mathcal{A}[X]$. We denote $\mathcal{H}_T$ the module obtained and we also denote pK by a bounded linear operator on $\mathcal{H}$ defined as follows:

$$pK(x) = p(T)(Kx), \; (x \in \mathcal{H}_T).$$

A subspace $\mathbb{M} \subset \mathcal{H}$ is said to be T-invariant if $T(\mathbb{M}) \subset \mathbb{M}$. The lattice of all T-invariant subspaces in $\mathcal{H}$ will be denoted by $Lat(T,\mathcal{H})$.
Obviously, we have

$$Lat(T,\mathcal{H}) \subset Lat(p(T),\mathcal{H}), \; for\, all\, p \in \mathcal{A}[X].$$

Motivated by the work of [25], we introduce the concept of divisible submodule in Hilbert $\mathbb{C}^*$-module. We refer to [14, 17, 23], for further information.
In the sequel, we take $(\mathcal{A}[X])^* = \mathcal{A}[X] - \{0\}$.

Definition 1.10. ([17]) Let T be $\mathcal{A}$-lineair map and $\mathbb{M} \in Lat(T,\mathcal{H})$. We shall say that $\mathbb{M}$ is T-divisible if

$$\mathbb{M} = p\mathbb{M}, \; for\, all\, p \in (\mathcal{A}[X])^*.$$

More precisely, for all $x \in \mathbb{M}$ and $p \in (\mathcal{A}[X])^*$, there exists $y \in \mathbb{M}$ such that

$$x = py = p(T)y.$$

Example 1.11. let $\mathbb{E}$ be a complex vector space and T be a lineair map on $\mathbb{E}$. We suppose that there exists an irreducible polynomial q in $\mathbb{C}[X]$ such that

$$\mathbb{E} = \bigcup_{k\geq 1} N\left(q^k(T)\right).$$

If $\mathbb{M} \in Lat(T,\mathbb{E})$ such that $\mathbb{M} \subset q(T)\mathbb{M}$. Then $\mathbb{M}$ is T-divisible. Indeed
For $x \in \mathbb{M}$, there exists $k_0 \geq 1$ such that

$$q^{k_0}(T)(x) = 0.$$

If $p \in \mathbb{C}[X]$ is coprime with q, then there exist two polynomials u, v such that

$$up + vq^{k_0} = 1.$$

Thus

$$x = p(T)u(T)x.$$

So

$$\mathbb{M} \subset p(T)\mathbb{M}.$$

Then, $\mathbb{M}$ is T-divisible.
Otherwise, there exists $n \geq 1$ and $q' \in \mathbb{C}[X]$ coprime with q such that

$$p = q^n q'.$$

So, there exist $u, v \in \mathbb{C}[X]$ such that

$$uq' + vq^k = 1$$

hence

$$x = q'(T)\, u(T)(x).$$

Thus

$$\mathbb{M} \subset q'(\mathbb{M}).$$

Since

$$\mathbb{M} \subset q^n(T)\, \mathbb{M}.$$

Then

$$\mathbb{M} \subset p(T)\, \mathbb{M}$$

So, $\mathbb{M}$ is T-divisible.

Example 1.12. Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ such that

$$T(e_1) = e_2 \text{ and } T(e_2) = e_1,$$

and

$$N(T) \subset \bigoplus_{k \geq 3} \mathbb{C} e_k.$$

Consider

$$\mathbb{M} = \mathbb{C}e_1 + \mathbb{C}e_2$$

By a simple calculations, we conclude that

$$\mathbb{M} = (T - \lambda)\, \mathbb{M}, \;\; for\, all\, \lambda \in \mathbb{C}.$$

Since $\mathbb{C}[X]$ is a principal ring. Then

$$\mathbb{M} = p(T)\, \mathbb{M}$$

Consequently, $\mathbb{M}$ is T-divisible.

Now, Let us consider $\{\mathcal{H}_k\}_{1 \leq k \leq n}$ be a countable set of Hilbert $\mathcal{A}$-modules and the following map :

$$\bigoplus_{1 \leq k \leq n} T_k : \bigoplus_{1 \leq k \leq n} \mathcal{H}_k \longrightarrow \bigoplus_{1 \leq k \leq n} \mathcal{H}_k$$

defined as follow

$$\left(\bigoplus_{1 \leq k \leq n} T_k \right) \left(\bigoplus_{1 \leq k \leq n} a_k \right) = T_1 a_1 \oplus T_2 a_2 ... \oplus T_n a_n, \;\; for\, all\, a_k \in \mathcal{H}_k.$$

So, it is easy to check the following result

$$\bigoplus_{1 \leq k \leq n} Lat\,(T_k, \mathcal{H}_k) \subset Lat \left(\bigoplus_{1 \leq k \leq n} T_k, \bigoplus_{1 \leq k \leq n} \mathcal{H}_k \right).$$

Proposition 1.13. *[14]Let T_k be $\mathcal{A}$-linear map and $\mathbb{M}_k$ be T_k-divisible. Then $\bigoplus_{1 \leq k \leq n} (\mathbb{M}_k, T_k)$ is $\left(\bigoplus_{1 \leq k \leq n} T_k \right)$-divisible, for every $1 \leq k \leq n$.*

Example 1.14. Let $\mathbb{E}_j$ be a complex vector space and T_j be a linear map on $\mathbb{E}_j$ such that, for each $1 \leq j \leq n$.

$$\mathbb{E}_j = N\left((T_j - \lambda_j)^{k_j}\right), (\lambda_j \in \mathbb{C}).$$

Suppose $\mathbb{M}_j \in Lat(T_j, \mathbb{E}_j)$ such that

$$\mathbb{M}_j \subset (T - \lambda_j I)\mathbb{M}_j.$$

Thus, similarly of the example 1.3, we obtain that $\mathbb{M}_j$ is T_j-divisible. Therefore $\bigoplus_{1\leq j\leq n}(\mathbb{M}_j, A_j)$ is $\left(\bigoplus_{1\leq j\leq n} A_j\right)$-divisible.

In addition, the concept of semi-regularity was originated by Kato's classical treatment [18] of perturbation theory and its has benefited from the work of many authors in the last years, in particular from the work of Mbekhta and Ouahab [26].

Definition 1.15. [26] An operator $T \in End^*_{\mathcal{A}}(\mathcal{H})$ is said to be semi-regular if $R(T)$ is closed and $N\left(T^k\right) \subset R(T)$, for every $k \geq 1$.

Obviously, all surjective and injective with closed range operators are semi-regular operators. Some examples of semi-regular operators may be found in [22].

Proposition 1.16. *[2] Assume that $T \in End^*_{\mathcal{A}}(\mathcal{H})$ is semi-regular and $G \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $TGT = T$. Then*

$$T^n G^n T^n = T^n, \ for\ all\ n \geq 1.$$

Xu and Sheng [32] showed that a bounded adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range.

Definition 1.17. ([30]) The pseudo-inverse of an operator $T \in End^*_{\mathcal{A}}(\mathcal{H})$ with closed range is defined as the unique operator $T^\dagger \in End^*_{\mathcal{A}}(\mathcal{H})$ such that

$$TT^\dagger u = u, \quad for\ all\ u \in R(T).$$

The main purpose of this manuscript is to describe some algebraic structure of the set of K-frames for divisible submodule in Hilbert $\mathbb{C}^*$-modules and we mainly give some operators preserving K-frames.

2 Maint results

In the sequel, we will assume that $K \in End^*_{\mathcal{A}}(\mathcal{H}_T)$ with closed range such that $KT = TK$ and we fix the following notation

$$\mathbb{D}_T = \{\mathbb{M} \subset \mathcal{H}_T : \ \mathbb{M}\ is\ T - divisible\}.$$

The next lemma will be useful as a tool in the our paper.

Proposition 2.1. *Let $\mathbb{M} \in \mathbb{D}_T$ and $p \in (Z(\mathcal{A})[X])^*$. Then there exists $\lambda > 0$ such that*

$$\lambda\langle K^*x, K^*x\rangle \leq \langle K^*px, K^*px\rangle, \ (x \in \mathbb{M}).$$

Proof. Let $x \in \mathbb{M}$ and $q \in (Z(\mathcal{A})[X])^*$. Then, there exists $y \in \mathbb{M}$ such that

$$x = qy = q(T)y.$$

So, we have

$$K(x) = K(qy) = Kq(T)y.$$

Thus

$$R(K) \subset R(Kq(T)).$$

Since

$$R(Kq(T)) = K(R(q(T))).$$

Then

$$R(K) = R(Kq(T))$$

Using lemma 1.9, there exists $\lambda > 0$ such that

$$\lambda\langle K^*x, K^*x\rangle \leq \langle (Kq(T))^* x, (Kq(T))^* x\rangle.$$

Now, let $p \in (Z(\mathcal{A})[X])^*$ such that $p(T) = (q(T))^*$. Then

$$\lambda\langle K^*x, K^*x\rangle \leq \langle K^*px, K^*px\rangle.$$

Which completes the proof. □

Theorem 2.2. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}_T$ and $\mathbb{M} \in \mathbb{D}_T$. Then $\{px_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathbb{M}$ with frame operator S_p defined by*

$$S_p = \sum_{0\leq i,k\leq n} a_{i,k}T^iST^k, \ \ (a_{i,k} \in Z(\mathcal{A}))\,.$$

for all $p \in (Z(\mathcal{A})[X])^*$.

Proof. Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}_T$ with frame bounds λ and μ. Then, for all $x \in \mathbb{M}$

$$\lambda\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq \mu\langle x, x\rangle.$$

So, there exists y $\in \mathbb{M}$ such that

$$x = qy = q(T)\,y, \ for\,all\,q \in (Z(\mathcal{A})[X])^*,$$

thus

$$\lambda\langle K^*qy, K^*qy\rangle \leq \sum_{j\in\mathbb{J}}\langle qy, x_j\rangle\langle x_j, qy\rangle \leq \mu\langle qy, qy\rangle, \ for\,all\,y \in \mathbb{M}.$$

Hence

$$\lambda\langle K^*qy, K^*qy\rangle \leq \sum_{j\in\mathbb{J}}\langle y, q(T)^* x_j\rangle\langle q(T)^* x_j, y\rangle \leq \mu\langle q(T)\,y, q(T)\,y\rangle, \ for\,all\ \ y \in \mathbb{M}.$$

From Proposition 2.1, there exists $\alpha > 0$ such that

$$\alpha\langle K^*x, K^*x\rangle \leq \langle K^*qy, K^*qy\rangle.$$

So, there exists $\lambda' = \alpha\lambda > 0$ and $\mu' = \mu \,||\, q(T)\,||^2 > 0$ such that

$$\lambda'\langle K^*y, K^*y\rangle \leq \sum_{j\in\mathbb{J}}\langle y, q(T)^* x_j\rangle\langle q(T)^* x_j, y\rangle \leq \mu'\langle y, y\rangle, \ for\,all\ \ y \in \mathbb{M}.$$

Now, let $p \in (Z(\mathcal{A})[X])^*$ such that $p(T) = q(T)^*$, we can write

$$\lambda'\langle K^*y, K^*y\rangle \leq \sum_{j\in\mathbb{J}}\langle y, px_j\rangle\langle px_j, y\rangle \leq \mu'\langle y, y\rangle, \ for\,all\,y \in \mathbb{M}.$$

Therefore, $\{px_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathbb{M}$.
On the other hand, let S be the frame operator for $\{x_j\}_{j\in\mathbb{J}}$. Then, for $x \in \mathbb{M}$

$$S(x) = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle.x_j$$

So, we have

$$p(T)\,Sp^*(T)\,y = \sum_{j\in\mathbb{J}}\langle y, p(T)\,x_j\rangle p(T)\,x_j, \ \ for\,all\,y \in \mathbb{M}.$$

Then, the frame operator for $\{px_j\}_{j\in\mathbb{J}}$ is defined as follow

$$S_p = p(T)\, Sp^*(T).$$

By letting $p(X) = \sum_{k=0}^{n} a_k X^k$ and $a_{i,k} = a_i a_k^*$, we also obtain

$$S_p = \sum_{0\leq i,k\leq n} a_{i,k} T^i S T^k.$$

This complete the proof. □

Example 2.3. Let $\{e_i\}_{i\geq 1}$ be an orthonormal basis of $l^2(\mathbb{C})$ with inner product defined by:

$$\Big\langle (a_n), (b_n) \Big\rangle = \sum_{n\geq 1} a_n \bar{b_n},\ (a_n) \in \mathbb{C}.$$

Let $K \in \mathcal{B}\left(l^2(\mathbb{C})\right)$ defined as follows

$$K(x_1, x_2, ..) = (x_1, x_2, 0, 0, .., 0).$$

Since $K = K^*$, then

$$\langle K^*x, K^*x\rangle = \mid x_1 \mid^2 + \mid x_2 \mid^2.$$

Now, Let $\{\theta_i\}_{i\in\mathbb{I}}$ be a sequence of $l^2(\mathbb{C})$ such that

$$\theta_1 = e_1,\ \theta_{2k+1} = (0,0,0,..)\ \ and\ \theta_{2k} = \frac{1}{\sqrt{2k}} e_{2k},\ for\, k \neq 0$$

For $(x_1, x_2, ..) \in l^2(\mathbb{C})$, we have

$$\sum_{j\geq 1} \langle x, \theta_j\rangle.\langle \theta_j, x\rangle = \mid x_1 \mid^2 + \sum_{j\geq 1} \frac{\mid x_{2j} \mid^2}{2j}.$$

Thus

$$\frac{1}{8}\langle K^*x, K^*x\rangle \leq \sum_{j\geq 1} \langle x, \theta_j\rangle.\langle \theta_j, x\rangle \leq \langle x, x\rangle.$$

Then $\{\theta_j\}_{j\in\mathbb{I}}$ is a K-frame for $\mathcal{H}$.
Let $T \in \mathcal{B}\left(l^2(\mathbb{C})\right)$ defined by

$$T(x_1, x_2, ..) = (x_2, x_1, 0, 0, .., 0)$$

and Let $\mathbb{M} \subset l^2(\mathbb{C})$ defined by

$$\mathbb{M} = \mathbb{C}e_1 + \mathbb{C}e_2.$$

Using the Example 1.4, we obtain that $\mathbb{M}$ is T-divisible.
For $\lambda \in \mathbb{C}$ and $j \geq 2$, we have

$$(T-\lambda)\,\theta_1 = e_2 - \lambda e_1\,;\quad (T-\lambda)\,\theta_2 = \frac{1}{\sqrt{2}}\,(e_1 - \lambda e_2)\ \ ; (T-\lambda)\,\theta_{2j} = -\frac{\lambda}{\sqrt{2}} e_{2j}.$$

For $m = m_1 e_1 + m_2 e_2 \in \mathbb{M}$, A straightforward calculation gives

$$\sum_{j\geq 1} \langle m, (T-\lambda)\,\theta_j\rangle.\langle (T-\lambda)\,\theta_j, m\rangle = \langle m, (T-\lambda)\,\theta_1\rangle.\langle (T-\lambda)\,\theta_1, m\rangle + \sum_{j\geq 1} \langle m, (T-\lambda)\,\theta_{2j}\rangle.\langle (T-\lambda)\,\theta_{2j}, m$$

$$= \mid \lambda m_1 \mid^2 + \mid m_2 \mid^2 + \frac{1}{2}\left(\mid \lambda m_2 \mid^2 + \mid m_1 \mid^2\right)$$

$$= \left(\mid \lambda \mid^2 + \frac{1}{2}\right) \mid m_1 \mid^2 + \left(1 + \frac{1}{2} \mid \lambda \mid^2\right) \mid m_2 \mid^2.$$

Therefore

$$\frac{1}{2}\langle K^{*}m,K^{*}m\rangle\leq\sum_{j\geq1}\langle m,(T-\lambda)\,\theta_{j}\rangle.\langle(T-\lambda)\,\theta_{j},m\rangle\leq\left(1+\mid\lambda\mid^{2}\right)\langle m,m\rangle,\ \ (m\in\mathbb{M})\,.$$

Since $\mathbb{C}[X]$ is a principal ring, there exist $\alpha,\beta>0$ such that

$$\alpha\langle K^{*}m,K^{*}m\rangle\leq\sum_{j\geq1}\langle m,p\,(T)\,\theta_{j}\rangle.\langle p\,(T)\,\theta_{j},m\rangle\leq\beta\langle m,m\rangle.$$

where $p\,(T)=a\prod_{1\leq i\leq n}\left(T-\lambda_{i}\right)^{n_{i}},\ a\in\mathbb{C}\ and\ n_{i}\in\mathbb{N}$.

Example 2.4. Let $T\in End^{*}_{\mathcal{A}}(\mathcal{H})$ be self-adjoint such that $\|\ T\ \|<1$ and $\mathbb{M}\in\mathbb{D}_{T}$. We define the exponential operator of T by :

$$e^{T}=\sum_{k\in\mathbb{N}}\frac{1}{k!}T^{k}.$$

If $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}_T$. Then $\{e^{T}\left(x_j\right)\}_{j\in\mathbb{J}}$ is a K-frame for $\mathbb{M}$.

In [1], we Recall that the left $\mathcal{A}$-module $\mathcal{R}$ is finitely generated if there exist $r_1,r_2,...,r_n$ in $\mathcal{R}$ such that for any x in $\mathcal{R}$, there exist $a_1,a_2,...,a_n$ in $\mathcal{A}$ with

$$x=\sum_{k=1}^{n}r_k a_k.$$

Proposition 2.5. *The following set*

$$\Gamma=\{S_p:\ p\in\left(Z\left(\mathcal{A}\right)[X]\right)^{*}\}\cup\{0\}.$$

is a finitely generated module over $Z\left(\mathcal{A}\right)$.

Proof. Let us consider $p,q\in\left(Z\left(\mathcal{A}\right)[X]\right)^{*}$ such that

$$S_p=\sum_{0\leq i,k\leq m}a_{i,k}T^{i}ST^{k},$$

and

$$S_q=\sum_{0\leq i',k'\leq m}b_{i',k'}T^{i'}ST^{k'}.$$

If $n<m$, we put

$$a_{i,k}=0,\ \ for\,n\leq i,k\leq m.$$

Then

$$S_p+S_q=\sum_{0\leq i,k\leq m}\left(a_{i,k}+b_{i,k}\right)T^{i}ST^{k}=\sum_{0\leq i,k\leq m}c_{i,k}T^{i}ST^{k}.$$

Then, Γ is a finitely generated module over $Z\left(\mathcal{A}\right)$. □

The following proposition shows that the sum of two K-frames under certain conditions is again a K-frame.

Proposition 2.6. *Let $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ be two K-frames for $\mathcal{H}$ with frame operator S_1 and S_2 respectively and let the corresponding analysis operators be Φ and ψ respectively. Let α_1,α_2 be a lower K-frame bound of $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ respectively. If there exists $\beta\in\mathbb{R}$ such that $\phi^{*}\psi+\psi^{*}\phi\geq\beta KK^{*}$ and $2\alpha+\beta>0$, where $\alpha=\inf\{\alpha_1,\alpha_2\}$. Then $\{x_j+y_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$.*

Proof. Let $x \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in\mathbb{J}}\langle x, x_j+y_j\rangle.\langle x_j+y_j, x\rangle &= \langle S_1(x), x\rangle + \langle S_2(x), x\rangle + \sum_{j\in\mathbb{J}}\langle x, y_j\rangle.\langle x_j, x\rangle + \sum_{j\in\mathbb{J}}\langle x, x_j\rangle.\langle y_j, x\rangle. \\ &= \langle S_1(x), x\rangle + \langle S_2(x), x\rangle + \langle \phi^*\psi x, x\rangle + \langle \psi^*\phi x, x\rangle.\end{aligned}$$

Since

$$\phi^*\psi + \psi^*\phi \geq \beta KK^*.$$

Hence

$$\sum_{j\in\mathbb{J}}\langle x, x_j+y_j\rangle.\langle x_j+y_j, x\rangle \geq (2\alpha+\beta)\, KK^*.$$

Applied the Minkowski's inequality, it is easy to check that $\{x_j+y_j\}_{j\in\mathbb{J}}$ is a bessel sequence for $\mathcal{H}$. By Lemma 1.7, we deduce that $\{x_j+y_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$.

□

Now, we will describe the algebraic structure of K-frames for $\mathbb{D}_T$.

Theorem 2.7. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}_T$. Then the following set*

$$\Gamma = \{x_j : j \in \mathbb{J}\} \cup \{0\}$$

is a left $(Z(\mathcal{A})[X])^*$-module.

Proof. It follows immediately from Theorem 2.2 and Proposition 2.6. □

In the following corollary, we give some new K-frames for $\mathcal{H}_{T_1} \bigoplus \mathcal{H}_{T_2}$.

Corollary 2.8. *Let $\{x_j\}_{j\in\mathbb{J}}$ and $\{y_j\}_{j\in\mathbb{J}}$ be two K-frames for $\mathcal{H}_{T_1}$ and $\mathcal{H}_{T_2}$, respectively. Then $\{X^n x_j \bigoplus X^p y_j\}_{j\in\mathbb{J}}$ is a K-frame for $(\mathbb{M}_1 \bigoplus \mathbb{M}_2, T_1 \bigoplus T_2)$, where $\mathbb{M}_k \in \mathbb{D}_{T_k}$, for all n, $p \in \mathbb{N}$ and k=1, 2.*

Proof. Result from Proposition 1.13 and Theorem 2.7. □

The tensor product of Hilbert spaces is a certain linear space of operators which was represented by Folland in [12], Kadison and Ringrose in [19].

Definition 2.9. [24]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. We define the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ as follows

$$\begin{array}{cccccc} u\otimes v & : & \mathcal{H}_2' & \longrightarrow & \mathcal{H}_1 \\ & & f & \longmapsto & f(v)\,u. \end{array}$$

For more details see [24, 29].

Theorem 2.10. *Let $\{x_j\}_{j\in\mathbb{J}}$ be K_1-frame for $\mathcal{H}_{T_1}$ with frame bounds λ_1 and μ_1. Let $\{y_j\}_{j\in\mathbb{J}'}$ be K_2-frame for $\mathcal{H}_{T_2}$ with frame bounds λ_2 and μ_2 and $\mathbb{M}_i \in \mathbb{D}_{T_i}$. Then, $\{(p \bigotimes q)(x_j \bigotimes y_j)\}_{j\in\mathbb{J}, j'\in\mathbb{J}'}$ is a $(K_1 \bigotimes K_2)$-frame for $(\mathbb{M}_1 \bigotimes \mathbb{M}_2)$ with frame operator defined as follows*

$$S_{p\otimes q}(z) = (S_p \otimes S_q)(z),\ \ z \in \left(\mathbb{M}_1 \bigotimes \mathbb{M}_2\right).$$

for all $p, q \in (Z(\mathcal{A})[X])^*$.

Proof. By Theorem 2.1, we have

$$\lambda_1\langle K_1^* x, K_1^* x\rangle \leq \sum_{j\in\mathbb{J},}\langle x, px_j\rangle\langle px_j, x\rangle \leq \mu_1\langle x, x\rangle,\ for\ all\ x \in \mathbb{M}_1$$

$$\lambda_2\langle K_2^* y, K_2^* y\rangle \leq \sum_{j\in\mathbb{J}}\langle y, qy_{j'}\rangle\langle qy_{j'}, y\rangle \leq \mu_2\langle y, y\rangle,\ for\ all\ y \in \mathbb{M}_2.$$

Thus

$$\lambda_1 \langle K_1^* x, K_1^* x\rangle \otimes \lambda_2 \langle K_2^* y, K_2^* y\rangle \leq \sum_{j\in\mathbb{J}, j'\in\mathbb{J}'} \langle x, px_j\rangle\langle px_j, x\rangle \otimes \langle y, qy_{j'}\rangle\langle qy_{j'}, y\rangle$$
$$\leq \mu_1\langle x, x\rangle \otimes \mu_2\langle y, y\rangle.$$

Then

$$\lambda_1\lambda_2 \langle K_1^* x \otimes K_2^* y, K_1^* x \otimes K_2^* y\rangle \leq \sum_{j\in\mathbb{J}, j'\in\mathbb{J}'} \langle x, px_j\rangle \otimes \langle y, qy_{j'}\rangle\langle px_j, x\rangle \otimes \langle qy_{j'}, y\rangle$$
$$\leq \mu_1\mu_2\langle x\otimes y, x\otimes y\rangle,$$

Hence

$$\lambda_1\lambda_2 \langle (K_1\otimes K_2)^* (x\otimes y), (K_1\otimes K_2)^* (x\otimes y)\rangle \leq \sum_{j\in\mathbb{J}, j'\in\mathbb{J}'} \langle x, px_j\rangle \otimes \langle y, qy_{j'}\rangle\langle px_j, x\rangle \otimes \langle qy_{j'}, y\rangle$$
$$\leq \mu_1\mu_2\langle x\otimes y, x\otimes y\rangle,$$

Consequently, we have, for all $x \otimes y \in \mathbb{M}_1 \otimes \mathbb{M}_2$.

$$\lambda_1\lambda_2 \langle (K_1\otimes K_2)^* (x\otimes y), (K_1\otimes K_2)^* (x\otimes y)\rangle \leq \sum_{j\in\mathbb{J}, j'\in\mathbb{J}'} \langle x\otimes y, px_j \otimes qy_{j'}\rangle\langle px_j \otimes qy_{j'}, x\otimes y\rangle$$
$$\leq \mu_1\mu_2\langle x\otimes y, x\otimes y\rangle.$$

Since

$$(p\otimes q)(x_j \otimes y_{j'}) = px_j \otimes qy_{j'}.$$

Then, $\{(p\otimes q)(x_j\otimes y_{j'})\}_{j\in\mathbb{J}, j'\in\mathbb{J}'}$ is a $(K_1 \otimes K_2)$ - frame for $\mathbb{M}_1 \otimes \mathbb{M}_2$.
On the other hand, we have

$$\left(S_{px_j} \otimes S_{qy_{j'}}\right)(x\otimes y) = S_{px_j} x \otimes S_{qy_{j'}} y$$
$$= p(T_1) S_1 p^* (T_1) x \otimes q(T_2) S_2 q^* (T_2) y.$$
$$= (p(T_1) S_1 \otimes q(T_2) S_2)(p^*(T_1) \otimes q^*(T_2))(x\otimes y).$$
$$= (p(T_1) \otimes q(T_2))(S_1\otimes S_2)(p(T_1)\otimes q(T_2))^* (x\otimes y).$$

Since, $S_1 \otimes S_2$ is the frame operator for $\{(x_j \otimes y_{j'})\}_{j\in\mathbb{J}, j'\in\mathbb{J}'}$. Then

$$\left(S_{px_j} \otimes S_{qy_{j'}}\right)(x\otimes y) = S_{px_j \otimes qy_{j'}} (x\otimes y).$$

So, for all $z \in \mathbb{M}_1 \otimes \mathbb{M}_2$, we have

$$S_{px_j \otimes qy_{j'}} (z) = S_{px_j} \otimes S_{qy_{j'}} (z).$$

This complete the proof. □

An immediate consequence is

Corollary 2.11. *The subspace generated by*

$$\mathcal{V} = \{x_j \otimes y_{j'} : i \in \mathbb{J}, j' \in \mathbb{J}'\}$$

is a left $(Z(\mathcal{A})[X])^* \bigotimes (Z(\mathcal{A})[X])^*$*- module.*

Theorem 2.12. *Let* $\{x_j\}_{j\in\mathbb{J}}$ *be a* K*-frame for* $\mathcal{H}_T$ *and* $p \in (Z(\mathcal{A})[X])^*$ *. Then* $\{x_j\}_{j\in\mathbb{J}}$ *is a* (pK)*-frame for* $\mathcal{H}_T$*.*

Proof. Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}_T$ with frame bounds λ and μ. Then, for all $x \in \mathcal{H}_T$

$$\lambda\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq \mu\langle x, x\rangle.$$

Since

$$R\left(Kp\left(T\right)\right) \subset R\left(K\right).$$

Thus, by Lemma 1.9, there exists $\alpha > 0$ such that

$$\alpha\langle\left(Kp\left(T\right)\right)^* x, \left(Kp\left(T\right)\right)^* x\rangle \leq \langle K^*x, K^*x\rangle.$$

Therefore

$$\alpha\lambda\langle\left(pK\right)^* x, \left(pK\right)^* x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq \mu\langle x, x\rangle,\ for\,all\,p \in \left(Z\left(\mathcal{A}\right)[X]\right)^*.$$

Which end this proof. □

In the next theorem, we give some new K-frames for $\mathcal{H}$.

Theorem 2.13. *Let $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}$ and $K, L \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $KLK = K$. Then $\{(KL)x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$.*

Proof. Suppose that $\{x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$ with frame bounds λ and μ. Then, for all $x \in \mathcal{H}$

$$\lambda\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, x_j\rangle\langle x_j, x\rangle \leq \mu\langle x, x\rangle.$$

Since

$$KLK = K,$$

hence

$$\lambda\langle K^*L^*K^*x, K^*L^*K^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle L^*K^*x, x_j\rangle\langle x_j, L^*K^*x\rangle \leq \mu\langle L^*K^*x, L^*K^*x\rangle.$$

By taking $\mu' = \mu \parallel KL \parallel^2$, we have

$$\lambda\langle K^*x, K^*x\rangle \leq \sum_{j\in\mathbb{J}}\langle x, \left(KL\right)x_j\rangle\langle\left(KL\right)x_j, x\rangle \leq \mu'\langle x, x\rangle.$$

Then $\{\left(KL\right)x_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathcal{H}$. □

Corollary 2.14. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ be semi-regular and $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}$. If L is such that $KLK = K$, then $\{(K^nL^n)x_j\}_{j\in\mathbb{J}}$ is a K^n-frame for $\mathcal{H}$, for every $n \geq 1$.*

Proof. It follows immediately from Proposition 1.16 and Theorem 2.13. □

Corollary 2.15. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ be semi-regular and $\{x_j\}_{j\in\mathbb{J}}$ be a K-frame for $\mathcal{H}$. Then $\{(K^nK^{\dagger n})x_j\}_{j\in\mathbb{J}}$ is a K^n-frame for $\mathcal{H}$, for every $n \geq 1$.*

Proof. The proof is straightforward. □

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