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Fixed point theorems on generalized $\alpha - \eta$ extended Z-contractions in extended b-metric spaces

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Abstract. The aim of this paper is to introduce generalized $\alpha - \eta$ extended Z-contractions in the context of extended b-metric spaces and obtain fixed points for such contractions. Some interesting examples are given to strengthen the proven theory. As an application, we establish the existence and uniqueness for solutions of functional integral equations.

1 Introduction

The study of analysis of new spaces and their properties were introduce and discussed by many authors in several ways, we refer[2,9, 15,16, 19, 20,21, 22, 23]. In the class of all, we mention the concept of b-metric space[4].

Definition 1.1 [4]. Let A be a non-empty set and a mapping σ_b :A×A→ R^+ is said to be *b-metric*, if there exists $B > 1$, such that σ satisfies the following for all ν, μ and $\varrho \in A$

(1)$\sigma_b(\mu, \nu) = 0$ if and only if $\mu = \nu$,

(2)$\sigma_b(\mu, \nu) = \sigma_b(\nu, \mu)$,

(3) $\sigma_b(\mu, \nu) \leq B[\sigma_b(\mu, \varrho) + \sigma_b(\varrho, \nu)]$.

Then the pair (A, σ_b) is said to a b-metric space shortly BMS.

For more literature on b- metric space we refer[1,3,4,10,11,13].

Recently, new type of generalized b- metric space namely extended b-metric space introduced by Kamran *et. al.* [12].

Definition 1.2 [12]. Let A be a non-empty set and ξ :A×A→ $[1, \infty)$. A mapping $\sigma_\xi : A \times A \to R^+$ is said to be an *extended b-metric* , if σ_ξ satisfies the following for all μ, ν and $\varrho \in A$

(1)$\sigma_\xi(\mu, \nu) = 0$ if and only if $\mu = \nu$,

(2)$\sigma_\xi(\mu, \nu) = \sigma_\xi(\nu, \mu)$,

(3) $\sigma_\xi(\mu, \nu) \leq \xi(\mu, \nu)[\sigma_\xi(\mu, \varrho) + \sigma_\xi(\varrho, \nu)]$.

Then the pair (A, σ_ξ) is said to be an extended b-metric space.

Definition 1.3[12]. Let (A, σ_ξ) be an extended b-metric space with ξ.

(1) A sequence $\{\wp_n\}$ in (A, σ_ξ) is called σ_ξ *convergent to* $\wp^*$ if
$\lim_{n\to+\infty} \sigma_\xi(\wp_n, \wp^*) = 0$. In this case, $lim_{n\to\infty}\wp_n = \wp^*$.

(2) A sequence $\{\wp_n\}$ in (A, σ_ξ) is called σ_ξ $Cauchy$ if
$\lim_{m,n\to+\infty} \sigma_\xi(\wp_n, \wp_m) = 0$.

(3) An extended b-metric space (A, σ_ξ) is said to be a *complete* if every Cauchy sequence in A is convergent to some point in A.

Definition 1.4[20]. Let $s \geq 1$ be a real number. A function $\phi : [0, \infty) \to [0, \infty)$ is called a *(b)-comparison function* if ϕ is increasing and there exist $k_o \in N$, $a \in [0,1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq as^k\phi^k(t)+v_k$, for $k \geq k_0$ and any $t \geq 0$.

Lemma 1.5[20]. Let $\phi:[0,\infty)\to[0,\infty)$ be a (b)- comparison function. Then

(1) the series $\sum_{k=1}^{\infty} s^k\phi^k(t)$ converges for any $t\in[0,\infty)$;

(2) the function $b_s:[0,\infty)\to[0,\infty)$ defined as $b_s=\sum_{k=1}^{\infty} s^k\phi^k(t)$ is increasing and is continuous at $t=0$.

Any (b)-comparison function ϕ satisfies $\phi(t)<t$ and $\lim_{n\to\infty}\phi^n(t)=0$ for each $t>0$.

The collection of all (b)- comparison functions denoted by Φ.

Remark 1.6[3,8,10,11] We remark that, a b-metric is not a continuous functional in general and thus so is an extended b-metric.

Here we give an example of discontinous extended b- metric space.

Example 1.7. Let $X=\{0,2\}$, $Y=\{\frac{1}{n}/n\in\mathbb{N}\}$ and $A=X\cup Y$. We define $\xi:A\times A\to[0,\infty)$ by $\xi(\mu,\nu)\ =\mu+\nu+1$ and $\sigma_\xi:A\times A\to[0,\infty)$ by

$$\sigma_\xi(\mu,\nu)\ =\ \begin{cases} 0 & if\ \mu=\nu \\ 3 & if\ \mu\neq\nu,\{\mu,\nu\}\subset X \\ 1 & if\ \mu\neq\nu,\{\mu,\nu\}\subset Y \\ \nu & if\ \mu\in X,\ \nu\in Y \\ \mu & if\ \mu\in Y,\ \nu\in X \end{cases}$$

Then (A,σ_ξ) is complete extended $b-$ metric space with respect to the functional ξ. We observe that $lim_{n\to\infty}\frac{1}{n}\ =\ 0$, and $lim_{n\to\infty}\sigma_\xi(\frac{1}{n},\frac{1}{5})\ =\ 1\ \neq\ \sigma_\xi(0,\frac{1}{5})\ =\ \frac{1}{5}$, which shows that σ_ξ is not continuous.

Throughout this paper we assume that all extended b-metric spaces are continuous.

Lemma 1.8 [6]. Let (A,σ_ξ) be an extended BMS and T be a selfmap on A. If there exists a sequence $\{p_n\}_{n\in N}$ such that $p_n>1$ such that

$\xi(\wp_n,\wp_{n+1})<p_n$ for all $n\in\mathbb{N}$ and $m>n$. Furthermore,

$$0<\sigma_\xi(\wp_n,\wp_{n+1})\leq\phi(\sigma_\xi(\wp_n,\wp_{n-1})) \tag{1.8.1}$$

for all $n\in\mathbb{N}$, $\phi\in\Phi$ then the sequence $\{\wp_n\}$ defined by $\wp_n=T\wp_{n-1}$ for all $n\in\mathbb{N}$ is a Cauchy sequence in A.

Proof. Let $\{\wp_n\}$ be a sequence defined by $\wp_n\ =\ T\wp_{n-1}$. By employing inequality (1.8.1) recursively, we derive that

$$\sigma_\xi(\wp_n,\wp_{n+1})\leq\phi^n(\sigma_\xi(\wp_0,\wp_1)).$$

In view of property ϕ, we get

$$\lim_{n\to\infty}\sigma_\xi(\wp_n,\wp_{n+1})=0.$$

In view of condition (3) of Definition 1.2, we have

$$\begin{aligned}\sigma_\xi(\wp_n,\wp_m) &\leq \xi(\wp_n,\wp_m)[\sigma_\xi(\wp_n,\wp_{n+1})+\sigma_\xi(\wp_{n+1},\wp_m)]\\ &\leq \xi(\wp_n,\wp_m)\sigma_\xi(\wp_n,\wp_{n+1})+\xi(\wp_n,\wp_m)\xi(\wp_{n+1},\wp_m)[\sigma_\xi(\wp_{n+1},\wp_{n+2})\\ &\qquad +\sigma_\xi(\wp_{n+2},\wp_m)]\\ &\leq \xi(\wp_1,\wp_m)\xi(\wp_2,\wp_m)\xi(\wp_3,\wp_m)......\xi(\wp_n,\wp_m)\phi^n(\sigma_\xi(\wp_0,\wp_1))\\ &\quad +\xi(\wp_1,\wp_m)\xi(\wp_2,\wp_m)\xi(\wp_3,\wp_m)......\xi(\wp_{n+1},\wp_m)\phi^{n+1}(\sigma_\xi(\wp_0,\wp_1))\\ &\quad +......+\xi(\wp_1,\wp_m)\xi(\wp_2,\wp_m)\xi(\wp_3,\wp_m)......\xi(\wp_{m-1},\wp_m)\phi^{m-1}(\sigma_\xi(\wp_0,\wp_1)).\end{aligned}$$

Choose for all $n\in\mathbb{N}$, $S_n=\sum_{j=1}^{n}\phi^j(\sigma_\xi(\wp_0,\wp_1))\prod_{i=1}^{j}\xi(\wp_i,\wp_m)$.

We deduce that $\xi(\wp_n, \wp_m) \leq S_{m-1} - S_{n-1}$ for all $m > n$.

Consider the series $\sum_{n=1}^{\infty} \phi^n(\sigma_\xi(\wp_0, \wp_1)) \prod_{i=1}^{n} \xi(\wp_i, \wp_m)$.

Let $p = max\{p_1, p_2, ...p_n\}$. We have

$$a_n = \phi^n(\sigma_\xi(\wp_0, \wp_1)) \prod_{i=1}^{j} \xi(\wp_i, \wp_m) \leq \phi^n(\sigma_\xi(\wp_0, \wp_1))p^n.$$

From Lemma 1.5, we have that the series $\sum_{n=1}^{\infty} \phi^n(\sigma_\xi(\wp_0, \wp_1))p^j$ converges.

Using comparison criteria for the convergence of series, we obtain $\sum_{n=1}^{\infty} \phi^n(\sigma_\xi(\wp_0, \wp_1)) \prod_{i=1}^{j} \xi(\wp_i, \wp_m)$ converges, and hence

$$\lim_{n,m\to\infty} \sigma_\xi(\wp_n, \wp_m) = 0,$$

thus $\{\wp_n\}$ is a Cauchy sequence.

The concepts of α orbital admissible maps and triangular α- orbital admissible maps are defined as follows

Definition 1.9[18]. Consider $A \neq \phi$ and $\alpha : A \times A \to [0, \infty)$. Then $T : A \to A$ is said to be α-*orbital admissible mapping* if for all $\nu \in A$, $\alpha(\nu, T\nu) \geq 1$ implies $\alpha(T\nu, T^2\nu) \geq 1$.

Definition 1.10[18]. Let T be a self map of a nonempty set A and $\alpha : A \times A \to [0, \infty)$. Then T is said to be *triangular* α*- orbital admissible mapping* if

(i) T is α- orbital admissible

(ii) $\alpha(\nu, \mu) \geq 1$ and $\alpha(\mu, T\mu) \geq 1$ implies $\alpha(\nu, T\mu) \geq 1$ for all $\nu, \mu \in A$.

Definition 1.11[6]. Let $T : A \to A$ and $\alpha, \eta : A \times A \to [0, \infty)$, then T is said to be α*-orbital admissible mapping with respect to* η if for all $\nu \in A$

$$\alpha(\nu, T\nu) \geq \eta(\nu, T\nu) \ implies \ \alpha(T\nu, T^2\nu) \geq \eta(T\nu, T^2\nu)$$

Definition 1.12[6]. Let A be a nonempty set. Let $T : A \to A$ and $\alpha, \eta : A \times A \to [0, \infty)$. Now, T is said to be *triangular* α*-orbital admissible mapping with respect to* η if

(i) T is α-orbital admissible mapping with respect to η.

(ii) $\alpha(\nu, \mu) \geq \eta(\nu, \mu)$ and $\alpha(\mu, T\mu) \geq \eta(\mu, T\mu) \Rightarrow \alpha(\nu, T\mu) \geq \eta(\nu, T\mu)$, for all $\nu, \mu \in A$.

Lemma 1.13[6]. Let T be a triangular α-orbital admissible mapping with respect to η. Assume that there exists $u_1 \in A$ such that $\alpha(u_1, Tu_1) \geq \eta(u_1, Tu_1)$. We define a sequence $\{u_n\}$ by $u_{n+1} = Tu_n$. Then $\alpha(u_m, u_n) \geq \eta(u_m, u_n)$ for all $m, n \in N$ with $m < n$.

Definition 1.14[6]. Let (A, σ_ξ) be a b-metric space and $\alpha, \eta : A \times A \to [0.\infty)$. A mapping $T : A \to A$ is said to be $\alpha - \eta$ continuous if every sequence $\{u_n\}$ in A with $\alpha(u_n, u_{n+1}) \geq \eta(u_n, u_{n+1})$ for all $n \in N$ and $u_n \to u$ as $n \to \infty$ implies $Tu_n \to Tu$ as $n \to \infty$.

On the other side, Khojastech *et. al.*,[14] introduced a new class of mappings called simulation functions and showed many results in the literature are simple consequences of the other obtained results.

Definition 1.15[14]. A function $\zeta : [0, \infty) \times [0, \infty) \to (-\infty, \infty)$ is called a *simulation function* if ζ satisfies the following conditions.

(i) $\zeta(0, 0) = 0$

(ii) $\zeta(t, s) < s - t$, for all $s, t > 0$.

(iii) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0, \infty)$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

The following are examples of simulation functions.

Example 1.16[7]. Let $\zeta : [0, \infty) \times [0, \infty) \to R$ be defined by

(i) $\zeta(t, s) = \lambda s - t$ for all $s, t \in [0, \infty)$ where $\lambda \in [0, 1)$

(ii) $\zeta(t,s)=\frac{k}{r}s-t$ for all $s,t\in[0,\infty)$ where $k\in[0,1)$ and $r\in(1,\infty)$

(iii) $\zeta(t,s)=\psi(s)-\phi(t)$ for all $s,t\in[0,\infty)$ where $\psi,\phi:[0,\infty)\to[0,\infty)$ such that $\psi(r)=\phi(r)=0$ if and only if $r=0$ and $\psi(t)<t<\phi(t)$ for all $t>0$.

(iv) $\zeta(t,s)=\frac{s}{1+s}-t$ for all $s,t\in[0,\infty)$

(v) $\zeta(t,s)=\psi(s)-t$ for all $s,t\in[0,\infty)$ where $\psi(t)<t$.

For more examples on simulation function we refer[11,17,24].

Recently, Chifu and Karpinar[5] introduced admissible extended Z-contraction mappings in extended b-metric spaces and obtained fixed points for such contractions.

Definition 1.17[5]. Let (A,σ_ξ) be an extended b-metric space with the function $\xi:A\times A\to[1,\infty)$. A mapping $T:A\to A$ is called an *admissible extended Z contraction* if there exists a simulation function ζ such that for all $\mu,\nu\in A$

$$\zeta(\alpha(\nu,\mu)\sigma_\xi(T\nu,T\mu),\phi(M_\xi(\nu,\mu))\geq 0 \tag{1.17.1}$$

where $\phi\in\Phi$ and $M_\xi(\nu,\mu)=max\{\sigma_\xi(\nu,\mu),\sigma_\xi(\nu,T\nu),\sigma_\xi(\mu,T\mu)\}$.

Theorem 1.18[5]. Let (A,σ_ξ) be an extended b-metric space with the function $\xi:A\times A\to[1,\infty)$. Suppose that there exists a sequence $\{q_n\}_{n\in\mathbb{N}}$ such that $\xi(\wp_n,\wp_m)<q_n$ for all $m>n$. Furthermore, if $T:A\to A$ is admissible extended Z- contraction satisfying:

(i) T is triangular α-orbital admissible mapping

(ii) there exists $\wp_0\in A$ such that $\alpha(\wp_0,T\wp_0)\geq 1$

(iii) T is a continuous mapping;

or

(iv) if $\{u_n\}$ is a sequence in A such that $\alpha(u_n,u_{n+1})\geq 1$ for all n and $u_n\to u\in A$ as $n\to\infty$, then there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\alpha(u_{n(k)},u)\geq 1$ for all k.

Then T has a fixed point $\wp^*\in X$ and $\{T^n\wp_0\}$ is converges to $\wp^*$.

Motivated by the results of Chifu and Karpinar[5] and some other similar results, in this paper, we introduce generalized $\alpha-\eta$ extended Z contraction involving rational expressions and obtained fixed points for such contractions. We provide some interesting examples to strengthen the proven theory . We also apply our results to examine the existence of functional integral equations solutions.

2 Main Results

Definition 2.1. Let (A,σ_ξ) be an extended b-metric space with the function $\xi:A\times A\to[1,\infty)$. A mapping $T:A\to A$ is called an *generalized $\alpha-\eta$ extended Z contraction* if there exists a simulation function ζ such that $\alpha(\nu,\mu)\geq\eta(\nu,\mu)$ implies

$$\zeta(\sigma_\xi(T\nu,T\mu),\phi(M_\xi(\nu,\mu))\geq 0 \tag{2.1.1}$$

where $\phi\in\Phi$ and

$$M_\xi(\nu,\mu)=max\{\sigma_\xi(\nu,\mu),\frac{\sigma_\xi(\nu,T\nu)\sigma_\xi^2(\nu,T\mu)+\sigma_\xi(\mu,T\mu)\sigma_\xi^2(\mu,T\nu)}{\sigma_\xi^2(\nu,T\mu)+\sigma_\xi^2(\mu,T\nu)},$$
$$\frac{\sigma_\xi(\nu,T\nu)\sigma_\xi(\nu,T\mu)+\sigma_\xi^2(\nu,\mu)+\sigma_\xi(\nu,T\nu)\sigma_\xi(\nu,\mu)}{\sigma_\xi(\nu,T\nu)+\sigma_\xi(\nu,T\mu)+\sigma_\xi(\nu,\mu)}\}$$

for all $\nu,\mu\in A$ with $\sigma_\xi^2(\nu,T\mu)+\sigma_\xi^2(\mu,T\nu)>0$ and $\sigma_\xi(\nu,T\nu)+\sigma_\xi(\nu,T\mu)+\sigma_\xi(\nu,\mu)>0$.

Theorem 2.2. Let (A,σ_ξ) be an extended b-metric space with the function $\xi:A\times A\to[1,\infty)$. Suppose that there exists a sequence $\{q_n\}$; $q_n>1$ such that $\xi(\wp_n,\wp_m)<q_n$ for all $n\in\mathbb{N}$ and $m>n$. Assume that $T:A\to A$ is a generalized $\alpha-\eta$ extended Z contraction satisfying:

(i) T is a triangular α-orbital admissible mapping with respect to η

(ii) there exists $\wp_1\in A$ such that $\alpha(\wp_1,T\wp_1)\geq\eta(\wp_1,T\wp_1)$, and

(iii) T is $\alpha - \eta$ continuous mapping.

Then T has a fixed point $\wp^* \in X$ and $\{T^n \wp_1\}$ is converges to $\wp^*$.

Proof. Let $\wp_1 \in A$ as in (iii) of our assumption, *i.e.*, $\alpha(\wp_1, T\wp_1) \geq \eta(\wp_1, T\wp_1)$, we define a sequence $\{\wp_n\}$ in A by

$$\wp_{n+1} = T\wp_n \tag{2.2.1}$$

for all $n \in \mathbb{N}$. Suppose that $\wp_{n_0} = \wp_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $\wp_{n_0} = T\wp_{n_0}$ which implies $\wp_{n_0}$ is a fixed point of T which completes the proof. Hence suppose that $\wp_n \neq \wp_{n+1}$ for all $n \in \mathbb{N}$.

On the account of Lemma 1.13, we have

$$\alpha(\wp_n, \wp_{n+1}) \geq \eta(\wp_n, \wp_{n+1}). \tag{2.2.2}$$

for all $n \in \mathbb{N}$.

On using condition (2.1.1) with $\nu = \wp_n$ and $\mu = \wp_{n-1}$, we have

$$\zeta(\sigma_\xi(\wp_n, \wp_{n+1}), M_T(\wp_n, \wp_{n-1})) \geq 0, \tag{2.2.3}$$

where

$$\begin{aligned} M_T(\wp_n, \wp_{n-1}) &= max\{\sigma_\xi(\wp_n, \wp_{n-1}), \frac{\sigma_\xi(\wp_n,\wp_{n+1})\sigma_\xi^2(\wp_n,\wp_n)+\sigma_\xi(\wp_{n-1},\wp_n)\sigma_\xi^2(\wp_{n-1},\wp_{n+1})}{\sigma_\xi^2(\wp_n,\wp_n)+\sigma_\xi^2(\wp_{n-1},\wp_{n+1})}, \\ &\quad \frac{\sigma_\xi(\wp_n,\wp_{n+1})\sigma_\xi(\wp_n,\wp_n)+\sigma_\xi^2(\wp_n,\wp_{n-1})+\sigma_\xi(\wp_n,\wp_{n+1})\sigma_\xi(\wp_n,\wp_{n-1})}{\sigma_\xi(\wp_n,\wp_{n+1})+\sigma_\xi(\wp_n,\wp_n)+\sigma_\xi(\wp_n,\wp_{n-1})}\} \\ &= max\{\sigma_\xi(\wp_n, \wp_{n-1}), \sigma_\xi(\wp_n, \wp_{n-1}), \sigma_\xi(\wp_n, \wp_{n+1})\} \end{aligned} \tag{2.2.4}$$

Suppose that $\sigma_\xi(\wp_n, \wp_{n-1}) < \sigma_\xi(\wp_n, \wp_{n+1})$, then from (2.2.3) and (2.2.4), it follows that

$$\begin{aligned} 0 &\leq \zeta(\sigma_\xi(\wp_n, \wp_{n+1}), \phi(\sigma_\xi(\wp_n, \wp_{n+1})) \\ &< \phi(\sigma_\xi(\wp_n, \wp_{n+1}) - \sigma_\xi(\wp_n, \wp_{n+1})). \end{aligned}$$

Consequently,

$$\begin{aligned} \sigma_\xi(\wp_n, \wp_{n+1}) &\leq \phi(\sigma_\xi(\wp_n, \wp_{n+1})) \\ &< \sigma_\xi(\wp_n, \wp_{n+1}), \end{aligned}$$

a contradiction. Therefore

$$\sigma_\xi(\wp_n, \wp_{n-1}) > \sigma_\xi(\wp_n, \wp_{n+1}) \tag{2.2.5}$$

Similarly, we can prove that

$$\sigma_\xi(\wp_{n-1}, \wp_{n-2}) > \sigma_\xi(\wp_n, \wp_{n-1}). \tag{2.2.6}$$

Hence from (2.2.5) and (2.2.6), we conclude that

$$\sigma_\xi(\wp_n, \wp_{n+1}) > \sigma_\xi(\wp_{n+1}, \wp_{n+2})$$

for all $n \in N$.

Thus from (2.1.1), we have,

$$\begin{aligned} 0 &\leq \zeta(\sigma_\xi(\wp_{n+1}, \wp_n), \phi(\sigma_\xi(\wp_n, \wp_{n-1})) \\ &< \phi(\sigma_\xi(\wp_n, \wp_{n-1})) - \sigma_\xi(\wp_n, \wp_{n+1}) \end{aligned}$$

which implies $\sigma_\xi(\wp_{n+1}, \wp_n) \leq \phi(\sigma_\xi(\wp_n, \wp_{n-1})) \leq \phi^n(\sigma_\xi(\wp_0, \wp_1))$. (2.2.7)

Taking limits as $n \to \infty$ and using the property of ϕ, we get

$$lim_{n\to\infty}\sigma_\xi(\wp_n, \wp_{n+1}) = 0. \tag{2.2.8}$$

By Lemma 1.8 and condition (2.2.7), we have $\{\wp_n\}$ is a Cauchy sequence in A. Since (A, σ_ξ) ia complete extended b-metric space, there exists $\wp \in A$ such that

$$lim_{n\to\infty}\sigma_\xi(\wp_n, \wp) = 0. \tag{2.2.9}$$

Since T is continuous, we derive that

$$lim_{n\to\infty}\sigma_\xi(T\wp_n, T\wp) = 0 \tag{2.2.10}$$

this implies $\wp = T\wp$.

Hence the theorem.

Theorem 2.3. Let (A, σ_ξ) be an extended b-metric space with the function $\xi : A \times A \to [1, \infty)$. Suppose that there exists a sequence $\{p_n\}$, $p_n > 1$ for all $n \in \mathbb{N}$ such that $\xi(\wp_n, \wp_m) < p_n$. Further, suppose that $T : A \to A$ is generalized $\alpha - \eta$ extended Z contraction satisfying: conditions (i), (ii) of Theorem 2.2 along with the following condition

(iii) if $\{\wp_n\}$ is a sequence in A such that $\alpha(\wp_n, \wp_{n+1}) \geq \eta(\wp_n, \wp_{n+1})$ for all $n \in \mathbb{N}$ and $\wp_n \to \wp$ as $n \to \infty$, then there exist a subsequence $\{\wp_{n_k}\}$ of $\{\wp_n\}$ such that $\alpha(\wp_{n_k}, \wp^*) \geq \eta(\wp_{n_k}, \wp^*)$ for all $k \in \mathbb{N}$.

T has a fixed point $\wp^* \in A$ and $\{T^n\wp_1\}$ is converges to $\wp^*$ is a fixed point of T.

Proof. By using the proof of Theorem 2.2, we obtain the sequence $\{\wp_n\}$ defined by $\wp_{n+1} = T\wp_n$ converges to $\wp^* \in A$ and $\alpha(\wp_n, \wp_{n+1}) \geq \eta(\wp_n, \wp_{n+1})$ for all $n \in N$. By our assumption (iii), there exist a subsequence $\{\wp_{n_k}\}$ of $\{\wp_n\}$ such that $\alpha(\wp_{n_k}, \wp^*) \geq \eta(\wp_{n_k}, \wp^*)$ for all $k \in N$. Now on using condition (2.1.1) with $\mu = \wp^*$ and $\nu = \wp_{n_k}$, we have

$$0 \leq \zeta(\sigma_\xi(T\wp_{n_k}, T\wp^*), \phi(M_\xi(\wp_{n_k}, \wp^*)))$$
$$< \phi(M_\xi(\wp_{n_k}, \wp^*) - \sigma_\xi(T\wp_{n_k}, T\wp^*)$$

which implies

$$\sigma_\xi(T\wp_{n_k}, T\wp^*) < \phi(M_\xi(\wp_{n_k}, \wp^*)) \tag{2.3.1}$$

$$M_\xi(\wp_{n_k}, \wp^*) = max\{\sigma_\xi(\wp_{n_k}, \wp^*), \frac{\sigma_\xi(\wp_{n_k}, T\wp_{n_k})\sigma_\xi^2(\wp_{n_k}, T\wp^*) + \sigma_\xi(\wp^*, T\wp^*)\sigma_\xi^2(\wp^*, T\wp_{n_k})}{\sigma_\xi^2(\wp_{n_k}, T\wp^*) + \sigma_\xi^2(\wp^*, T\wp_{n_k})},$$
$$\frac{\sigma_\xi(\wp_{n_k}, T\wp_{n_k})\sigma_\xi(\wp_{n_k}, T\wp^*) + \sigma_\xi^2(\wp_{n_k}, \wp^*) + \sigma_\xi^2(\wp_{n_k}, T\wp_{n_k})\sigma_\xi(\wp_{n_k}, \wp^*)}{\sigma_\xi(\wp_{n_k}, T\wp_{n_k}) + \sigma_\xi(\wp_{n_k}, T\wp^*) + \sigma_\xi(\wp_{n_k}, \wp^*)}\}.$$

Letting lim sup as $k \to \infty$, in the above inequality, using (2.2.8) and (2.2.9), we have

$$limsup_{k\to\infty} M_\xi(\wp_{n_k}, \wp^*) = 0. \tag{2.3.2}$$

Thus from (2.3.1) and (2.3.2), we get

$limsup_{k\to\infty}\sigma_\xi(T\wp_{n_k}, T\wp^*) \leq limsup_{k\to\infty}(M_\xi(\wp_{n_k}, \wp^*)) = 0$ which implies $\sigma_\xi(\wp^*, T\wp^*) = 0$. Hence $T\wp^* = \wp^*$.

Theorem 2.4. In addition to the hypotheses of Theorem 2.3, assume the following:

(H) for all $\mu \neq \nu \in A$, there exists $x \in A$ such that $\alpha(\mu, x) \geq \eta(\mu, x)$, $\alpha(\nu, x) \geq \eta(\nu, x)$ and $\alpha(x, Tx) \geq \eta(x, Tx)$, then T has a unique fixed point.

Proof. Suppose that u^* , v^* be two fixed points of T with $u^* \neq v^*$.

Then by our assumption, there exists $x \in A$ such that $\alpha(\mu, x) \geq \eta(\mu, x)$, $\alpha(\nu, x) \geq \eta(\nu, x)$ and $\alpha(x, Tx) \geq \eta(x, Tx)$.

By applying Theorem 2.1, we deduce that $\{T^n x\}$ converges to a fixed point say z^*.

Since T is triangular α-orbital admissible map with respect to η, we have

$\alpha(x, T^n x) \geq \eta(x, T^n x)$ and hence

$$\alpha(u^*, T^n x) \geq \eta(u^*, T^n x) \; and \; \alpha(v^*, T^n x) \geq \eta(v^*, T^n x) \tag{2.4.1}$$

Now, $\sigma_\xi(u^*, T^n x) \leq M(\sigma_\xi(u^*, T^n x))$

$$\leq max\{\sigma_\xi(u^*, T^n x), \frac{\sigma_\xi(u^*, Tu^*)\sigma_\xi^2(u^*, T^{n+1}x) + \sigma_\xi(T^n x, T^{n+1}x)\sigma_\xi^2(T^n x, Tu^*)}{\sigma_\xi^2(u^*, T^{n+1}x) + \sigma_\xi^2(T^n x, Tu^*)},$$
$$\frac{\sigma_\xi(u^*, Tu^*)\sigma_\xi(u^*, T^{n+1}x) + \sigma_\xi(u^*, Tu^*)\sigma_\xi(u^*, T^n x) + \sigma_\xi^2(u^*, T^n x)}{\sigma_\xi(u^*, Tu^*) + \sigma_\xi(u^*, T^{n+1}x) + \sigma_\xi(u^*, T^n x)}\}$$
$$= max\{\sigma_\xi(u^*, T^n x), \frac{\sigma_\xi(T^n x, T^{n+1}x)\sigma_\xi^2(T^n x, Tu^*)}{\sigma_\xi^2(u^*, T^{n+1}x) + \sigma_\xi^2(T^n x, Tu^*)}, \frac{\sigma_\xi^2(u^*, T^n x)}{\sigma_\xi(u^*, T^{n+1}x) + \sigma_\xi(u^*, T^n x)}\}.$$

On taking limit supremum as $n \to \infty$, we have

$$\sigma_\xi(z^*, u^*) \leq limsup_{n\to\infty} M(\sigma_\xi(u^*, T^n x)) \leq \sigma_\xi(z^*, u^*)$$

Therefore

$$limsup_{n\to\infty} M(\sigma_\xi(u^*, T^n x)) = \sigma_\xi(u^*, z^*). \tag{2.4.2}$$

We now show that $u^* = z^*$.

Now from (2.1.1), (2.4.1) and (2.4.2), we have

$$0 \leq \xi(\sigma_\xi(Tu^*, T^{n+1}x)), \phi(M(\sigma_\xi(u^*, T^n x)))$$

which implies

$$\sigma_\xi(u^*, T^{n+1}x) \leq \phi(M(\sigma_\xi(u^*, T^n x))).$$

Taking limsup as $n \to \infty$, in the above inequality, we have

$$\begin{aligned}\sigma_\xi(u^*, z^*) &\leq limsup_{n\to\infty}\sigma_\xi(M(\sigma_\xi(u^*, T^n x))) \\ &\leq \phi(\sigma_\xi(u^*, z^*)) \\ &< \sigma_\xi(u^*, z^*),\end{aligned}$$

a contradiction. Therefore $u^* = z^*$.

Similarly, we can prove that $v^* = z^*$. Thus, it follows that $u^* = v^*$.

Hence T has a unique fixed point.

3 Corollaries and Examples

Corollary 3.1. Let (A, σ_ξ) be an extended b-metric space with the function $\xi : A \times A \to [1, \infty)$ and suppose that there exists a sequence $\{q_n\}$; $q_n > 1$, such that $\xi(\wp_n, \wp_m) < q_n$ for all $n \in \mathbb{N}$ and $m > n$. Assume that $T : A \to A$ is a generalized α extended Z contraction $i.\ e.$, $\alpha(\mu, \nu) \geq 1$ implies

$$\zeta(\sigma_\xi(T\nu, T\mu), \phi(M_\xi(\nu, \mu)) \geq 0 \tag{3.1.1}$$

where $\phi \in \Phi$ and

$$M_\xi(\nu,\mu) = max\{\sigma_\xi(\nu,\mu), \frac{\sigma_\xi(\nu,T\nu)\sigma_\xi^2(\nu,T\mu)+\sigma_\xi(\mu,T\mu)\sigma_\xi^2(\mu,T\nu)}{\sigma_\xi^2(\nu,T\mu)+\sigma_\xi^2(\mu,T\nu)}, \frac{\sigma_\xi(\nu,T\nu)\sigma_\xi(\nu,T\mu)+\sigma_\xi^2(\nu,\mu)+\sigma_\xi(\nu,T\nu)\sigma_\xi(\nu,\mu)}{\sigma_\xi(\nu,T\nu)+\sigma_\xi(\nu,T\mu)+\sigma_\xi(\nu,\mu)}\}$$

for all $\nu, \mu \in A$ with $\sigma_\xi^2(\nu, T\mu) + \sigma_\xi^2(\mu, T\nu) > 0$ and $\sigma_\xi(\nu, T\nu) + \sigma_\xi(\nu, T\mu) + \sigma_\xi(\nu, \mu) > 0$. Furthermore, suppose that

(i) T is a triangular α-orbital admissible mapping.

(ii) there exists $\wp_1 \in A$ such that $\alpha(\wp_1, T\wp_1) \geq 1$, and

(iii) T is continuous

or

(iv) if $\{\wp_n\}$ is a sequence in A such that $\alpha(\wp_n, \wp_{n+1}) \geq 1$ for all $n \in N$ and $\wp_n \to \wp$ as $n \to \infty$, then there exist a subsequence $\{\wp_{n_k}\}$ of $\{\wp_n\}$ such that $\alpha(\wp_{n_k}, \wp^*) \geq 1$.

Then T has a fixed point $\wp^* \in A$ and $\{T^n\wp_1\}$ is converges to $\wp^*$.

Moreover, for all $\mu, \nu \in Fix(T)$, we have $\alpha(\mu, \nu) \geq 1$, where $Fix(T)$ denotes the set of fixed points of T, then T has a unique fixed point.

Proof. Proof follows by choosing $\eta(\mu, \eta) = 1$ in Theorem 2.2,Theorem 2.3 and Theorem 2.4 respectively.

Corollary 3.2. Let (A, σ_ξ) be an extended b-metric space with the function $\xi : A \times A \to [1, \infty)$ and $\alpha, \eta : A \times A \to [0, \infty)$ be two mappings. Further, suppose that there exists a sequence $\{q_n\}$; $q_n > 1$, for all $n \in N$ such that $\xi(\wp_n, \wp_m) < q_n$ for all $m > n$. Consider, $T : A \to A$ and $\phi \in \Phi$ such that $\alpha(\mu, \nu) \geq \eta(\mu, \nu)$ implies

$$\sigma_\xi(T\nu, T\mu) \leq \phi(M_\xi(\nu, \mu)) \tag{3.2.1}$$

where

$$M_\xi(\nu,\mu) = max\{\sigma_\xi(\nu,\mu), \frac{\sigma_\xi(\nu,T\nu)\sigma_\xi^2(\nu,T\mu)+\sigma_\xi(\mu,T\mu)\sigma_\xi^2(\mu,T\nu)}{\sigma_\xi^2(\nu,T\mu)+\sigma_\xi^2(\mu,T\nu)}, \frac{\sigma_\xi(\nu,T\nu)\sigma_\xi(\nu,T\mu)+\sigma_\xi^2(\nu,\mu)+\sigma_\xi(\nu,T\nu)\sigma_\xi(\nu,\mu)}{\sigma_\xi(\nu,T\nu)+\sigma_\xi(\nu,T\mu)+\sigma_\xi(\nu,\mu)}\}$$

for all $\nu, \mu \in A$ with $\sigma_\xi^2(\nu, T\mu) + \sigma_\xi^2(\mu, T\nu) > 0$ and $\sigma_\xi(\nu, T\nu) + \sigma_\xi(\nu, T\mu) + \sigma_\xi(\nu, \mu) > 0$.

Further, suppose that

(i) T is a triangular α-orbital admissible mapping with respect to η.

(ii) there exists $\wp_1 \in A$ such that $\alpha(\wp_1, T\wp_1) \geq 1$, and

(iii) T is $\alpha-\eta$ continuous mapping,

or

(iv) if $\{\wp_n\}$ is a sequence in A such that $\alpha(\wp_n, \wp_{n+1}) \geq 1$ for all $n \in N$ and $\wp_n \to \wp$ as $n \to \infty$, then there exists a subsequence $\{\wp_{n_k}\}$ of $\{\wp_n\}$ such that $\alpha(\wp_{n_k}, \wp^*) \geq 1$.

Then T has a fixed point $\wp^* \in A$ and $\{T^n\wp_1\}$ is converges to $\wp^*$.

Proof. Proof follows by choosing $\zeta(t,s) = s - t$ in Theorem 2.2. Theorem 2.3 and Theorem 2.4.

Example 3.3 Let $A = [0,1]$, we define by $\xi : A \times A \to [0,\infty)$ by

$$\xi(\mu,\eta) = \begin{cases} 1 & if\ \mu = \eta \\ 2 + (\mu-\eta)^2 & if \mu \neq \eta, \end{cases}$$

and $\sigma_\xi : A \times A \to [0,\infty)$ by

$$\sigma_\xi(\mu,\eta) = \begin{cases} 0 & if\ \mu = \eta \\ (\mu-\eta)^2 & if \mu \neq \eta. \end{cases}$$

Clearly, $\sigma_\xi(\mu,\eta)$ forms an extended B-metric space with respect to ξ.

We define a (b)-comparison $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = \frac{t}{4}$.

It is easy to see that ϕ is decreasing and

$$\phi^n(t)\Pi_{i=1}^{j}\xi(\wp_i, \wp_m) = t(\frac{1}{4})^n 3^n = t(\frac{3}{4})^n < \infty.$$

Hence ϕ is an extended comparison function.

We now define $T : A \to A$ by

$$T\mu = \begin{cases} \frac{\mu+1}{4}\ if\ \mu \in [0,\frac{1}{2}] \\ 1 - \frac{\mu}{2}\ if\ \mu \in (\frac{1}{2}, 1]. \end{cases}$$

Further, suppose that $\alpha, \eta : A \times A \to [0,\infty)$ by

$$\alpha(\mu,\nu) = \begin{cases} 5 + e^{\mu\nu} & if\ \mu,\nu \in [0,\frac{1}{2}] \\ 0 & otherwise, \end{cases} \quad \text{and} \quad \eta(\mu,\nu) = \begin{cases} 2 + e^{\mu\nu} & if\ \mu,\nu \in [0,\frac{1}{2}] \\ 3 & otherwise. \end{cases}$$

When $\mu,\nu \in [0,\frac{1}{2}]$, we have $\alpha(T\mu, TT\mu) \geq \eta(T\mu, TT\mu)$.
Hence T- is α-orbital admissible with respect to η.

Suppose that $\alpha(\mu,\nu) \geq \eta(\mu,\nu)$ and $\alpha(\nu, T\nu) \geq \eta(\mu, T\mu)$, then $\mu,\nu \in [0,\frac{1}{2}]$ which implies that $\alpha(\mu, T\nu) \geq \eta(\mu, T\nu)$. Hence T is triangular α- orbital admissible with respect to η.

Consider a sequence $\wp_n \in N$ such that $\wp_n \to \wp^*$ as $n \to \infty$ and $\alpha(\wp_n, \wp_{n+1}) \geq \eta(\wp_n, \wp_{n+1})$ for all $n \in \mathbb{N}$, then $\{\wp_n\} \subseteq [0,\frac{1}{2}]$ for all $n \in \mathbb{N}$.

Then $\lim_{n\to\infty} T\wp_n = \lim_{n\to\infty} \frac{\wp_n+1}{4} = \lim_{n\to\infty} \frac{\wp_n}{4} + \frac{1}{4} = \frac{\wp^*}{4} + \frac{1}{4} = T\mu$, hence T is $\alpha-\eta$ continuous .

We now verify the inequality (2.1.1) with simulation function
$\zeta : [0,\infty) \times [0,\infty) \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{s}{2} - t$

$$\begin{aligned}
\zeta(\sigma_\xi(T\mu, T\nu), \phi(M_\xi(\sigma_\xi(\mu,\nu)))) &= \zeta(\sigma_\xi(\tfrac{\mu+1}{4}, \tfrac{\nu+1}{4}), \tfrac{1}{4}M_\xi(\sigma_\xi(\mu,\nu))) \\
&= \tfrac{1}{8}M\sigma_\xi(\mu,\nu) - \tfrac{1}{16}\sigma_\xi(\mu,\nu) \\
&\geq \tfrac{\sigma_b(\mu,\nu)}{8} - \tfrac{1}{16}\sigma_\xi(\mu,\nu) \\
&= \tfrac{\sigma_\xi(\mu,\nu)}{16} \geq 0.
\end{aligned}$$

Hence T satisfies all the postulates of Theorem 2.3. We note that $\frac{1}{3}$ and $\frac{2}{3}$ are two fixed points of T.

Here we observe that condition (H) fails to hold, for let $\mu = \frac{3}{4}$ and $\nu = 1$, then there no $x \in A$ such that $\alpha(\frac{3}{4}, x) \geq \eta(\frac{3}{4}, x)$, $\alpha(1, x) \geq \eta(1, x)$ and $\alpha(x, Tx) \geq \eta(x, Tx)$.

Example 3.4 Let $A = [0, \frac{6}{5}]$, we define by $\xi : A \times A \to [0, \infty)$ by

$$\xi(\mu, \eta) = \begin{cases} 1 & if\ \mu = \eta \\ 2 + (\mu - \eta)^2 & if \mu \neq \eta, \end{cases}$$

and $\sigma_\xi : A \times A \to [0, \infty)$ by

$$\sigma_\xi(\mu, \eta) = \begin{cases} 0 & if\ \mu = \eta \\ (\mu + \eta)^3 & if \mu \neq \eta. \end{cases}$$

Clearly, $\sigma_\xi(\mu, \eta)$ forms an extended B-metric space with respect to ξ.

We define a (b)-comparison $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{5t}{43}$.

It is easy to see that ϕ is decreasing and

$$\phi^n(t)\Pi_{i=1}^{j}\xi(\wp_i, \wp_m) = t(\frac{5}{43})^n(\frac{86}{25})^n = t(\frac{2}{5})^n < \infty.$$

Hence ϕ is an extended comparison function.

We now define that $T : A \to A$ by

$$T\mu = \begin{cases} \frac{\mu}{4}\ if\ \mu \in [0, 1] \\ \mu - \frac{1}{5}\ if\ \mu \in (1, \frac{6}{5}]. \end{cases}$$

Further, suppose that $\alpha, \eta : A \times A \to [0, \infty)$ by

$$\alpha(\mu, \nu) = \begin{cases} e^{\mu+\nu} & if\ \mu, \nu \in [0, 1] \\ 0 & otherwise, \end{cases} \quad \text{and} \quad \eta(\mu, \nu) = \begin{cases} e^{\frac{\mu+\nu}{2}} & if\ \mu, \nu \in [0, 1] \\ 3 & otherwise. \end{cases}$$

When $\mu, \nu \in [0, 1]$, we have $\alpha(T\mu, TT\mu) \geq \eta(T\mu, TT\mu)$.
Hence T- is α-orbital admissible with respect to η.

Suppose that $\alpha(\mu, \nu) \geq \eta(\mu, \nu)$ and $\alpha(\nu, T\nu) \geq \eta(\mu, T\mu)$, then $\mu, \nu \in [0, 1]$ which implies that $\alpha(\mu, T\nu) \geq \eta(\mu, T\nu)$. Hence T is triangular α- orbital admissible with respect to η. Also, T is $\alpha - \eta$ continuous .

We now verify the inequality (2.1.1) with a simulation function $\zeta : [0, \infty) \times [0, \infty) \to (-\infty, \infty)$ by $\zeta(t, s) = \frac{2s}{3} - t$

$$\begin{aligned} \zeta(\sigma_\xi(T\mu, T\nu), \psi(M_\xi(\sigma_\xi(\mu, \nu)))) &= \zeta(\sigma_\xi(\tfrac{\mu}{4}, \tfrac{\nu}{4}), \tfrac{5}{43}M_\xi(\sigma_b(\mu, \nu))) \\ &= \tfrac{10}{129}M\sigma_\xi(\mu, \nu) - \tfrac{1}{64}\sigma_b(\mu, \nu) \\ &\geq \tfrac{10\sigma_\xi(\mu, \nu)}{129} - \tfrac{1}{64}\sigma_\xi(\mu, \nu) \\ &= \tfrac{511\sigma_\xi(\mu, \nu)}{8256} \geq 0. \end{aligned}$$

Hence T satisfies the inequality (2.1.1). Also, since for any $\mu \neq \nu \in A$, we have $\alpha(\mu, 0) \geq \eta(\mu, 0)$, $\alpha(\nu, 0) \geq \eta(\nu, 0)$ and $\alpha(0, T0) \geq \eta(0, T0)$, T satisfies condition (H). Hence T satisfies all the hypotheses of Theorem 2.3 and $\mu = 0$ is the unique fixed point of T.

4 Applications

We now give existence theorem for the following functional integral equations.

$$\mu(t)=g(t,\mu(t))+\int_0^1 f(t,s,\mu(t))dt \tag{4.1.1}$$

where $g:[0,1]\times R\to R$ and $f:[0,1]\times[0,1]\times R\to R$ are continuous functions.

Let $A=C([0,1])$ of real functions on $[0,1]$ and $\xi: A\times A\to[1,\infty)$ be defined by $\xi(\mu(t),\nu(t))=1+\mu(t)+\nu(t)$.

We define a mapping $\sigma_\xi: A\times A\to[0,\infty)$ by

$$\sigma_\xi(\mu(t),\nu(t)) = \begin{cases} 0 & if\ \mu(t)=\nu(t) \\ (\mu(t))^2+(\nu(t))^2 & if\ \mu(t)\neq\nu(t). \end{cases}$$

It is easy to see that (A,σ_ξ) is a complete extended b-metric space with the function ξ.

Theorem 4.1 Consider the the integral equation (4.1.1) such that : for all $\mu(t),\nu(t)\in R$ and $0\le t,s<1$,

(i) $|g(t,\mu(t))+g(t,\nu(t))|\le\sqrt{\frac{1}{9}M_\xi(\mu(t),\nu(t))}$

(ii) $|f(t,s,\mu(t))+f(t,s,\nu(t))|\le\sqrt{\frac{1}{81}M_\xi(\mu(t),\nu(t))}$,

where $M_\xi(\mu(t),\nu(t))$

$$=max\{\sigma_\xi(\nu(t),\mu(t)),\frac{\sigma_\xi(\nu(t),T\nu(t))\sigma_\xi^2(\nu(t),T\mu(t))+\sigma_\xi(\mu(t),T\mu(t))\sigma_\xi^2(\mu(t),T\nu(t))}{\sigma_\xi^2(\nu(t),T\mu(t))+\sigma_\xi^2(\mu(t),T\nu(t))},$$
$$\frac{\sigma_\xi(\nu(t),T\nu(t))\sigma_\xi(\nu(t),T\mu(t))+\sigma_\xi^2(\nu(t),\mu(t))+\sigma_\xi(\nu(t),T\nu(t))\sigma_\xi(\nu(t),\mu(t))}{\sigma_\xi(\nu(t),T\nu(t))+\sigma_\xi(\nu(t),T\mu(t))+\sigma_\xi(\nu(t),\mu(t))}\}$$

(iii) there exists $\wp_1\in A$ such that $\alpha(\wp_1,T\wp_1)\ge\eta(\wp_1,T\wp_1)$

.

(iv) T is a $\alpha-\eta$ continuous mapping.

Then the integral equation defined by (4.1.1) has a solution.

Proof. We define $T:A\to A$ by

$$T(\mu(t))=g(t,\mu(t))+\int_0^1 f(t,s,\mu(t))dt \tag{4.2.1}$$

where $g:[0,1]\times R\to R$ and $f:[0,1]\times[0,1]\times R\to R$ are continuous functions. It is clear that, finding solution of integral equation is to investigate fixed point in A.

We now define $\alpha: A\times A\to[0,\infty)$ and $\eta: A\times A\to[0,\infty)$ by $\alpha(\mu(t),\nu(t))=\eta((\mu(t),\nu(t))=1$. Therefore T is triangular α - orbital admissible map with respect to η.

Consider

$$\begin{aligned}\sigma_\xi(T\mu(t),T\nu(t)) &= |T\mu(t)+T\nu(t))|^2\\
&\le(|g(t,\mu(t))+\textstyle\int_0^1 f(t,s,\mu(s))ds+g(t,\nu(t))+\int_0^1 f(t,s,\nu(s))ds|)^2\\
&\le(|g(t,\mu(t))+g(t,\nu(t))|+|\textstyle\int_0^1 f(t,s,\mu(s))ds+\int_0^1 f(t,s,\nu(s))ds|)^2\\
&\le(||g(t,\mu(t))+g(t,\nu(t))|+|\textstyle\int_0^1|f(t,s,\mu(s))+f(t,s,\nu(s))ds|)^2\\
&\le[\sqrt{\tfrac{1}{9}M_\xi(\mu(t),\nu(t))}+\sqrt{\tfrac{1}{81}M_\xi(\mu(t),\nu(t))}]^2\\
&=\tfrac{16}{81}M_\xi(\mu(t),nu(t)))=\tfrac{1}{2}\phi(M_\xi(\mu(t),\nu(t))),\end{aligned}$$

which implies

$$0\le\frac{1}{2}\phi(M_\xi(\mu(t),\nu(t))-\sigma_\xi(T\mu(t),T\nu(t)))$$
$$0\le\zeta(\sigma_\xi(T\mu(t),T\nu(t)),\phi(M_\xi(\mu(t),\nu(t))).$$

This proves that generalized $\alpha-\eta$ extended Z-contraction. Thus all the conditions of Theorem 2.4 are verified, with a simulation function $\zeta(t,s)=\frac{s}{2}-t$ and (b)- comparison function $\phi(t)=\frac{8t}{81}$, hence T has unique fixed point, which is a solution of integral equation (4.1.1).

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