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FIXED POINT THEOREMS IN C^*-ALGEBRA VALUED ASYMMETRIC METRIC SPACES

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Abstract In this work, we introduce the concept of C^*-algebra valued asymmetric metric space, the concept of forward and the concept of backward C^*-valued asymmetric contractions. We discuss the existence and uniqueness of fixed points for a self-mapping defined on a C^*-algebra valued asymmetric metric space, and we give an application.

1 Introduction

The scientific starting point of the fixed point theory was set up in the 20th century. The fundamental outcome of this theory is the Picard-Banach-Caccioppoli contraction principle which brought into crucial and relevant fields of research: the theory of functional equations, integral equations, physic, economy, ...

Many researchers have dealt with the theory of fixed point in two ways: the first affirms the conditions on the mapping whereas the second takes the set as a more general structure. Indeed the fixed point theorem is established in several cases such as asymmetric metric spaces which generalize metric spaces. These spaces are introduced by Wilson [1] and have been studied by J. Collins and J. Zimmer . Other interesting results in asymmetric metric spaces have also been demonstrated by Aminpour, Khorshidvandpour and Mousavi [10]. This research has contributed to interesting applications, for example in rate-independent plasticity models [8], shape memory alloys [9], material failure models [7]. In mathematics, we find other applications such as the study of asymmetric metric spaces to prove the existence and uniqueness of Hamilton-Jacobi equations [7].

Recently, in a more general context, Zhenhua Ma, Lining Jiang and Hongkai Sun introduced the notion of C^*-algebra valued metric spaces and analogous to the Banach contraction principle and established a fixed point theorem for C^*-valued contractive mappings [3]. These results were generalized by Samina Batul and Tayyab Kamran in [5] by introducing the concept of C^*-valued contractive type condition. M.Mlaiki et al. [4] define the C^*-algebra valued partial b-metric spaces. In [11], G. Kalpana and Z. S. Tasneem introduce the definition of a C^*-algebra valued rectangular b-metric spaces and interpret the notion of C^*-algebra valued triple controlled metric type spaces and derive certain fixed point theorems for Banach and Kannan type contraction mappings of the underlying spaces [12].

In this paper, we first introduce the notion of C^*-algebra valued asymmetric metric spaces and we establish a fixed point theorem analogous to the results presented in [5]. Some examples are provided to illustrate our results. Finally, existence and uniqueness results for a type of operator equation is given.

2 Preliminaries

In this section, we give some basic definitions. $\mathbb{A}$ will denote a unitary C* -algebra with a unit $I_{\mathbb{A}}$. An involution on $\mathbb{A}$ is a conjugate linear map $a \mapsto a^*$ on $\mathbb{A}$ such that
$a^{**} = a$ and $(ab)^* = b^*a^*$ for all a and b in $\mathbb{A}$.

A Banach *-algebra is a algebra provided with a involution and a complete multiplicative norm such that $\|a^*\| = \|a\|$ for all a in $\mathbb{A}$.

A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$. $\mathbb{A}_h$ will denote the set of all self-adjoint elements a (i.e., satisfying $a^* = a$), and $\mathbb{A}^+$ will be the set of positive elements of $\mathbb{A}$, i.e., the elements $a \in \mathbb{A}_h$ having the spectrum $\sigma(a)$ contained in $[0, +\infty)$. Note that $\mathbb{A}^+$ is a (closed) cone in the normed space $\mathbb{A}$ [2], which infers a partial order $\preceq$ on $\mathbb{A}_h$ by $a \preceq b$ if and only if $b - a \in \mathbb{A}^+$. When $\mathbb{A}$ is a unitary C*-algebra, then for any $x \in \mathbb{A}_+$ we have $|x| = (x^*x)^{\frac{1}{2}}$. We will use the following results.

Lemma 2.1. *[2] Suppose that $\mathbb{A}$ is a unitary C^*-algebra with a unit $I_{\mathbb{A}}$*

(i) $\mathbb{A}^+ = \{a^*a : a \in \mathbb{A}\}$*;*

(ii) if $a, b \in \mathbb{A}_h$, $a \preceq b$, and $c \in \mathbb{A}$, then $c^*ac \preceq c^*bc$;

(iii) for all $a, b \in \mathbb{A}_h$, if $0_{\mathbb{A}} \preceq a \preceq b$ then $\|a\| \leq \|b\|$;

(iv) $0 \preceq a \preceq I_{\mathbb{A}} \Leftrightarrow \|a\| \leq 1$.

Lemma 2.2. *[2] Suppose that $\mathbb{A}$ is a unitary C^*-algebra with a unit $I_{\mathbb{A}}$.*

(i) if $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I_{\mathbb{A}} - a$ is invertible and $\left\|a(I_{\mathbb{A}} - a)^{-1}\right\| < 1$;

(ii) suppose that $a, b \in \mathbb{A}$ with $a, b \succeq 0_{\mathbb{A}}$ and $ab = ba$, then $ab \succeq 0_{\mathbb{A}}$;

(iii) by $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq 0_{\mathbb{A}}$ and $I_{\mathbb{A}} - a \in \mathbb{A}'_+$ is a invertible operator, then $(I_{\mathbb{A}} - a)^{-1}b \succeq (I_{\mathbb{A}} - a)^{-1}c$.

3 Main results

To begin with, let us start from some basic definitions.

Definition 3.1. Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:

(i) $0_{\mathbb{A}} \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_{\mathbb{A}} \Leftrightarrow x = y$;

(ii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra valued asymmetric metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued asymmetric metric space.

It is obvious that C^*-algebra-valued asymmetric metric spaces generalize the concept of C^*-algebra valued b-metric spaces [**?**].

Example 3.2. Let $\mathbb{A} = M_{2\times 2}(\mathbb{R})$ and $X = \mathbb{R}$. Define $d : \mathbb{R} \times \mathbb{R} \to M_{2\times 2}(\mathbb{R})$ by

$$d(x,y) = \begin{cases} \begin{bmatrix} x-y & 0 \\ 0 & 0 \end{bmatrix} & \text{if } x \geqslant y \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & y-x \end{bmatrix} & \text{if } x < y \end{cases}$$

with $\left\| \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \right\| = \left(\sum_{i=1}^{4} |x_i|^2 \right)^{\frac{1}{2}}$ where x_i are real numbers. Then $(X, \mathbb{A}, d)$ is a C^*-algebra valued asymmetric metric space, where partial ordering on $\mathbb{A}_+$ is given as

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \succeq \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} \Leftrightarrow x_i \geq y_i \geq 0 \text{ for } i = 1, 2, 3, 4.$$

It is clear that $0_{\mathbb{A}} \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_{\mathbb{A}} \Leftrightarrow x = y$.
We will verify triangular inequality. Let x, y and z in $\mathbb{R}$ then we have six cases:

(i) let $x \leqslant y$ then $d(x, y) = \begin{bmatrix} 0 & 0 \\ 0 & y-x \end{bmatrix}$

a. If $x \leqslant y \leqslant z$

$$d(x,z)+d(z,y)=\begin{bmatrix} 0 & 0 \\ 0 & z-x \end{bmatrix}+\begin{bmatrix} z-y & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} z-y & 0 \\ 0 & z-x \end{bmatrix}\succeq d(x,y).$$

b. If $x \leqslant z \leqslant y$

$$d(x,z)+d(z,y)=\begin{bmatrix} 0 & 0 \\ 0 & z-x \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & y-z \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & y-x \end{bmatrix}\succeq d(x,y).$$

c. If $z \leqslant x \leqslant y$

$$d(x,z)+d(z,y)=\begin{bmatrix} x-z & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & y-z \end{bmatrix}=\begin{bmatrix} x-z & 0 \\ 0 & y-z \end{bmatrix}\succeq d(x,y).$$

(ii) Let $x \geqslant y$ then $d(x,y)=\begin{bmatrix} x-y & 0 \\ 0 & 0 \end{bmatrix}$

a. If $x \geqslant y \geqslant z$

$$d(x,z)+d(z,y)=\begin{bmatrix} x-z & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & y-z \end{bmatrix}=\begin{bmatrix} x-z & 0 \\ 0 & y-z \end{bmatrix}\succeq d(x,y).$$

b. If $x \geqslant z \geqslant y$

$$d(x,z)+d(z,y)=\begin{bmatrix} x-z & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} z-y & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} x-y & 0 \\ 0 & 0 \end{bmatrix}\succeq d(x,y).$$

c. If $z \geqslant x \geqslant y$

$$d(x,z)+d(z,y)=\begin{bmatrix} 0 & 0 \\ 0 & z-x \end{bmatrix}+\begin{bmatrix} z-y & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} z-y & 0 \\ 0 & z-x \end{bmatrix}\succeq d(x,y).$$

Note that $d(1,2) \neq d(2,1)$.

Example 3.3. Let $\mathbb{A}=L^\infty(\mathbb{R})$ and $X=\mathbb{R}$. Define $d: X\times X \to L^\infty(\mathbb{R})$ by $d(x,y)=f_{x,y}$

$$f_{x,y}:\mathbb{R}\longrightarrow\mathbb{R} \quad , f_{x,y}(t)=\begin{cases} (x-y)\,t & if x \geqslant y \\ (y-x)\,\dfrac{T-t}{T} & if x<y \end{cases}$$

where $T\in\mathbb{R}^+$ and $f_{x,y}$ is a T-periodic function, we have:

(i) $0_{\mathbb{A}} \preceq d(x,y)$ for all $x,y\in X$ and $d(x,y)=0_{\mathbb{A}} \Leftrightarrow x=y$;

(ii) We will verify triangular inequality. Let x,y and z in $\mathbb{R}$. For $t\in[0,T[$ we have six cases:

a. If $x \leqslant y \leqslant z$

$$\begin{cases} d(x,y)(t)=f_{x,y}(t)=(y-x)\,\dfrac{T-t}{T} \\ d(x,z)(t)+d(z,y)(t)=(z-x)\,\dfrac{T-t}{T}+(z-y)\,t\succeq(y-x)\,\dfrac{T-t}{T}=d(x,y)(t). \end{cases}$$

b. If $z \leqslant x \leqslant z \leqslant y$

$$\begin{cases} d(x,y)(t)=(y-x)\,\dfrac{T-t}{T} \\ d(x,z)(t)+d(z,y)(t)=(x-z)\,t+(y-z)\,\dfrac{T-t}{T}\succeq(y-x)\,\dfrac{T-t}{T}=d(x,y)(t). \end{cases}$$

c. If $x \leqslant z \leqslant y$

$$\begin{cases} d(x,y)(t)=(y-x)\,\dfrac{T-t}{T} \\ d(x,z)(t)+d(z,y)(t)=(z-x)\,\dfrac{T-t}{T}+(y-z)\,\dfrac{T-t}{T}=(y-x)\,\dfrac{T-t}{T}\succeq d(x,y)(t). \end{cases}$$

d. If $y \leqslant x \leqslant z$

$$\begin{cases} d(x,y)(t)=f_{x,y}(t)=(x-y)\,t \\ d(x,z)(t)+d(z,y)(t)=(z-x)\,\dfrac{T-t}{T}+(z-y)\,t\succeq(x-y)\,t=d(x,y)(t). \end{cases}$$

e. If $z \leqslant y \leqslant x$

$$\begin{cases} d(x,y)(t) = (x-y)\,t \\ d(x,z)(t) + d(z,y)(t) = (x-z)\,t + (y-z)\,\dfrac{T-t}{T} \succeq (x-y)\,t = d(x,y)(t). \end{cases}$$

f. If $y \leqslant z \leqslant x$

$$\begin{cases} d(x,y)(t) = (x-y)\,t \\ d(x,z)(t) + d(z,y)(t) = (x-z)\,t + (z-y)\,t = (x-y)\,t \succeq d(x,y)(t). \end{cases}$$

Note that $d(\frac{T}{2},0)(t) = \frac{T}{2}t$ and $d(0,\frac{T}{2})(t) = \frac{T-t}{2}$ for all $t \in [0,T[$

In what follows, we define in the same way the forward convergence and the backward convergence in [1] but in a more general context.

Definition 3.4. Let $(X, d, \mathbb{A})$ be a C^*-algebra valued asymmetric metric space, $x \in X$ and $\{x_n\}$ a sequence in X.

(i) one say $\{x_n\}$ forward converges to x with respect to $\mathbb{A}$ and we write $x_k \xrightarrow{f} x$, if and only if for given $\epsilon \succ 0_{\mathbb{A}}$, there exists $k \in \mathbb{N}$ such that for all $n \geqslant k$

$$d\left(x, x_n\right) \preceq \epsilon.$$

(ii) one say $\{x_n\}$ backward converges to x with respect to $\mathbb{A}$ and we write $x_n \xrightarrow{b} x$, if and only if for given $\epsilon \succ 0_{\mathbb{A}}$, there exists $k \in \mathbb{N}$ such that for all $n \geqslant k$

$$d\left(x_n, x\right) \preceq \epsilon.$$

(iii) one say $\{x_n\}$ converges to x if $\{x_n\}$ forward converges and backward converges to x.

Example 3.5. $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} y - x & \text{if } y \geqslant x \\ 1 & \text{if } y < x \end{cases}.$$

Let $x \in \mathbb{R}^+$ and let $x_n = x\left(1 + \frac{1}{n}\right)$. Then $x_n \xrightarrow{f} x$ but $x_n \overset{b}{\nrightarrow} x$. This example asserts that the existence of a forward limit does not imply the existence of a backward limit.

Lemma 3.6. *Let $(X, \mathbb{A}, d)$ a C^*-algebra valued asymmetric metric space. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.*

Proof. Fix $\varepsilon \succ 0_{\mathbb{A}}$. By assumption, $x_n \xrightarrow{f} x$ so there exists $N_1 \in \mathbb{N}$ such that $d\left(x, x_n\right) \preceq \frac{\varepsilon}{2}$ for all $n \geqslant N_1$. Also, $x_n \xrightarrow{b} y$, so there exists $N_2 \in \mathbb{N}$ such that $d\left(x_n, y\right) \preceq \frac{\varepsilon}{2}$ for all $n \geqslant N_2$. Then for all $n \geqslant N := \max\left\{N_1, N_2\right\}, d\left(x, y\right) \preceq d\left(x, x_n\right) + d\left(x_n, y\right) \preceq \varepsilon$. As ε was arbitrary, we deduce that $d\left(x, y\right) = 0$, which implies $x = y$ □

Definition 3.7. Let $(X, \mathbb{A}, d)$ a C^*-algebra valued asymmetric metric space and $\{x_n\}_n$ a sequence in X.

(i) One say that $\{x_n\}$ forward Cauchy sequence (with respect to $\mathbb{A}$), if for given $\epsilon \succ 0_{\mathbb{A}}$, there exists k belonging to $\mathbb{N}$ such that for all $n > p \geqslant k$

$$d\left(x_p, x_n\right) \preceq \epsilon.$$

(ii) One say that $\{x_n\}$ backward Cauchy sequence (with respect to $\mathbb{A}$), if for given $\epsilon \succ 0_{\mathbb{A}}$, for all $n > p \geqslant k$

$$d\left(x_n, x_p\right) \preceq \epsilon.$$

Definition 3.8. Let $(X, d, \mathbb{A})$ a C^*-algebra valued asymmetric metric space. X is said to be forward (backward) complete if every forward (backward) Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, forward (backward) converges to $x \in X$.

Definition 3.9. Let $(X, d, \mathbb{A})$ a C^*-algebra valued asymmetric metric space. X is said to be complete if X is forward and backward complete.

Example 3.10. we take the example(3.2), $(\mathbb{R}, L^\infty(\mathbb{R}), d)$ is a complete C^*-algebra valued asymmetric metric space.

Indeed, it suffices to verify the completeness. Let $\{x_n\}$ in $\mathbb{R}$ be a Cauchy sequence with respect to $L^\infty(\mathbb{R})$. Then for a given $\varepsilon > 0$, there is a natural number N such that for all $n, p \geq N$

$$\|d(x_n, x_p)\|_\infty = \|f_{x_n,x_p}\|_\infty < \varepsilon,$$

since

$$\|f_{x_n,x_p}\|_\infty = \begin{cases} (x_n - x_p)\,T & \text{if } x_n \geqslant x_p \\ (x_p - x_n) & \text{if } x_p > x_n \end{cases}$$

then $\{x_n\}$ is a Cauchy sequence in the space $\mathbb{R}$. Thus, there is x in $\mathbb{R}$ such that $\{x_n\}$ converges to x. For $\epsilon > 0$ there exists number k belonging to $\mathbb{N}$ such that $|x_n - x| \leq \varepsilon$ if $n \geq k$. It follows that :

$$\|d(x, x_n)\|_\infty \vee \|d(x_n, x)\|_\infty \leq \varepsilon\, max\{1, T\},$$

therefore, the sequence $\{x_n\}$ converges to x in $\mathbb{R}$ with respect to $L^\infty(\mathbb{R})$, that is, $(\mathbb{R}, L^\infty(\mathbb{R}), d)$ is complete with respect to $L^\infty(\mathbb{R})$.

Definition 3.11. Let $(X, d, \mathbb{A})$ be C^*-algebra valued asymmetric metric space. A mapping $T : X \to X$ is said forward (respectively backward) C^*-algebra valued contractive mapping on X, if there exists a in $\mathbb{A}$ with $\|a\| < 1$ such that

$$d(Tx, Ty) \preceq a^* d(x, y) a,$$

$$(\text{respectively } \; d(Tx, Ty) \preceq a^* d(y, x) a)$$

for each $x, y \in X$.

Example 3.12. Let $\mathbb{A} = M_{2\times2}(\mathbb{R})$ and $X = \mathbb{R}$. Define $d : \mathbb{R} \times \mathbb{R} \to M_{2\times2}(\mathbb{R})$ by

$$d(x, y) = \begin{cases} \begin{bmatrix} x - y & 0 \\ 0 & 0 \end{bmatrix} & \text{if } x \geqslant y \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{4}(y - x) \end{bmatrix} & \text{if } x < y, \end{cases}$$

then $(X, \mathbb{A}, d)$ is a C^*-algebra valued asymmetric metric space, where the norm and the partial ordering on $\mathbb{A}^+$ are given as example 3.1.

Consider $T : X \to X$ by $Tx = \frac{1}{4}x$. Then,

$$d(Tx, Ty) = d(\frac{1}{4}x, \frac{1}{4}y) \begin{cases} \begin{bmatrix} \frac{1}{4}(x - y) & 0 \\ 0 & 0 \end{bmatrix} & \text{if } x \geqslant y \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{16}(y - x) \end{bmatrix} & \text{if } x < y \end{cases}$$

it follows that

$$d(Tx, Ty) \preceq a^* d(x, y) a.$$

Indeed

$$d(Tx,Ty)=\begin{cases}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x-y & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\preceq a^*d(x,y)a, \text{ if } x\geqslant y\\ \\ \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \frac{1}{4}(y-x)\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\preceq a^*d(x,y)a, \text{ if } x<y\end{cases}$$

where $a=\begin{bmatrix}\frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{3}}\end{bmatrix}$.

Next, we prove asymmetric version of C^*-algebra valued contractive mapping [3].

Theorem 3.13. *If $(X,\mathbb{A},d)$ is a complete C^*-algebra-valued asymmetric metric space and T is a forward C^*-algebra valued contractive mapping, then there exists a unique fixed point in X.*

Proof. One suppose that $\mathbb{A}\neq 0_{\mathbb{A}}$. Choose $x\in X$.

Notice that in a C^*-algebra, if $a,b\in\mathbb{A}_+$ and $a\preceq b$, then for any $x\in\mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax\preceq x^*bx$. Thus

$$\begin{aligned}d\left(T^{n+1}x,T^nx\right)&=d\left(T(T^nx),T(T^{n-1}x)\right)\\&\preceq a^*d\left(T^nx,T^{n-1}x\right)a\\&\preceq (a^*)^2d\left(T^{n-1}x,T^{n-2}x\right)a^2\\&\preceq\cdots\\&\preceq (a^*)^n d\left(Tx,x\right)a^n.\end{aligned}$$

Take $n+1>p$

$$\begin{aligned}d\left(T^{n+1}x,T^px\right)&\preceq d\left(T^{n+1}x,T^nx\right)+d\left(T^nx,T^{n-1}x\right)+\cdots+d\left(T^{p+1}x,T^px\right)\\&\preceq\sum_{k=p}^{n}(a^*)^k d\left(Tx,x\right)a^k\\&=\sum_{k=p}^{n}(a^*)^k d\left(Tx,x\right)^{\frac{1}{2}}d\left(Tx,x\right))^{\frac{1}{2}}a^k\\&=\sum_{k=p}^{n}\left(d\left(Tx,x\right)^{\frac{1}{2}}a^k\right)^*\left(d\left(Tx,x\right)^{\frac{1}{2}}a^k\right)\\&=\sum_{k=p}^{n}\left|d\left(Tx,x\right)^{\frac{1}{2}}a^k\right|^2\\&\preceq\left\|\sum_{k=p}^{n}\left|d\left(Tx,x\right)^{\frac{1}{2}}a^k\right|^2\right\|I_{\mathbb{A}}\\&\preceq\left\|d\left(Tx,x\right)^{\frac{1}{2}}\right\|^2\sum_{k=p}^{n}\|a\|^{2k}I_{\mathbb{A}}\\&\preceq\left\|d\left(Tx,x\right)^{\frac{1}{2}}\right\|^2\frac{\|a\|^{2p}}{1-\|a\|^2}I_{\mathbb{A}}\to 0_{\mathbb{A}}\quad(p\to\infty).\end{aligned}$$

In the same way we prove

$$d\left(T^px,T^{n+1}x\right)\preceq\left\|d\left(x,Tx\right)^{\frac{1}{2}}\right\|^2\frac{\|a\|^{2p}}{1-\|a\|^2}I_{\mathbb{A}}\to 0_{\mathbb{A}}\quad(p\to\infty).$$

Therefore $\{x_n\}$ is a forward and backward Cauchy sequence. By the completeness of $(X, \mathbb{A}, d)$, there exists an $x_0 \in X$ such that $\{T^n x\}$ converges to x_0 with respect to $\mathbb{A}$.

One has

$$\begin{aligned} \theta &\preceq d(Tx_0, x_0) \preceq d\left(Tx_0, T^{n+1}x\right) + d\left(T^{n+1}x, x_0\right) \\ &\preceq a^* d\left(x_0, T^n x\right) a + d\left(T^{n+1}x, x_0\right) \to 0_{\mathbb{A}} \quad (n \to \infty). \end{aligned}$$

Hence, $Tx_0 = x_0$, therefore x_0 is a fixed point of T.
Now suppose that $y (\neq x_0)$ is another fixed point of T, since

$$0_{\mathbb{A}} \preceq d(x_0, y) = d(Tx_0, Ty) \preceq a^* d(x_0, y) a$$

we have

$$\begin{aligned} 0 &\leq \|d(x_0, y)\| = \|d(Tx_0, Ty)\| \\ &\leq \|a^* d(x_0, y) a\| \\ &\leq \|a^*\| \, \|d(x_0, y)\| \|a\| \\ &= \|a\|^2 \|d(x_0, y)\| \\ &< \|d(x_0, y)\|, \end{aligned}$$

which is impossible. So $d(x_0, y) = 0_{\mathbb{A}}$ and $x_0 = y$, which implies that the fixed point is unique. □

Definition 3.14. (Forward T-orbitally lower semi-continuous) A function $G : X \to \mathbb{A}$ is said to be forward T-orbitally lower semi continuous at x_0 with respect to $\mathbb{A}$ if the sequence $\{x_n\}$ in $\mathcal{O}_T(x)$ is such that $x_n \xrightarrow{f} x$ with respect to $\mathbb{A}$ implies

$$\|G(x_0)\| \leqslant \liminf \|G(x_n)\|$$

where $\mathcal{O}_T(x) = \{T^n x \mid n \in \mathbb{N}\}$.

Definition 3.15. (Forward Contractive Type Mapping) Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued asymmetric metric space. A mapping $T : X \to X$ is said to be a forward C^*-valued contractive type mapping if there exists an $x \in X$ and an $a \in \mathbb{A}$ such that

$$d\left(Ty, T^2 y\right) \preceq a^* d(y, Ty) a$$

with $\|a\| < 1$ for every $y \in \mathcal{O}_T(x)$.

Theorem 3.16. *Let $(X, \mathbb{A}, d)$ be a forward complete C^*-algebra valued asymmetric metric space and $T : X \to X$ be a forward C^*-algebra valued contractive type mapping.*
Then

(i) $\exists x_0 \in X$ such that the sequence $T^n x$ in $\mathcal{O}_T(x)$ forward converges to x_0,

(ii) x_0 is a fixed point of T if and only if the map $G(x) = d(x, Tx)$ is forward T-orbitally lower semi continuous at x_0 with respect to $\mathbb{A}$.

Proof. We assume that $\mathbb{A}$ is a nontrivial C^*-algebra.

(i) Since the above forward contractive condition holds for each element of $\mathcal{O}_T(x)$ and $\|a\| < 1$, it follows that:

$$d\left(T^n x, T^{n+1} x\right) \preceq (a^*)^n \, d(x, Tx) a^n.$$

Then for $p < n$, we have from the triangular inequality that

$$\begin{aligned} d\left(T^p x, T^{n+1} x\right)) &\preceq d\left(T^p x, T^{p+1} x\right) + d\left(T^{p+1} x, T^{p+2} x\right) + \ldots . d\left(T^n x, T^{n+1} x\right)) \\ &\preceq \sum_{k=p}^{n} \left(a^*\right)^k d(x, Tx) a^k \\ &\preceq \sum_{k=p}^{n} \left\| d\left(x, Tx\right)^{\frac{1}{2}} \right\|^2 \left\| a^k \right\|^2 .1_{\mathbb{A}} \\ &\preceq \left\| d\left(x, Tx\right)^{\frac{1}{2}} \right\|^2 \sum_{k=p}^{n} \|a\|^{2k} .1_{\mathbb{A}} \\ &\preceq \left\| d\left(x, Tx\right)^{\frac{1}{2}} \right\|^2 \frac{\|a\|^{2p}}{1 - \|a\|^2} .1_{\mathbb{A}} \to 0_{\mathbb{A}} \quad (p \to \infty). \end{aligned}$$

This shows that $\{T^n x\}$ is a forward Cauchy sequence in X with respect to $\mathbb{A}$. By forward completeness of $(X, \mathbb{A}, d)$, there exists some $x_0 \in X$ such that

$$T^n x \xrightarrow{f} x_0$$

with respect to $\mathbb{A}$.

(ii) one suppose that $Tx_0 = x_0$ and $\{T^n x\}$ is a sequence in $\mathcal{O}_T(x)$ with $T^n x \xrightarrow{f} x_0$ with respect to A, then

$$\begin{aligned} \|G\left(x_0\right)\| &= \|d\left(x_0, Tx_0\right)\| \\ &= 0 \\ &\leq \liminf \|G\left(T^n x\right)\|. \end{aligned}$$

Reciprocally, if G is forward T-orbitally lower semi continuous at x_0 then

$$\begin{aligned} \|G\left(x_0\right)\| = \|d\left(x_0, Tx_0\right)\| &\leq \liminf \| G\left(T^n x\right)\| \\ &= \liminf \left\| d\left(T^n x, T^{n+1} x\right) \right\| \\ &\leq \liminf \|a\|^{2n} \|d(x, Tx)\| \\ &= 0 \end{aligned}$$

as a result $d\left(x_0, Tx_0\right) = 0_{\mathrm{A}}$, proving T has a fixed point.

□

Example 3.17. $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x,y) = \begin{cases} x - y & \text{if } x \geqslant y \\ 1 & \text{if } x < y \end{cases}$$

We consider $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such as $Tx = \dfrac{x}{4}$

$$d(Tx,Ty) = \begin{cases} \frac{1}{4}(x-y) & x \geqslant y \\ 1 & x < y \end{cases}$$

T is not a forward C^*-valued contractive mapping.
If $x < y$, we know $d(Tx,Ty) \leqslant a^* d(x,y) a$, then

$$\begin{aligned} d(Tx,Ty) &\leqslant a^* d(x,y) a \\ 1 &\leqslant a^2 \\ 1 &\leqslant \|a\| \end{aligned}$$

therefore contradiction.

We prove that T is forward C^*-valued contractive type mapping.
Let $x > 0$. We have

$$\begin{cases} d\left(Ty, T^2y\right) = d\left(\frac{y}{4}, \frac{y}{16}\right) = \frac{3y}{16} \\ d(y, Ty) = \frac{3y}{4} \end{cases}$$

then, there exists a in $\mathbb{A}$ such that $d\left(Ty, T^2y\right) \preceq a^* d(y, Ty)a$ for every $y \in \mathcal{O}_T(x)$ with $\|a\| = |\frac{1}{\sqrt{2}}| < 1$.

Define $G : X \to A$ by

$$G(x) = d(x, Tx)$$

so

$$\liminf_{x \to 0} G(x) = G(0) = 0,$$

then G is forward T-orbitally lower semi continuous at zero and 0 is a fixed point of T.

Example 3.18. Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{A} = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$ by

$$d(x,y) = \begin{cases} \begin{bmatrix} x-y & 0 \\ 0 & 0 \end{bmatrix} & \text{if } x \geqslant y \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & y-x \end{bmatrix} & \text{if } x < y \end{cases}$$

with partial ordering and norm on $\mathbb{A}$ are given as example 3.1.

We consider $T : \mathbb{R} \to \mathbb{R}$ such that

$$Tx = \begin{cases} \frac{x}{4} & x \geqslant 0 \\ 1 & x < 0 \end{cases}$$

Then for $y \in \mathcal{O}_T(x), x \geq 0$

$$\begin{aligned} d\left(Ty, T^2y\right) &= \begin{bmatrix} \frac{y}{4} - \frac{y}{16} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3y}{16} & 0 \\ 0 & 0 \end{bmatrix} \\ &\preceq \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{3y}{4} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \\ &= a^* d(Ty, y)a, \end{aligned}$$

where

$$a = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \textit{ and } \|a\| = \frac{\sqrt{2}}{\sqrt{3}} < 1.$$

Theorem 3.19. *Let $(X, \mathbb{A}, d)$ be a forward complete C^*-algebra valued asymmetric metric space and $T : X \to X$ be a mapping which satisfies for all $y \in \mathcal{O}_T(x)$*

$$d\left(Ty, T^2y\right) \preceq a\, d\left(y, T^2y\right)$$

with $\|a\| \leq \frac{1}{2}$ and $a \in \mathbb{A}'_+$, then

(i) $\exists x_0 \in X$ such that the sequence $T^n x$ forward converges to x_0,

(ii) x_0 is a fixed point of T if and only if $G(x) = d(x, Tx)$ is forward T-orbitally lower semi continuous at x_0 with respect to $\mathbb{A}$.

Proof. Assume that $\mathbb{A} \neq \{0_{\mathrm{A}}\}$.

$$\begin{aligned} d\left(T^n x, T^{n+1} x\right) &= d\left(T(T^{n-1} x), T^{n+1} x\right) \\ &\preceq ad\left(T^{n-1} x, T^{n+1} x\right) \\ &\preceq a\left[d\left(T^{n-1} x, T^n x\right) + d\left(T^n x, T^{n+1} x\right)\right] \\ &\preceq ad\left(T^{n-1} x, T^n x\right) + ad\left(T^n x, T^{n+1} x\right). \end{aligned}$$

Thus

$$d\left(T^n x, T^{n+1} x\right) - ad\left(T^n x, T^{n+1} x\right) \preceq ad\left(T^{n-1} x, T^n x\right)$$

which implies that

$$(1_{\mathrm{A}} - a)\, d\left(T^n x, T^{n+1} x\right) \preceq ad\left(T^{n-1} x, T^n x\right).$$

Since $a \in \mathbb{A}'_+$ with $\|a\| < \frac{1}{2}$, by Lemma(2.2) we have $(1_{\mathrm{A}} - a)^{-1} \in \mathbb{A}'_+$ and also

$$a\,(1_{\mathrm{A}} - a)^{-1} \in \mathbb{A}'_+ \text{ with } \left\|a\,(1_{\mathbb{A}} - a)^{-1}\right\| < 1.$$

Therefore

$$d\left(T^n x, T^{n+1} x\right) \preceq a\,(1_{\mathbb{A}} - a)^{-1}\, d\left(T^{n-1} x, T^n x\right).$$

Let's consider $h = a\,(1_{\mathbb{A}} - a)^{-1}$ then

$$d\left(T^n x, T^{n+1} x\right) \preceq hd\left(T^{n-1} x, T^n x\right).$$

Let $\{T^n x\}$ be a sequence in $\mathcal{O}_T(x)$. Then from the triangular inequality, for $m < n$ we have

$$\begin{aligned} d\left(T^m x, T^{n+1} x\right) &\preceq \sum_{k=m}^{n} \left\|h^{k/2}\right\|^2 \left\|d(x, Tx)^{1/2}\right\|^2 1_{\mathbb{A}} \\ &\preceq \left\|d(x, Tx)^{1/2}\right\|^2 \sum_{k=m}^{n} \left\|h^{k/2}\right\|^2 1_{\mathbb{A}} \\ &\preceq \left\|d(x, Tx)^{1/2}\right\|^2 \frac{\|h\|^m}{1 - \|h\|} 1_{\mathbb{A}} \\ &\longrightarrow 0_{\mathbb{A}} \text{ as } m \longrightarrow \infty. \end{aligned}$$

This proves that $\{T^n x\}$ is a forward Cauchy sequence in X with respect to $\mathbb{A}$. Since $(X, \mathbb{A}, d)$ is a forward complete C^*-algebra valued asymmetric metric space, there exists $x_0 \in X$ such that $T^n x \xrightarrow{f} x_0$.

If $Tx_0 = x_0$ and $\{x_n\}$ is a sequence in $\mathcal{O}_T(x)$ such that $T^n x \xrightarrow{f} x_0$ with respect to A, then

$$\begin{aligned} \|G\left(x_0\right)\| &= \|d\left(x_0, Tx_0\right)\| \\ &= 0 \\ &\leq \liminf \|G\left(x_n\right)\|. \end{aligned}$$

Conversely, if G is T-orbitally lower semi continuous at x_0 then

$$\begin{aligned} \|G\left(x_0\right)\| = \|d\left(x_0, Tx_0\right)\| &\leq \liminf \|G\left(T^n x\right)\| \\ &= \liminf \left\|d\left(T^n x, T^{n+1} x\right)\right\| \\ &\leq \liminf \|h\|^n \|d(x, Tx)\| \\ &= 0 \end{aligned}$$

this implies that

$$d\left(x_0, Tx_0\right) = 0_{\mathbb{A}}$$

thus T has a fixed point. □

4 Application

In this section, we will apply our theorem to prove the existence of solution of integral equation. Let G be the multiplicative group $]0;1]$ with its left invariant Haar measure μ. Defined by:

$$H = L^2(G) = \left\{ f : G \to \mathbb{R} \quad | \int_G |f(t)|^2 \, d\mu(t) < \infty \right\} \text{ which's an Hilbert space}$$

$$X = L^\infty(G) = \{ f : G \to \mathbb{R} \quad | \quad \|f\|_\infty < \infty \} \text{ which's a Banach algebra.}$$

Let $B(H)$ the set of all bounded linear operators on the Hilbert space H. Note that $B(H)$ is a unitary C^*-algebra. We define an asymmetric metric as:

$$d : X \times X \to B(H)$$
$$(f, g) \to d(f, g)$$

with

$$d(f,g) = \begin{cases} \pi_{\frac{1}{2}(f-g)\chi_{\{f>g\}}} + \pi_{(g-f)\chi_{\{g>f\}}} & if \quad f \neq g \\ 0 \qquad if \quad f = g \end{cases}$$

where π_f is the multiplication operator given by :

$$\pi_f : X \to X$$
$$\psi \to f.\psi$$

and

$$\chi_A(t) = \begin{cases} 1 \quad if & x \in A \\ 0 \quad if & x \in A^c \end{cases}$$

It is knwon that $\|\pi_f\| = \|f\|_\infty$.

Here $(X, B(H), d)$ is a complete C^*-valued asymmetric metric space with respect to $B(H)$.

Let

$$K : \quad G \times G \times \mathbb{R} \to \mathbb{R}$$
$$(x, y, t) \to \alpha.x \frac{t}{y^2 + k} \qquad (\alpha > 0, k > 0).$$

Let

$$T : X \to X$$
$$f \to Tf$$
$$Tf(x) = \int_G K(x, y, f(y)) d\mu(y), \; x \in G.$$

Choose f_0 defined as follows:

$$f_0 : G \to \mathbb{R}$$
$$x \to x$$

then

$$\begin{aligned} Tf_0(x) &= \int_G K\left(x, y, f_0(y)\right) d\mu(y) \\ &= \int_0^1 \alpha x \frac{y}{y^2 + k} d\mu(y) \\ &= \alpha \cdot x \int_0^1 \frac{y}{y^2 + k} d\mu(y) \\ &= \frac{\alpha x}{2} \ln\left(\frac{1}{k} + 1\right) \\ &> f_0(x) \qquad (\forall x \in X). \end{aligned}$$

In addition, using simple calculation, we find that

$$T^{n+1} f_0(x) > T^n f_0(x) \qquad (\forall x \in X, \quad \forall n \in \mathbb{N}).$$

If we take $g = T f_0$, then

$$\begin{aligned}
\left\|d\left(T f_0, T^2 f_0\right)\right\| &= \|d(T f_0, T g)\| \\
&= \|\pi_{T g - T f_0}\| \\
&= \sup_{\|\psi\|_2=1} \langle (T g - T f_0)\,\psi, \psi\rangle, \qquad \text{for any } \psi \in H \\
&= \sup_{\|\psi\|_2=1} \int_G \alpha \int_G x \frac{g(y) - f_0(y)}{y^2 + k} d\mu(y) \psi(x)^2 d\mu(x) \\
&\leq \alpha \|g - f_0\|_\infty \sup_{\|\psi]=1} \int_G x \psi(x)^2 d\mu(x) \int_G \frac{1}{y^2 + k} d\mu(y) \\
&\leq \alpha \|g - f_0\|_\infty \frac{\arctan \frac{1}{\sqrt{k}}}{\sqrt{k}} \\
&\leq \frac{\alpha}{k} \|g - f_0\|_\infty.
\end{aligned}$$

For $\dfrac{\alpha}{k} < 1$, we must take

$$\alpha < \frac{1}{e^{\frac{\alpha}{2}} - 1} \Leftrightarrow \alpha e^{\frac{\alpha}{2}} - \alpha - 1 < 0$$

which is possible because

$$\lim_{x \to +\infty} \alpha e^{\frac{\alpha}{2}} - \alpha - 1 = +\infty$$

and $\alpha \to \alpha e^{\frac{\alpha}{2}} - \alpha - 1$ is a continuous function, which take -1 at $\alpha = 0$.
We will have

$$\left\|d\left(T f_0, T^2 f_0\right)\right\| \leq \lambda \|d(f_0, T f_0)\|$$

with $\lambda = \alpha \frac{\arctan \frac{1}{\sqrt{k}}}{\sqrt{k}}$ and $\lambda < 1$.

Therefore the condition of the Theorem 3.2 is verified which ensures the forward convergence of $T^n f_0$ to $\tilde{f}$ in X with respect $\mathbb{A}$. It remains to verify that $\tilde{f}$ is a fixed point. It will suffice to verify that G is forward T-orbitally lower-semi-continuous at $\tilde{f}$.

$$\begin{aligned}
\left\|G\left(\tilde{f}\right)\right\| = \left\|d\left(\tilde{f}, T\tilde{f}\right) \leq \liminf \right\| G\left(T^n f_0\right) \| \\
&= \liminf \left\|d\left(T^n f_0, T^{n+1} f_0\right)\right\| \\
&\leqslant \liminf \left(\frac{\alpha}{2} \ln\left(\frac{1}{k} + 1\right)\right)^n \left(\frac{\alpha}{2} \ln\left(\frac{1}{k} + 1\right) - 1\right) (= +\infty).
\end{aligned}$$

Thus the integral equation $f(x) = \int_G K\left(x, y, f(y)\right) d\mu(y)$ admits a solution.

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