Palestine Journal of Mathematics

Vol. 12(1)(2023) , 872–882

INVESTIGATIONS ON WEIGHTED BI UNIQUE RANGE SETS OVER NON-ARCHIMEDEAN FIELD

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Communicated by Harikrishnan Panackal
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MSC 2010 Classifications: Primary 12J25, 11E95; Secondary 30D35.

Keywords and phrases: Non-Archimedean, meromorphic function, bi-unique range set.

Abhijit Banerjee is thankful to PRG programme for financial assistance. Sayantan Maity is thankful to Council of Scientific and Industrial Research (India) for providing fellowship under File No: 09/106(0191)/2019-EMR-I.

Abstract In this paper, we have studied on weighted bi-URSM corresponding to a most generalized form of a polynomial over a non-Archimedean field. The exhibition of our results are devoid of any extra suppositions. Our paper is the latest form of in-continuation of a number of existing results [14], [15].

1 Introduction and Motivation

We assume that readers are familiar with the basic Nevanlinna theory over the field of complex numbers. We now shortly recall Nevanlinna theory over non-Archimedean field.

In what follows, throughout our paper we consider $\mathbb{F}$ to be an algebraically closed non-Archimedean field with characteristic zero such that it is complete with respect to a non-trivial non-Archimedean absolute value. We denote by log and ln as the real logarithm of base $p > 1$ and e respectively. Let $A_{[r}(\mathbb{F})$ be the set of all power series whose radius of convergence is greater than or equal to r. We denote the collection of all entire functions on $\mathbb{F}$ by $\mathcal{A}(\mathbb{F})\big(= A_{[\infty}(\mathbb{F})\big)$ and the collection of all meromorphic functions on $\mathbb{F}$ by $\mathcal{M}(\mathbb{F})$ and $\widetilde{\mathbb{F}} = \mathbb{F} \cup \{\infty\}$.

Let z be a solution of $f(z) = a$, the multiplicity of z is denoted by $w(a, f; z)$. For $f \in \mathcal{M}(\mathbb{F})$ and $a \in \widetilde{\mathbb{F}}$ we define

$$E_f(a) = \big\{(z, w(a, f; z)) : z \text{ is solution of } f(z) = a\big\}.$$

Now for $f \in \mathcal{M}(\mathbb{F})$ and $S \subset \widetilde{\mathbb{F}}$, define

$$E_f(S) = \cup_{a\in S} \left\{(z, w(a, f; z)) : z \text{ is solution of } f(z) = a\right\}.$$

In [20], Meng-Liu introduced the notion of weighted sharing of values over non-Archimedean field.

Let k be a non-negative integer or ∞. The set of all a-points of f with multiplicity m is counted m times if $m \leq k$ and counted $k + 1$ times if $m > k$, is denoted by $E^k_f(a)$. For two function $f, g \in \mathcal{M}(\mathbb{F})$ if $E^k_f(a) = E^k_g(a)$, then we say f, g share the value a with weight k. We say that f and g share the value a CM (IM) if $E^\infty_f(a) = E^\infty_g(a)$ $(E^0_f(a) = E^0_g(a))$.

Inspired from the definition of weighted sharing of sets as introduced in [18], we demonstrate the analogous definition over non-Archimedean field as follows:

We say f, g share the set S with weight k if $E^k_f(S) = E^k_g(S)$ for a set $S \subset \widetilde{\mathbb{F}}$. We write f, g share (S, k) to mean that f, g share the set S with weight k. In particular if $S = \{a\}$, then we write f, g share (a, k). We say that f and g share the set S CM (IM) if $E^\infty_f(S) = E^\infty_g(S)$ $(E^0_f(S) = E^0_g(S))$.

It was Gross-Yang [11], who first used the terminology "unique range sets for entire functions (URSE)". Later on, the analogous definition for meromorphic function (URSM) was also introduced in the literature (see p. 438, [19]). Next we recall some well known terminologies and definitions.

Definition 1.1. [19] Let f, g be two meromorphic functions over $\mathbb{C}$ and $S \subset \mathbb{C} \cup \{\infty\}$. If $E_f^{\infty}(S) = E_g^{\infty}(S)$ implies $f \equiv g$ then S is called a unique range set for meromorphic functions or in short URSM.

Definition 1.2. [5] Let f, g be two meromorphic functions over $\mathbb{C}$ and $S \subset \mathbb{C} \cup \{\infty\}$. If $E_f^k(S) = E_g^k(S)$ implies $f \equiv g$ then S is called a unique range set for meromorphic functions with weight k or in brief URSMk.

Definition 1.3. [3] Let f, g be two meromorphic functions over $\mathbb{C}$ then a pair of sets $S, T \subset \mathbb{C}$ such that $S \cap T = \varnothing$ is called bi-URSM if $E_f^{\infty}(S) = E_g^{\infty}(S)$, $E_f^{\infty}(T) = E_g^{\infty}(T)$ implies $f \equiv g$.

Definition 1.4. [19] Let $P(z)$ be a polynomial in $\mathbb{C}$. If for any two non-constant meromorphic functions f and g, the condition $P(f) \equiv P(g)$ implies $f \equiv g$, then P is called a uniqueness polynomial for meromorphic functions. We say P is UPM in short.

Khoai-Yang [17] introduced the notion of strong uniqueness polynomial for meromorphic functions or in short SUPM.

Definition 1.5. [17] Let $P(z)$ be a polynomial in $\mathbb{C}$. If for any two non-constant meromorphic functions f and g, the condition $P(f) \equiv cP(g)$ implies $f \equiv g$, where c is a non-zero constant, then P is called a strong uniqueness polynomial for meromorphic functions or SUPM in brief.

In *Definitions 1.1-1.5* replacing $\mathbb{C}$ by $\mathbb{F}$, the definitions of URSM, URSMk, bi-URSM, UPM and SUPM over a non-Archimedean field can be given analogously.

The notion of weighted bi-URSM over $\mathbb{C}$ was introduced by Banerjee ([3], p. 122). Analogously we define weighted bi-URSM over non-Archimedean field as follows:

Definition 1.6. A pair of finite, disjoint sets S and T in $\mathbb{F}$ is called bi-unique range sets for meromorphic functions with weights p, k if for any two non-constant meromorphic functions f and g, $E_f^p(S) = E_g^p(S)$, $E_f^k(T) = E_g^k(T)$ imply $f \equiv g$. We say S, T, are bi-URSMp, k in short. As usual, if both $p = k = \infty$, we say S, T, are bi-URSM.

Fujimoto [10] introduced the following definition and called it as “Property H" which was latter characterized as “Critical Injection Property".

Definition 1.7. [5] Let $P(z)$ be a polynomial such that $P'(z)$ has l distinct zero namely $z_1, z_2, \ldots, z_l$. If $P(z_i) \neq P(z_j)$ for $i \neq j$, $i, j \in \{1, 2, \ldots, l\}$, then $P(z)$ is said to satisfy the critical injection property.

Over the non-Archimedean field the same definitions of critical injection property can be given.

For basic terminologies of value distribution theory over non-Archimedean field, readers can make a glance on [1], [2], [20]. Here we recall a few of them.

For a real constant ρ such that $0 < \rho \leq r$, the counting function $N(r, a; f)$ of $f \in \mathcal{M}(\mathbb{F})$ is defined as follows:

$$N(r, a; f) = \frac{1}{\ln p} \int_{\rho}^{r} \frac{n(t, a; f)}{t}\, dt,$$

where $n(t, a; f)$ is the number of solutions (CM) of $f(z) = a$ in the disk $D_t = \{z \in \mathbb{F} : |z| \leq t\}$. For $l \in \mathbb{Z}^+$, define

$$N_l(r, a; f) = \frac{1}{\ln p} \int_{\rho}^{r} \frac{n_l(t, a; f)}{t}\, dt,$$

where $n_l(t, a; f) = \sum_{|z| \leq r} \min\{l, w(a, f; z)\}$. Thus $N_1(r, a; f)$ denotes the counting function of a-points of f where multiplicity is counted only once, in short we call it “reduced counting function”.

Definition 1.8. For $a \in \widetilde{\mathbb{F}}$ we denote by $N(r, a; f \mid = 1)$ the counting function of simple a-points of f. For $k \in \mathbb{Z}^+$ we denote by $N(r, a; f \mid \leq k)(N(r, a; f \mid \geq k))$ the counting function of those a-points of f whose multiplicities are not greater(less) than k where each a-point is counted according to its multiplicity. $N_1(r, a; f \mid \leq k)(N_1(r, a; f \mid \geq k))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

2 Background and Main results

Several interesting results on URSM over $\mathbb{F}$ have been obtained (see [7], [12], [14], [21]). We notice that, in [14] and [21], the authors considered a pair of sets S, T, where one of them contains only one element and proved $\{S, T\}$ is a bi-URSM. In this regard, we would like to mention a very recent work of Khoai-An [15] where the authors introduced the following polynomial:

$$P_{KA}(z) = (m+n+1)\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{m+n+1-j}z^{m+n+1-j}a^j + 1 = Q^*(z) + 1, \tag{2.1}$$

where $a \in \mathbb{F} \setminus \{0\}$ and $Q^*(a) \neq -1, -2$. Khoai-An [15] obtained the following result:

Theorem A. [15] Let f, g be two non-constant meromorphic functions on $\mathbb{F}$, $P_{KA}(z)$ be defined by (2.1) with conditions $Q^*(a) \neq -1, -2$ and let $\min\{m, n\} \geq 2$. Let $S = \{z \in \mathbb{F} \mid P_{KA}(z) = 0\}$.

(i) If $m + n \geq 9$ and $E_f^{\infty}(S) = E_g^{\infty}(S)$, then $f \equiv g$.

(ii) If $E_f^{\infty}(S) = E_g^{\infty}(S)$, $E_f^{\infty}(\infty) = E_g^{\infty}(\infty)$, then $f \equiv g$.

Thus we see that, (ii) of *Theorem A* gives the existence of bi-URSM, where one set is $\{\infty\}$ and this case is rather easy to tackle. So the natural question arises about the existence of bi-URSM S, T, where each set S and T contains at least two elements from $\mathbb{F}$. In this perspective Khoai-Hoa [16] obtained the next result. To state the next result, the following definition is needed:

Definition 2.1. [16] A statement $\mathcal{S}(a_1, \ldots, a_n)$ is said to be held for a generic set $\{a_1, \ldots, a_n \in \mathbb{F}\}$ if there exists a proper algebraic subset $\sum \subset \mathbb{F}^n$ such that $\mathcal{S}(a_1, \ldots, a_n)$ holds for all $a_1, \ldots, a_n \in \mathbb{F}$ whenever $(a_1, \ldots, a_n) \notin \sum$.

Theorem B. [16] For $n \geq 3$ and a generic set $\{a_1, a_2; b_1, b_2, \ldots, b_n\}$ of elements in $\widetilde{\mathbb{F}}$, the couple $S = \{a_1, a_2\}$, $T = \{b_1, \ldots, b_n\}$ is a bi-URSM pair for meromorphic functions in $\mathbb{F}$.

It can be noticed that the proof of *Theorem B* is based on algebraic and geometric approaches. Moreover, from *Theorem B* it follows the existence of bi-URSM $\{S, T\}$, where S and T have at least two elements. However, the authors of [16] can not give an explicit bi-URSM.

Question 2.1. Can it be possible to find an explicit bi-URSM, where both the sets contain at least two elements?

Using the well known Nevanlinna's value distribution theory over non-Archimedean field we will try to find the answer of *Question 2.1*, which is the prime motivation to write this paper.

Very recently in [6], Banerjee-Maity introduced a new polynomial of degree $m + n + 1$ in the following manner:

$$\begin{aligned} P(z) &= \sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{m+n+1-j}z^{m+n+1-j}a^j \\ &\quad + \sum_{i=1}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\frac{(-1)^{i+j}}{m+n+1-i-j}z^{m+n+1-i-j}a^j b^i + c \\ &= Q(z) + c, \end{aligned} \tag{2.2}$$

where a and b are distinct such that $a \in \mathbb{F} \setminus \{0\}$, $b \in \mathbb{F}$, $c \in \mathbb{F} \setminus \{0, -Q(a), -Q(b)\}$. It is easy to verify that

$$P'(z) \quad = \quad (z-a)^n (z-b)^m.$$

Note 2.1. From *Note B* of [6], we get that $P(z)$ is a generalization of famous Frank-Reinders polynomial [9].

Note 2.2. The set of all zeros of $P'(z)$ is $\{a, b\}$. $P(z)$ have only simple zeros since $c \in \mathbb{F} \setminus \{-Q(a), -Q(b)\}$.

Note 2.3. From *Remark 1.10* of [6], we see that, $P(z)$ is a critically injective polynomial.

Our first result gives an affirmative answer to *Question 2.1* under more relaxed sharing hypothesis namely weighted sharing.

Theorem 2.1. Let f, g be two non-constant meromorphic functions on $\mathbb{F}$ and m, n be two positive integers such that $\min\{m, n\} \geq 2$. Consider the polynomial (2.2) such that $P(a) \neq -1$ and $S = \{a, b\}$, $T = \{z \mid P(z) = 0\}$. Now

(i) when $P(b) \neq 1$, $n \geq m + 2$, or

(ii) when $P(b) = 1$,

then for both cases *(i)* and *(ii)* the couple of sets S, T is a bi-URSM$0, 3$.

Note that when $P(b) = 1$, then the minimum cardinality of the set T is 5.

Note 2.4. Take $n = m = 2$, $a = 1$, $b = 0$, $c = 1$ and set

$$P(z) = \frac{z^5}{5} - \frac{z^4}{2} + \frac{z^3}{3} + 1.$$

Denote $S = \{1, 0\}$, $T = \{z \mid P(z) = 0\}$. Then by *Theorem 2.1*, $\{S, T\}$ is a bi-URSM, where T contains 5 elements.

Question 2.2. In *Theorem 2.1*, can it be possible to remove the condition "$P(a) \neq -1$"?

Question 2.3. In *Theorem 2.1*, can it be possible to reduce the weight of the set T?

In order to answer the *Questions 2.2* and *2.3*, we obtain the following result:

Theorem 2.2. Let f, g be two non-constant meromorphic functions on $\mathbb{F}$ and m, n be two positive integers such that $m \geq 2$, $n \geq m+2$. Consider the polynomial $P(z)$ as (2.2) and $S = \{a, b\}$, $T = \{z \mid P(z) = 0\}$, then the couple of sets S, T is a bi-URSM$0, 2$.

3 Lemmas

Lemma 3.1. [13] Let $f(z)$ be a non-constant meromorphic function on $\mathbb{F}$ and $a_1, a_2, \ldots, a_n \in \widetilde{\mathbb{F}}$ are distinct points. Then

$$(n-2)T(r, f) \leq \sum_{i=1}^{n} N_1(r, a_i; f) - N^0(r, 0; f') - \log r + O(1),$$

where $N^0(r, 0; f')$ denotes the counting function of zeros of f' which are not a_i $(i = 1, 2, \ldots, n)$ points of f.

The next lemma follows from the equivalence of (i) and (iv) of *Theorem 1* of Wang [21].

Lemma 3.2. [21] Let f, g be two non-constant meromorphic functions on $\mathbb{F}$ and $P(z)$ be a critically injective polynomial such that the derivative of $P(z)$ is of the form $(z-\alpha)^m(z-\beta)^n$ and let $\min\{m, n\} \geq 2$. If $P(f) \equiv P(g)$ then $f \equiv g$.

Lemma 3.3. [15] Let f, g be two non-constant meromorphic functions on $\mathbb{F}$ and $P(z)$ be a polynomial with no multiple zero and the derivative of $P(z)$ is of the form $(z-\alpha)^m(z-\beta)^n$, also let $\min\{m,n\} \geq 2$. Assume that there exist constant $c_1 \neq 0$ and c_2 such that

$$\frac{1}{P(f)} = \frac{c_1}{P(g)} + c_2,$$

then $c_2 = 0$.

From now onward we denote two non-constant meromorphic functions $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{F}$ such that $\mathcal{F} = \frac{P(f)+c}{c}$ and $\mathcal{G} = \frac{P(g)+c}{c}$, where $P(z)$ is defined as in (2.2). Besides this we also consider two functions $\mathcal{H}$ and Ψ as follows:

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right), \tag{3.1}$$

$$\Psi = \frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{G}'}{\mathcal{G}-1}. \tag{3.2}$$

Lemma 3.4. Let $\mathcal{H} \not\equiv 0$ and $\mathcal{F}, \mathcal{G}$ share $(1,1)$ then

$$N(r,1;\mathcal{F} \mid= 1) = N(r,1;\mathcal{G} \mid= 1) \leq N(r,\infty;\mathcal{H}) + O(1).$$

Proof. As $\mathcal{F}$ and $\mathcal{G}$ share $(1,1)$ so each simple 1-point of $\mathcal{F}$ is also simple 1-point of $\mathcal{G}$ and vice versa. Now each simple 1-point of $\mathcal{F}$ (i.e., simple 1-point of $\mathcal{G}$) is a zero of $\mathcal{H}$. Note that $m(r,\mathcal{H}) = O(1)$. Hence

$$N(r,1;\mathcal{F} \mid= 1) = N(r,1;\mathcal{G} \mid= 1) \leq N(r,0;\mathcal{H}) \leq T(r,\mathcal{H}) \leq N(r,\infty;\mathcal{H}) + O(1).$$

□

Lemma 3.5. Let $S = \{a,b\}$, $T = \{z \mid P(z) = 0\}$, where $P(z)$ is defined as in (2.2). Let $\mathcal{H} \not\equiv 0$ and f, g be any two non-constant meromorphic functions on $\mathbb{F}$ such that, $E_f^p(S) = E_g^p(S)$ for $0 \leq p \leq \infty$ and $E_f^0(T) = E_g^0(T)$, then

$$\begin{aligned} N(r,\infty;\mathcal{H}) \quad \leq \quad & \overline{N}_1(r,a;f) + \overline{N}_1(r,b;f) + \overline{N}_1(r,\infty;f) + \overline{N}_1(r,\infty;g) + \overline{N}_1^0(r,0;f') \\ & + \overline{N}_1^0(r,0;g') + \overline{N}_1^*(r,1;\mathcal{F},\mathcal{G}), \end{aligned}$$

where $\overline{N}_1^0(r,0;f')$ denotes the reduced counting function of those zeros of f' which are not zeros of $(\mathcal{F}-1)(f-a)(f-b)$ and $\overline{N}_1^0(r,0;g')$ denotes the similar counting function. $\overline{N}_1^*(r,1;\mathcal{F},\mathcal{G})$ denotes the reduced counting function of those 1-points of $\mathcal{F}$ whose multiplicities differ from the multiplicities of the corresponding 1-points of $\mathcal{G}$.

Proof. Note that, $\mathcal{F}' = \frac{P'(f)}{c} = \frac{(f-a)^n(f-b)^m f'}{c}$. As $E_f^p(S) = E_g^p(S)$ for $0 \leq p \leq \infty$ and $E_f^0(T) = E_g^0(T)$, so the lemma directly follows by calculating all the possible poles of $\mathcal{H}$. □

Lemma 3.6. Let $\mathcal{F}, \mathcal{G}$ shares $(1,k)$, where $1 \leq k < \infty$. Then

$$\begin{aligned} & \overline{N}_1(r,1;\mathcal{F}) + \overline{N}_1(r,1;\mathcal{G}) - N(r,1;\mathcal{F} \mid= 1) + \left(k - \frac{1}{2}\right)\overline{N}_1^*(r,1;\mathcal{F},\mathcal{G}) \\ \leq \quad & \frac{1}{2}\left[N(r,1;\mathcal{F}) + N(r,1;\mathcal{G})\right]. \end{aligned}$$

The *Lemma 3.6* can be considered as the non-Archimedean version of the *Lemma 2.10* of [4]. Proof of the lemma is omitted as it can be done proceeding similarly as *Lemma 2.10* of [4].

Lemma 3.7. Let S, T be defined as in Lemma 3.5 and m, n be two positive integers such that $\min\{m,n\} \geq 2$. Assume $\Psi \not\equiv 0$ and $E_f^p(S) = E_g^p(S)$, $E_f^k(T) = E_g^k(T)$. Then

$$\begin{aligned} & ((p+1)n + p)\left[\overline{N}_1(r,a;f \mid\geq p+1) + \overline{N}_1(r,b;f \mid\geq p+1)\right] \\ \leq \quad & \overline{N}_1^*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}_1(r,\infty;f) + \overline{N}_1(r,\infty;g) + O(1). \end{aligned}$$

Proof. The condition $E_f^k(T) = E_g^k(T)$ implies $\mathcal{F}$, $\mathcal{G}$ share $(1, k)$. Now from (3.2) we get

$$\Psi = \frac{(f-a)^n(f-b)^m f'}{c(\mathcal{F}-1)} - \frac{(g-a)^n(g-b)^m g'}{c(\mathcal{G}-1)}.$$

Let z_0 be a(or b) point of f with multiplicity r. As $E_f^p(S) = E_g^p(S)$ and $\min\{m, n\} \geq 2$, so when $r \leq p$ then z_0 is a zero of Ψ of multiplicity $\min\{nr + r - 1, mr + r - 1\} \geq 3r - 1$ and when $r > p$ then z_0 is a zero of Ψ of multiplicity at least $\min\{(p+1)n+p, (p+1)m+p\} \geq (3p+2)$. Thus we can deduce

$$\begin{aligned} & (3p+2)\left[N_1(r, a; f \mid \geq p+1) + N_1(r, b; f \mid \geq p+1)\right] \\ \leq\ & N(r, 0; \Psi) \\ \leq\ & T(r, \Psi) + O(1) \\ \leq\ & N(r, \infty; \Psi) + O(1) \\ \leq\ & N_1^*(r, 1; \mathcal{F}, \mathcal{G}) + N_1(r, \infty; f) + N_1(r, \infty; g) + O(1). \end{aligned}$$

□

Remark 3.1. In particular, for $p = 0$ in Lemma 3.7 we have

$$N_1(r, a; f) + N_1(r, b; f) \leq \frac{1}{2}\left[N_1^*(r, 1; \mathcal{F}, \mathcal{G}) + N_1(r, \infty; f) + N_1(r, \infty; g)\right] + O(1).$$

Lemma 3.8. Let $\mathcal{F}$, $\mathcal{G}$ share $(1, k)$, where $1 \leq k \leq \infty$. Then

$$N_1^*(r, 1; \mathcal{F}, \mathcal{G}) \leq \frac{1}{k}\left[N_1(r, a; f) + N_1(r, b; f)\right] + O(1).$$

Proof. By using lemma of logarithmic derivative

$$\begin{aligned} & m\left(r, \frac{f'}{(f-a)(f-b)}\right) \qquad (3.3) \\ =\ & m\left(r, \left\{\frac{f'}{(a-b)(f-a)} + \frac{f'}{(b-a)(f-b)}\right\}\right) \\ \leq\ & \max\left\{m\left(r, \frac{f'}{(a-b)(f-a)}\right), m\left(r, \frac{f'}{(b-a)(f-b)}\right)\right\} \\ =\ & O(1). \end{aligned}$$

Note that all zeros of $P(f)$ are simple, let us denote them by $\omega_j (j = 1, 2, \ldots, m+n+1)$. Using the fact $\mathcal{F}$, $\mathcal{G}$ share $(1, k)$ and equation (3.3) we have

$$\begin{aligned} N_1^*(r, 1; \mathcal{F}, \mathcal{G}) &\leq N_1(r, 1; \mathcal{F} \mid \geq k+1) \\ &\leq \frac{1}{k}\left[N(r, 1; \mathcal{F}) - N_1(r, 1; \mathcal{F})\right] \\ &\leq \frac{1}{k}\left[\sum_{j=1}^{m+n+1}\left(N(r, \omega_j; f) - N_1(r, \omega_j; f)\right)\right] \\ &\leq \frac{1}{k}N(r, 0; f' \mid f \neq a, b) \\ &\leq \frac{1}{k}N\left(r, 0; \frac{f'}{(f-a)(f-b)}\right) \\ &\leq \frac{1}{k}N\left(r, \infty; \frac{f'}{(f-a)(f-b)}\right) + O(1) \\ &\leq \frac{1}{k}\left[N_1(r, a; f) + N_1(r, b; f)\right] + O(1). \end{aligned}$$

□

Lemma 3.9. Let S, T be defined as in Lemma 3.5. Let $\Psi \not\equiv 0$ and $E_f^0(S) = E_g^0(S)$, $E_f^k(T) = E_g^k(T)$ (for $2 \leq k \leq \infty$) and $n \geq 2$. Then

$$N_1^*(r,1;\mathcal{F},\mathcal{G}) \leq \frac{1}{2k-1}\left[N_1(r,\infty;f) + N_1(r,\infty;g)\right] + O(1).$$

Proof. Combining *Lemma 3.8* and *Remark 3.1* we get,

$$\begin{aligned} N_1^*(r,1;\mathcal{F},\mathcal{G}) &\leq \frac{1}{k}\left[N_1(r,a;f) + N_1(r,b;f)\right] + O(1) \\ &\leq \frac{1}{2k}\left[N_1^*(r,1;\mathcal{F},\mathcal{G}) + N_1(r,\infty;f) + N_1(r,\infty;g)\right] + O(1). \end{aligned}$$

Thus we have

$$N_1^*(r,1;\mathcal{F},\mathcal{G}) \leq \frac{1}{2k-1}\left[N_1(r,\infty;f) + N_1(r,\infty;g)\right] + O(1).$$

□

Lemma 3.10. (Theorem 1.11, [6]) Let m, n be two positive integers such that $\min\{m,n\} \geq 2$. Consider the polynomial (2.2) such that $P(a) \neq -1$. Now

(i) when $P(b) \neq 1$, $n \geq m+2$, or

(ii) when $P(b) = 1$,

then for both the cases *(i)* and *(ii)*, $P(z)$ is a SUPM.

4 Proofs of the theorems

Proof of Theorem 2.1. Let f, g be two non-constant meromorphic functions on $\mathbb{F}$ such that $E_f^0(S) = E_g^0(S)$, $E_f^3(T) = E_g^3(T)$. Thus $\mathcal{F}$, $\mathcal{G}$ share $(1,3)$.
Case 1: First assume $\Psi \not\equiv 0$.
Sub-case 1.1: Suppose $\mathcal{H} \not\equiv 0$. Applying *Lemma 3.1, 3.4, 3.5* and *Lemma 3.6* for $k = 3$ we get

$$\begin{aligned} & (m+n+2)[T(r,f) + T(r,g)] \qquad (4.1) \\ \leq\ & N_1(r,1;\mathcal{F}) + N_1(r,a;f) + N_1(r,b;f) + N_1(r,\infty;f) \\ & + N_1(r,1;\mathcal{G}) + N_1(r,a;g) + N_1(r,b;g) + N_1(r,\infty;g) \\ & - N^0(r,0;f') - N^0(r,0;g') - 2\log r + O(1) \\ \leq\ & N(r,1;\mathcal{F} \mid= 1) - \frac{5}{2}N_1^*(r,1;\mathcal{F},\mathcal{G}) + \frac{1}{2}\left[N(r,1;\mathcal{F}) + N(r,1;\mathcal{G})\right] \\ & + 2[N_1(r,a;f) + N_1(r,b;f)] + N_1(r,\infty;f) + N_1(r,\infty;g) \\ & - N^0(r,0;f') - N^0(r,0;g') - 2\log r + O(1) \\ \leq\ & N(r,\infty;\mathcal{H}) - \frac{5}{2}N_1^*(r,1;\mathcal{F},\mathcal{G}) + \frac{1}{2}\left[N(r,1;\mathcal{F}) + N(r,1;\mathcal{G})\right] \\ & + 2[N_1(r,a;f) + N_1(r,b;f)] + N_1(r,\infty;f) + N_1(r,\infty;g) \\ & - N^0(r,0;f') - N^0(r,0;g') - 2\log r + O(1) \\ \leq\ & 3[N_1(r,a;f) + N_1(r,b;f)] + 2[N_1(r,\infty;f) + N_1(r,\infty;g)] \\ & + \frac{1}{2}\left[N(r,1;\mathcal{F}) + N(r,1;\mathcal{G})\right] - \frac{3}{2}N_1^*(r,1;\mathcal{F},\mathcal{G}) - 2\log r + O(1). \end{aligned}$$

Using *Remark 3.1*, from (4.1) we deduce

$$\begin{aligned}
&(m+n+2)[T(r,f)+T(r,g)] \qquad (4.2)\\
\leq\ & \frac{3}{2}[N_1^*(r,1;\mathcal{F},\mathcal{G})+N_1(r,\infty;f)+N_1(r,\infty;g)]+2[N_1(r,\infty;f)+N_1(r,\infty;g)]\\
&+\left(\frac{m+n+1}{2}\right)[T(r,f)+T(r,g)]-\frac{3}{2}N_1^*(r,1;\mathcal{F},\mathcal{G})-2\log r+O(1)\\
\leq\ & \left(\frac{3}{2}+2+\frac{m+n+1}{2}\right)[T(r,f)+T(r,g)]-2\log r+O(1).
\end{aligned}$$

Hence (4.2) implies

$$\left(\frac{m+n}{2}-2\right)[T(r,f)+T(r,g)]+2\log r\leq O(1). \tag{4.3}$$

From the condition $\min\{m,n\}\geq 2$ we have $m+n\geq 4$. Thus (4.3) gives a contradiction since $m+n\geq 4$.

Sub-case 1.2: Suppose $\mathcal{H}\equiv 0$. Integrating (3.1) two times and as $c\neq 0$ we obtain

$$\begin{aligned}
\frac{1}{\mathcal{F}-1} &\equiv \frac{A}{\mathcal{G}-1}+B \quad \text{(where } A,\ B \text{ are constants such that } A\neq 0).\\
\Longrightarrow \frac{c}{P(f)} &\equiv \frac{cA}{P(g)}+B.\\
\Longrightarrow \frac{1}{P(f)} &\equiv \frac{A}{P(g)}+\frac{B}{c}. \qquad (4.4)
\end{aligned}$$

Now applying *Lemma 3.3* for the equation (4.4) we get $\frac{B}{c}=0$. Consider a constant $A_1=\frac{1}{A}$. Thus we have $P(f)\equiv A_1P(g)$. Next applying *Lemma 3.10* we obtain $f\equiv g$.

Case 2: We assume $\Psi\equiv 0$. Integrating (3.2) we get

$$\begin{aligned}
\mathcal{F}-1 &\equiv A_2(\mathcal{G}-1)\\
\Longrightarrow P(f) &\equiv A_2P(g).
\end{aligned}$$

Next applying *Lemma 3.10* we get $f\equiv g$. This completes the proof of the theorem. □

Proof of Theorem 2.2. Let f, g be two non-constant meromorphic functions on $\mathbb{F}$ such that $E_f^0(S)=E_g^0(S)$, $E_f^2(T)=E_g^2(T)$. Thus $\mathcal{F}$, $\mathcal{G}$ share $(1,2)$.

Case 1: First assume $\Psi\not\equiv 0$.

Sub-case 1.1: Suppose $\mathcal{H}\not\equiv 0$. Applying *Lemma 3.1, 3.4, 3.5* and *Lemma 3.6* for $k=2$ we get

$$\begin{aligned}
&(m+n+2)[T(r,f)+T(r,g)] \qquad (4.5)\\
\leq\ & N_1(r,1;\mathcal{F})+N_1(r,a;f)+N_1(r,b;f)+N_1(r,\infty;f)\\
&+N_1(r,1;\mathcal{G})+N_1(r,a;g)+N_1(r,b;g)+N_1(r,\infty;g)\\
&-N^0(r,0;f')-N^0(r,0;g')-2\log r+O(1)\\
\leq\ & N(r,1;\mathcal{F}\mid=1)-\frac{3}{2}N_1^*(r,1;\mathcal{F},\mathcal{G})+\frac{1}{2}[N(r,1;\mathcal{F})+N(r,1;\mathcal{G})]\\
&+2[N_1(r,a;f)+N_1(r,b;f)]+N_1(r,\infty;f)+N_1(r,\infty;g)\\
&-N^0(r,0;f')-N^0(r,0;g')-2\log r+O(1)\\
\leq\ & N(r,\infty;\mathcal{H})-\frac{3}{2}N_1^*(r,1;\mathcal{F},\mathcal{G})+\frac{1}{2}[N(r,1;\mathcal{F})+N(r,1;\mathcal{G})]\\
&+2[N_1(r,a;f)+N_1(r,b;f)]+N_1(r,\infty;f)+N_1(r,\infty;g)\\
&-N^0(r,0;f')-N^0(r,0;g')-2\log r+O(1)\\
\leq\ & 3[N_1(r,a;f)+N_1(r,b;f)]+2[N_1(r,\infty;f)+N_1(r,\infty;g)]\\
&+\frac{1}{2}[N(r,1;\mathcal{F})+N(r,1;\mathcal{G})]-\frac{1}{2}N_1^*(r,1;\mathcal{F},\mathcal{G})-2\log r+O(1).
\end{aligned}$$

Using *Remark 3.1* and *Lemma 3.9* in (4.5) we deduce

$$\begin{aligned}
& (m+n+2)[T(r,f)+T(r,g)] \qquad (4.6)\\
\leq\ & \frac{3}{2}\left[N_1^*(r,1;\mathcal{F},\mathcal{G})+N_1(r,\infty;f)+N_1(r,\infty;g)\right]+2[N_1(r,\infty;f)+N_1(r,\infty;g)]\\
& +\left(\frac{m+n+1}{2}\right)[T(r,f)+T(r,g)]-\frac{1}{2}N_1^*(r,1;\mathcal{F},\mathcal{G})-2\log r+O(1)\\
\leq\ & N_1^*(r,1;\mathcal{F},\mathcal{G})+\left(\frac{3}{2}+2+\frac{m+n+1}{2}\right)[T(r,f)+T(r,g)]-2\log r+O(1)\\
\leq\ & \frac{1}{3}[N_1(r,\infty;f)+N_1(r,\infty;g)]+\left(\frac{m+n+8}{2}\right)[T(r,f)+T(r,g)]-2\log r+O(1)\\
\leq\ & \left(\frac{1}{3}+\frac{m+n+8}{2}\right)[T(r,f)+T(r,g)]-2\log r+O(1).
\end{aligned}$$

Hence (4.6) implies

$$\left(\frac{m+n}{2}-\frac{7}{3}\right)[T(r,f)+T(r,g)]+2\log r\leq O(1). \tag{4.7}$$

We have $m\geq 2$ and $n\geq m+2$, these two conditions imply $m+n\geq 6$. Thus (4.7) gives a contradiction since $m+n\geq 6$.

Sub-case 1.2: Suppose $\mathcal{H}\equiv 0$. Integrating (3.1) two times we obtain

$$\begin{aligned}
\frac{1}{\mathcal{F}-1} &\equiv \frac{A}{\mathcal{G}-1}+B \quad (\text{where } A, B \text{ are constants such that } A\neq 0)\\
\Longrightarrow \frac{1}{P(f)} &\equiv \frac{A}{P(g)}+\frac{B}{c} \quad (\text{as } c\neq 0). \qquad (4.8)
\end{aligned}$$

As $m\geq 2$ and $n\geq m+2$ so obviously $\min\{m,n\}\geq 2$. Now applying *Lemma 3.3* for the equation (4.8). We get $\frac{B}{c}=0$. Consider a constant $A_1=\frac{1}{A}$.

Sub-case 1.2.1: Let us assume $A_1\neq 1$. Now (4.8) implies

$$\begin{aligned}
P(f) &\equiv A_1P(g)\\
\Longrightarrow P(f)-c &\equiv A_1(P(g)-c)+c(A_1-1)\\
\Longrightarrow Q(f) &\equiv A_1Q(g)+c(A_1-1)\\
\Longrightarrow Q(f)-Q(b) &\equiv A_1Q(g)-(Q(b)-c(A_1-1)). \qquad (4.9)
\end{aligned}$$

Note that since $P(f)\equiv A_1P(g)$, therefore $T(r,f)=T(r,g)+O(1)$. Recall that the only zeros of $Q'(z)$ are a and b. So the only possible multiple zeros of $\phi(z):=A_1Q(z)-(Q(b)-c(A_1-1))$ are a and b. First assume b is the multiple zero of $\phi(z)$. Thus $\phi(b)=0$, i.e.,

$$\begin{aligned}
& A_1Q(b)=Q(b)-c(A_1-1)\\
\Longrightarrow\ & (A_1-1)(Q(b)+c)=0\\
\Longrightarrow\ & c=-Q(b),
\end{aligned}$$

a contradiction as we have $c\neq -Q(b)$. Next assume a is the multiple zero of $\phi(z)$. It is easy to see that $\phi(z)=(z-a)^{n+1}W_1(z)$, where $W_1(a)\neq 0$ and all zeros of $W_1(z)$ are simple namely α_j $(j=1,2,\ldots,m)$. Notice that, $Q(z)-Q(b)=(z-b)^{m+1}W_2(z)$, where $W_2(b)\neq 0$ and all zeros of $W_2(z)$ are simple. Let us denote them by β_j $(j=1,2,\ldots,n)$. Hence from (4.9)

$$N_1(r,b;f)+\sum_{j=1}^{n}N_1(r,\beta_j;f)=N_1(r,a;g)+\sum_{j=1}^{m}N_1(r,\alpha_j;g). \tag{4.10}$$

Next using the Second Fundamental Theorem, (4.10) and the fact $T(r,f) = T(r,g) + O(1)$ we get

$$\begin{aligned}(n-1)T(r,f) &\leq N_1(r,b;f) + \sum_{j=1}^{n} N_1(r,\beta_j;f) - \log r + O(1)\\ &= N_1(r,a;g) + \sum_{j=1}^{m} N_1(r,\alpha_j;g) - \log r + O(1)\\ &\leq (m+1)T(r,f) - \log r + O(1).\end{aligned}$$

Thus we have $(n-m-2)T(r,g) + \log r \leq O(1)$, this contradicts the given condition $n \geq m+2$. Hence we see neither a nor b are multiple zeros of $\phi(z)$, and hence all the zeros of $\phi(z)$ are simple say γ_j $(j = 1, 2, \ldots, m+n+1)$. From (4.9)

$$N_1(r,b;f) + \sum_{j=1}^{n} N_1(r,\beta_j;f) = \sum_{j=1}^{m+n+1} N_1(r,\gamma_j;g). \tag{4.11}$$

Using the Second Fundamental Theorem and the equation (4.11) we deduce

$$\begin{aligned}(m+n-1)T(r,g) &\leq \sum_{j=1}^{m+n+1} N_1(r,\gamma_j;g) - \log r + O(1)\\ &= N_1(r,b;f) + \sum_{j=1}^{n} N_1(r,\beta_j;f) - \log r + O(1)\\ &\leq (n+1)T(r,g) - \log r + O(1).\end{aligned}$$

Hence we obtain $(m-2)T(r,g) + \log r \leq O(1)$. Since $m \geq 2$, we get a contradiction.
Sub-case 1.2.2: Next assume $A_1 = 1$. Thus $P(f) \equiv P(g)$, and by *Lemma 3.2* we conclude $f \equiv g$.
Case 2: Now assume $\Psi \equiv 0$. Integrating (3.2) we get

$$\begin{aligned}\mathcal{F} - 1 &\equiv A_2(\mathcal{G} - 1)\\ \Longrightarrow P(f) &\equiv A_2 P(g).\end{aligned}$$

Proceeding similarly as done in Sub-case 1.2.1 we get a contradiction and next following the steps of Sub-case 1.2.2 we deduce $f \equiv g$.
Therefore by Case 1 and 2 we get that the couple of sets S, T is a bi-URSM$0, 2$.

□

5 An open question

In [8] (p.136), the authors presented the following example. Let $a, b, c, d \in \mathbb{F}$ be arbitrary distinct values. Set the function $h(z)$ which is different from the identity function such that,

$$h(z) = \frac{z(ab-cd) - ab(c+d) + cd(a+b)}{z(a+b-c-d) - ab + cd}.$$

We have $h(a) = b;\ h(b) = a;\ h(c) = d;\ h(d) = c$. So, if we denote by $S = \{a, b\},\ T = \{c, d\}$ then $\{S, T\}$ is not a bi-URSM. On the other hand, according to *Theorem B* there exist bi-URSM $\{S, T\}$, where S has 2 elements, and T has 3 elements.

Considering *Theorem B* and *Note 2.4*, the following question deserves further attention.

(I) Can it be possible to find an explicit bi-URSM $\{S, T\}$, where S has 2 elements, and T has 3 or 4 elements?

Unfortunately, the authors have no answer to the above question till now.

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Received: November 16th, 2021
Accepted: August 14th, 2022