**Palestine Journal of Mathematics**

**Vol. 12(1)(2023)** , 883–891 

# ON IDEMPOTENT GRAPH OF RINGS

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: 05C10, 15A03, 05C45.

Keywords and phrases: Clique, radius, planar graph, idempotent.

**Abstract** Let $R$ be a commutative ring with identity. The idempotent graph of a ring is defined with vertex set a ring $R$ and distinct vertices $x$ and $y$ in $R$ are adjacent if and only if $x + y \in Id(R)$, the set of idempotents of $R$ and it is denoted by $G_{Id}(R)$. In this paper, we determine the clique number of $G_{Id}(R)$. We find the radius, the independence number and the chromatic number of $G_{Id}(\mathbb{Z}_n)$. We determine which idempotent graphs of $\mathbb{Z}_n$ are planar and Hamiltonian.

## 1 Introduction

Throughout this paper, let $R$ denote a commutative ring with non zero identity element. In this paper, we study the idempotent graph of a ring $R$, introduced by Razzaghi and Sahebi [7] in the year 2020, which is defined as "the (undirected) graph with all the elements of $R$ as vertices and for distinct $x, y \in R$, the vertices $x$ and $y$ are adjacent if and only if $x + y \in Id(R)$ and it is denoted by $G_{Id}(R)$". In [7], the author gave the necessary and sufficient condition for $G_{Id}(R)$ to be connected and also derived the chromatic index, the diameter and the girth of the graph. In this paper, we find some more properties of the idempotent graph $G_{Id}(R)$ and in particular $G_{Id}(\mathbb{Z}_n)$. If $R$ is a direct product of copies of the field $\mathbb{Z}_2$, then the idempotent graph is a complete graph. The study of idempotent graphs are useful in characterizing the class of rings in which every element is sum of two idempotents. Such a ring $R$ is isomorphic to $R_1 \times R_2$, where $\frac{R_1}{J(R_1)}$ is Boolean with $J(R_1) = \{0\}$ or $J(R_1) = \{0, 2\}$ and $R_2$ is zero or a subdirect products of $\mathbb{Z}_3's$. For basic definition we refer the reader to [2].

In section 2, we compute the clique number of $G_{Id}(R)$ and show that the idempotent graph of $R$ is a complete graph or disjoint union of complete graphs. In section 3, we find the diameter, the independence number of $G_{Id}(\mathbb{Z}_n)$ and show that $G_{Id}(\mathbb{Z}_n)$ contains a Hamiltonian cycle. Further, we show that $G_{Id}(\mathbb{Z}_n)$ is planar if and only if $n$ has two distinct prime divisors.

## 2 Clique number of idempotent graph of $R$

In this section, we prove that the idempotent graph is either a complete graph or a disjoint union of complete graphs when $R$ is a ring of characteristic 2 and also we determine the clique number of $G_{Id}(R)$.

**Lemma 2.1.** *Let $e$ be an idempotent element of R. Then $e$ and $-e$ are idempotents if and only if* $2e = 0$.

**Proof.** Suppose $-e$ is an idempotent, then $(-e)^2 = e = -e$ which implies that $2e = 0$. Converse is trivial. □

**Proposition 2.2.** *If $Char(R) = 2$, then $G_{Id}(R)$ is $K_{|R|}$ or a disjoint union of $K_{|Id(R)|}$.*

**Proof.** If $Char(R) = 2$, then the set of idempotents forms a complete graph. For every $x \in R \backslash Id(R)$, $2x = 0$. Let $x + Id(R)$, $y + Id(R)$ be the distinct cosets. If $x + e_1$, $y + e_2$ are adjacent for some $e_1, e_2 \in Id(R)$, then $x + y \in Id(R)$ and $x - y = x + y - 2y \in Id(R)$ which implies that $x + Id(R) = y + Id(R)$. Therefore, $G_{Id}(R)$ is a disjoint union of $\frac{|R|}{|Id(R)|}$ numbers of $K_{|Id(R)|}$. $\square$

In the following proposition we find the clique number of a ring $R$.

**Proposition 2.3.** *If $R \cong R_1 \times R_2 \times ... \times R_k$, where $R_i's$ are local rings such that $(R_i, +) =< Id(R_i) >$ and $Char(R_i) \neq 2$ for all $i$, $1 \leq i \leq k$, then the clique number is $k + 1$.*

**Proof.** Let $R \cong R_1 \times R_2 \times ... \times R_k$, where $R_i's$ are local rings and $x$=$(a_1, a_2, ..., a_k)$ with $a_i \in R_i$ for all $i$, $1 \leq i \leq k$. The idempotent graph of $R_i$ is a line graph and for any element $a \in R_i$, $a$ will be adjacent to at most two vertices. Let $C = \{(a_1, a_2, ..., a_k)|a_i \in R_i\}$ be a clique.
Case 1: Suppose $a_k$=0 in any element of $C$. Then any other vertex adjacent to this vertex in $C$ should have $a_k$=$\{0, 1\}$. If $a_{k-1} \notin \{0, 1, a_{l,k-1}, a_{l-1,k-1}\}$, where $a_{l,k-1}$ and $a_{l-1,k-1}$ denotes the last two vertices of the line graph $G_{Id}(R_{k-1})$, then the degree of $a_{k-1}$ will be 2 in the line graph $G_{Id}(R_{k-1})$. Thus, $C$ contains at most two elements which is not a maximal clique. If $a_{k-1} = 1$, then in all other vertices in $C$, $a_{k-1} = 0$. If $a_{k-1} = a_{l-1,k-1}$, then in all other vertices in $C$, $a_{k-1} = a_{l,k-1}$. If $a_{k-1} = 0$ or $a_{l,k-1}$, then in $C$ at most one vertex can have $a_{k-1} = 1$ or $a_{l-1,k-1}$ and all other vertices should have $a_{k-1} = 0$ or $a_{l,k-1}$. This is true $\forall a_i \in R_i, i \leq k - 1$ and the graph $G_{Id}(R_i)$ is a line graph in which except the end vertices all other vertices have degree 2. In order to get a maximum clique, we need to take $k$ vertices from either end of the line graph and one vertex adjacent to the end vertex. Thus, all clique with the maximum number of elements will have same cardinality irrespective of the choice of $a_i$ from either end of the corresponding line graph. Without loss of generality, we consider the clique of maximum size as $C = \{(0, 0, ..., 0), (1, 0, ..., 0), (0, 1, ..., 0), ..., (0, 0, ..., 1)\}$ having $k + 1$ vertices. Suppose we attach a vertex $(a_1, a_2, ..., a_k)$ to $C$. Then $a_i$ are idempotents and hence, $a_i = 0$ *or* 1 for all $i$, $1 \leq i \leq k$. If two entries $a_i$ and $a_j$ are non zero, then $(a_1, a_2, ..., a_i, ..., a_k)$ is not adjacent to $(0, 0, ..., 1_i, ..., 0)$. Hence, the clique number is $k + 1$.
Case 2: Suppose the clique contains an element with $a_k$=$a_{l,k}$, then following the procedure as in case 1, without loss of generality we consider the clique of maximum size as $C = \{(0, 0, ..., a_{l-1,k}), (1, 0, ..., a_k), (0, 1, ..., a_k), ..., (0, 0, ..., 1, a_k), (0, 0, ..., 0, a_k)\}$ having $k$+1 vertices, where $a_{l-1,k} \in R_k$. Suppose we add a new vertex $(0, ..., a_i, ..., a_{l-1,k})$, where $a_i \neq 0$ to the above clique, then the resulting collection of $k + 2$ vertices does not form a complete graph.
Case 3: Suppose the clique contains an element whose $k^{th}$ component is any vertex of degree 2 in the line graph. Let $a_{i,k}$ and $a_{j,k}$ be the two vertices adjacent to $a_k$. Proceeding as in case 1, without loss of generality we consider the clique of maximum size as $C = \{(0, 0, ..., a_k), (1, 0, ..., a_{i,k}), (0, 1, ..., a_{i,k}), ..., (0, 0, ..., 1, a_{i,k}), (0, 0, ..., 0, a_{i,k})\}$ on $k + 1$ vertices. If a vertex is added to this clique, then it should be a vertex with its $k^{th}$ coordinate $a_{j,k}$ but $a_{j,k}$ is not adjacent to $a_{i,k}$. So, $\nexists$ a clique on $k + 2$ vertices. $\square$
In the following proposition, we prove that the idempotent graph of a ring is isomorphic to a disjoint union of its subgraphs.

**Proposition 2.4.** *Let $R$ be a finite ring and $(S, +) =< Idem(R) >$. Then $G_{Id}(R) = G_{Id}(S) \bigsqcup G_{Id}(S)$ if and only if $\forall$ $y \in R \backslash S$, $y = z + s$ for every $z \in R \backslash S$ and some $s \in S$.*

**Proof.** Suppose $G_{Id}(R) = G_{Id}(S) \bigsqcup G_{Id}(S)$. Then by theorem 3.1[7], $G_{Id}(S)$ is a connected induced subgraph of $R$. Since $R \backslash S$ is a connected component, $\exists$ a path between any two vertices. Let $y - y_1 - y_2 - ... - y_n - z$ be a path between $y$ and $z$ in $G_{Id}(R \backslash S)$ for some $y, z \in R \backslash S$. Then $y + y_1$, $y_1 + y_2$, $..., y_n + z \in Id(R)$ which implies that $y = -y_1 + e_1$, $y_1 = -y_2 + e_2$.... So, it follows that $y = z + s$ for some $s \in S$. Similarly, every element of $R \backslash S$ can be expressed in the form $y = z + s$ for every $z \in R \backslash S$ and for some $s \in S$.
Conversely, suppose $\forall$ $y \in R \backslash S$, $y = z + s$ for every $z \in R \backslash S$ and some $s \in S$. If $\exists$ $z \in R \backslash S$ such that $2z = 0$, then for $y, z \in R \backslash S$, $y = z + s_i$ for some $s_i \in S$ and $G_{Id}(R \backslash S) \cong G_{Id}(S)$. Otherwise, expressing an element $y \in R \backslash S$ as $y = z + s$ and the remaining elements $y_i$ of $R \backslash S$, as $y_i = -z + s_i$ for all $i$, $1 \leq i < |R \backslash S|$, $y$ will be adjacent to $y_i$ if and only if $s$ is adjacent to $s_i$. Similarly, expressing every element of $R \backslash S$ as $y = z + s$ and other elements as $-z + s_i$, we see that there is an edge in $G_{Id}(R \backslash S)$ if and only if there is an edges in $G_{Id}(S)$. Hence, $G_{Id}(R \backslash S) = G_{Id}(S)$. $\square$

## 3 The idempotent graph of the ring of integer modulo $n$

In this section, we continue to investigate some basic graph theoretic properties of $G_{Id}(\mathbb{Z}_n)$, for $n$ not a power of a prime. We find the diameter, the radius, the independence number and exhibit a Hamiltonian cycle. We show that the idempotent graph is non planar if $n$ has at least three distinct prime divisors.

In the following two lemmas, we find the degree of a vertex in $G_{Id}(\mathbb{Z}_n)$ and the girth of $G_{Id}(\mathbb{Z}_n)$.

**Lemma 3.1.** *Let $x$ be a vertex of $G_{Id}(\mathbb{Z}_n)$. Then the degree of $x$ is either $|Id(\mathbb{Z}_n)|$ or $|Id(\mathbb{Z}_n)|-1$.*

**Proof.** Let a vertex $x$ be adjacent to a vertex $y$ of $G_{Id}(\mathbb{Z}_n)$. Then $x+y=a$ for some $a \in Id(\mathbb{Z}_n)$ and hence $y = a - x$ . If $2x \in Id(\mathbb{Z}_n)$, then $x$ is adjacent to $a-x$ for any $a \in Id(\mathbb{Z}_n) \setminus \{2x\}$. Hence, the degree of $x$ is $|Id(\mathbb{Z}_n)|-1$. Next, if $2x \notin Id(\mathbb{Z}_n)$, then $x$ is adjacent to $a-x$ for any $a \in Id(\mathbb{Z}_n)$. Hence, the degree of $x$ is $|Id(\mathbb{Z}_n)|$. □

**Proposition 3.2.** *Girth of $G_{Id}(\mathbb{Z}_n)=3$.*

**Proof.** Let $e$ be a nontrivial idempotent. Then $1-e$ is also a non trivial idempotent. Therefore, $0-e-(1-e)-0$ is a cycle of length 3.□

Next, we find the diameter and the radius of $G_{Id}(\mathbb{Z}_n)$.

**Proposition 3.3.** *Diameter of $G_{Id}(\mathbb{Z}_n)$= max$\{p_1^{l_1}, ..., p_k^{l_k}\}-1$, where $p_i's$ are distinct prime divisors of $n$.*

**Proof.** Each component of $\mathbb{Z}_{p_i^{l_i}}$ forms the line graph. Let $x, y \in \mathbb{Z}_n$ such that $x=(a_1, a_2, ..., a_k)$ and $y=(b_1, b_2, ..., b_k)$, where $a_i,\ b_i \in \mathbb{Z}_{p_i^{l_i}}$. Then $x$ is adjacent to $y$ if and only if each coordinate $a_i+b_i$ is an idempotent. So d(x,y)=$max_{1\le i\le k}\{d(a_i,b_i)\}$ and $d(a_i,b_i)$ is the distance between $a_i$ and $b_i$ in the component graph $G_{Id}(\mathbb{Z}_{p_i^{l_i}})$. Now, $d(0,\frac{p_i^{l_i}+1}{2})$ or $d(0,\frac{p_i^{l_i}}{2})=p_i^{l_i}-1$ is the length of the longest path in the idempotent graph of $\mathbb{Z}_{p_i^{l_i}}$. Thus, diam $G_{Id}(\mathbb{Z}_n)$= max$\{p_1^{l_1}, ..., p_k^{l_k}\}-1$. □

**Proposition 3.4.** *Radius of* $G_{Id}(\mathbb{Z}_n) = \begin{cases} 2^{l-1}, & \textit{if max}\{p_1^{l_1}, ..., p_k^{l_k}\}=2^l. \\ \frac{max\{p_1^{l_1}, ..., p_k^{l_k}\}-1}{2}, & \textit{otherwise}. \end{cases}$

**Proof.** If max $\{p_1^{l_1}, ..., p_k^{l_k}\}=2^l$, then the vertex with its $i^{th}$ coordinate $2^{l-2}$ or $2^l-2^{l-2}$ for some $i$ has the least eccentricity. In this case, the least eccentricity is $2^{l-1}$. If max $\{p_1^{l_1}, ..., p_k^{l_k}\}$ is a power of odd prime, say $p_i^{l_i}$, then a vertex having one of its coordinate $p_i^{l_i}-(\frac{p_i^{l_i}-1}{4})$ will have the least eccentricity among all other vertices. In this case, the least eccentricity is $\frac{p_i^{l_i}-1}{2}$. □

In the next four propositions, we find the independence number of $G_{Id}(\mathbb{Z}_n)$ for $n$=$2p^k$, $n$=$p^kq^l$, where $p^k<q^l$ and $p,q\neq 2$, $n$=$2^kp^l$, $k\neq 1$ and $n$= $\Pi_{i=1}^{k}p_i^{n_i},\ i\geq 3$.

**Proposition 3.5.** *If $R\cong\mathbb{Z}_{2p^k}$, then $\alpha(G_{Id}(\mathbb{Z}_{2p^k}))=p^k-1$.*

**Proof.** Let $A$=$\{(0,0),(0,p^k-1),(0,p^k-2),...,(0,p^k-\frac{p^k-1}{2}),(1,p^k-1),(1,p^k-2),...,(1,\frac{p^k+1}{2}+1)\}$. Then in $A$ non-zero vertices are non idempotent, $(p^k-\frac{p^k-1}{2})+(\frac{p^k+1}{2}+1)$=$p^k+2$ and no pair of vertices within this set is adjacent. Hence, $A$ is an independent set. Next, if we adjoin any idempotent element to $A$, then it will be adjacent to $(0,0)$. Let $x\in\mathbb{Z}_2\times\mathbb{Z}_{p^k}\setminus\{A\}$ be a non idempotent element. If $x=(a,b)$ or $(1,p^k-\frac{p^k-1}{2})$, where $1<b<p^k-(\frac{p^k-1}{2})$, $\exists\, y=(0,p^k-b)$ or $y=(1,p^k-b)$ an element in $A$ such that $x+y$ is an idempotent. Now, $|A|$= $\frac{p^k+1}{2}+(\frac{p^k+1}{2})-2=p^k-1$.

Let $B$ be any independent set of cardinality $|A|+1$. Then for every $y\in B$, $y=(a,b)$, where $a\in\{0,1\}$ and $b\in\{0,1,...,p^k-1\}$. If $b\in\{1,2,...,\frac{p^k-1}{2}\}$, then $|B|$= $\frac{p^k-1}{2}\times 2=p^k-1$, which is a contradiction. Otherwise if $b \in \{0, p^k-1, ..., \frac{p^k+1}{2}\}$, then $|B|= \frac{p^k+1}{2} \times 2 - 2 = p^k - 1$, which is again a contradiction. Hence, the independence number $\alpha(G_{Id}(Z_{2p^k})) = p^k - 1$. □

**Proposition 3.6.** *If $R \cong \mathbb{Z}_{p^k q^l}$, then $\alpha(G_{Id}(\mathbb{Z}_{p^k q^l})) = \frac{(q^l-1)p^k+2}{2}$, where $p^k < q^l$ and $p, q \neq 2$.*

**Proof.** We consider the subsets $A$=$\{(0,q^l-1),(0,q^l-2),...,(0,\frac{q^l+1}{2}+1),(1,q^l-1),(1,q^l-2),...,(1,\frac{q^l+1}{2}+1),...,(p^k-1,q^l-1),(p^k-1,q^l-2),...,(p^k-1,\frac{q^l+1}{2}+1)\}$, $B$=$\{(0,0),(p^k-1,0),(p^k-2,0),...,(\frac{p^k+1}{2},0)\}$ and $C$=$\{(0,\frac{q^l+1}{2}),(p^k-1,\frac{q^l+1}{2}),(p^k-2,\frac{q^l+1}{2}),...,(\frac{p^k+1}{2},\frac{q^l+1}{2})\}$ of $V(G_{Id}(\mathbb{Z}_{p^k q^l}))$. Then, we claim that $A \cup B \cup C$ is an independent set of $V(G_{Id}(\mathbb{Z}_{p^k q^l}))$. Suppose $x = (a,b) \in A$, where $a \in \mathbb{Z}_{p^k}$, $b \in \mathbb{Z}_{q^l}$ and $\frac{q^l+1}{2} < b < q^l$. For any $y = (a_1,b_1) \in A$, $x$ is not adjacent to $y$, as $b + b_1 \ncong 0$ *or* $1$ $(mod\ q^l)$. Also, no vertex of the set $A$ is adjacent to a vertex of $B$. Let $x = (a,b) \in A$ and $y = (a_1,b_1) \in C$. Then $b + b_1 = \frac{q^l+1}{2} + (\frac{q^l+1}{2}+1) \geq q^l + 1 + 1 \cong 2$ $(mod\ q^l)$ shows that no vertex of $A$ is adjacent to any vertex of $C$. Next, the set $B$ is independent as for any $x = (a,b) \in B$, $a$ belongs to the set of non adjacent vertices of the idempotent graph of $\mathbb{Z}_{p^k}$ and $b = 0$. Similarly, $x = (a,b) \in B$ is not adjacent to a vertex $y = (a_1,b_1)$ in $C$, as $b + b_1 \ncong 0$ *or* $1$ $(mod\ q^l)$. For any $x = (a,b) \in C$, $a$ belongs to the set of non adjacent vertices of the idempotent graph of $\mathbb{Z}_{p^k}$ and so $C$ is an independent set.
Lastly, we prove that $A \cup B \cup C$ is the maximum independent set. Suppose $(x_1,y_1) \in V(\mathbb{Z}_{p^k q^l}) \setminus \{A \cup B \cup C\}$. If $(x_1,y_1)$ is an idempotent element, then $(x_1,y_1) \sim (0,0)$. Now, we assume that $(x_1,y_1)$ is a non idempotent element with at least one of $x_1$ or $y_1 > 1$. If $y_1 > 1$, then $1 < y_1 \leq \frac{q^l+1}{2}$. When $1 < y_1 < \frac{q^l+1}{2}$, $\exists$ a vertex $(1-x_1, q^l-y_1)$ in $A$ such that $(x_1,y_1) \sim (1-x_1,q^l-y_1)$. If $y_1 = \frac{q^l+1}{2}$, then $(x_1,y_1) \sim (1-x_1,y_1)$, where $(1-x_1,y_1) \in C$. When $x_1 > 1$ and $y_1 = 0$, the vertex $(1-x_1,0) \in B$ is such that $(x_1,y_1) \sim (1-x_1,0)$. If $x_1 > 1$ and $y_1 = 1$, then the vertex $(1-x_1,q^l-1) \in A$ is such that $(x_1,y_1) \sim (1-x_1,q^l-1)$. So, $|A \cup B \cup C| = (\frac{q^l+1}{2}-2)p^k + (\frac{p^k+1}{2})2 = \frac{(q^l-1)p^k+2}{2}$. Let $D$ be any independent set of cardinality $|A \cup B \cup C| + 1$. Then for every $y \in D$, $y = (a,b)$, where $a \in \mathbb{Z}_{p^k}$ and $b \in \mathbb{Z}_{q^l}$. If $b \in \{1,2,...,\frac{q^l-1}{2}\}$, then $|D|= \frac{q^l-1}{2} \times p^k < \frac{(q^l-1)p^k+2}{2}$, which is a contradiction. Otherwise, if $b \in \{0,q^l-1,...,\frac{q^l+1}{2}\}$, then $|D| = (\frac{q^l+1}{2}-2)p^k + (\frac{p^k+1}{2})2 = \frac{(q^l-1)p^k+2}{2}$, which is again a contradiction. Hence, the independence number $\alpha(G_{Id}(\mathbb{Z}_{p^k q^l})) = \frac{(q^l-1)p^k+2}{2}$. □

**Proposition 3.7.** *If $R \cong \mathbb{Z}_{2^k p^l}$, $k \neq 1$, then*
$$\alpha(G_{Id}(\mathbb{Z}_{2^k p^l})) = \begin{cases} 2^{k-1}(p^l-1), & if\ 2^k < p^l. \\ \frac{p^l(2^k-1)+1}{2}, & if\ p^l < 2^k. \end{cases}$$

**Proof.** Case 1. If $2^k < p^l$, we consider three subsets $A, B$ and $C$ of the set of vertices, where $A$=$\{(0,p^l-1),(0,p^l-2),...,(0,\frac{p^l+1}{2}+1),(1,p^l-1),(1,p^l-2),...,(1,\frac{p^l+1}{2}+1),...,(2^k-1,p^l-1),(2^k-1,p^l-2),...,(2^k-1,\frac{p^l+1}{2}+1)\}$,
$B$=$\{(1,0),(2,0),(3,0),...,(2^{k-1},0)\}$, $C$=$\{(1,\frac{p^l+1}{2}),(2,\frac{p^l+1}{2}),(3,\frac{p^l+1}{2}),...,(2^{k-1},\frac{p^l+1}{2})\}$. Then we claim that $A \cup B \cup C$ is an independent set of $V(G_{Id}(\mathbb{Z}_{2^k} \times \mathbb{Z}_{p^l}))$. Suppose $x = (a,b) \in A$, where $a \in \mathbb{Z}_{2^k}$, $b \in \mathbb{Z}_{p^l}$ and $\frac{p^l+1}{2} < b < p^l$. For any $y = (a_1,b_1) \in A$, $x$ is not adjacent to $y$ as $b + b_1 \ncong 0$ *or* $1$ $(mod\ p^l)$. Also, no vertex of the set $A$ is adjacent to any vertex of $B$. Let $x = (a,b) \in A$ and $y = (a_1,b_1) \in C$. Then $b + b_1 = \frac{p^l+1}{2} + (\frac{p^l+1}{2}+1) \geq p^l + 1 + 1 \cong 2$ $(mod\ p^l)$, showing that no vertex of $A$ is adjacent to any vertex of $C$. Next, the set $B$ is independent as for any $x = (a,b) \in B$, $a$ belongs to the set of non adjacent vertices of the idempotent graph of $\mathbb{Z}_{2^k}$ and $b = 0$. Similarly, $x = (a,b) \in B$ is not adjacent to a vertex $y = (a_1,b_1)$ in $C$ as $b + b_1 \ncong 0$ *or* $1$ $(mod\ p^l)$. For any $x = (a,b) \in C$, $a$ belongs to the set of non adjacent vertices of the idempotent graph of $\mathbb{Z}_{2^k}$, which shows that the set $C$ is independent.
Next, we prove that $A \cup B \cup C$ is the maximum independent set. Suppose $(x_1,y_1) \in V(\mathbb{Z}_{2^k p^l}) \setminus \{A \cup B \cup C\}$. If $(x_1,y_1)$ is an idempotent element, then $(x_1,y_1) = (0,0)$ *or* $(0,1)$ *or* $(1,1)$. If $(x_1,y_1) = (0,0)$ *or* $(0,1)$, then $(x_1,y_1) \sim (1,0)$ in B. If $(x_1,y_1) = (1,1)$, then $(1,1) \sim (0,p^l-1)$ in $A$. Now, suppose $(x_1,y_1)$ is a non idempotent element with at least one of $x_1$ or $y_1 > 1$. If

$y_1 > 1$, then $1 < y_1 \leq \frac{p^l+1}{2}$. When $1 < y_1 < \frac{p^l+1}{2}$, $\exists$ a vertex $(1-x_1, p^l - y_1)$ in $A$ such that $(x_1, y_1) \sim (1-x_1, p^l - y_1)$. If $y_1 = \frac{p^l+1}{2}$, then $(x_1, y_1) \sim (1-x_1, y_1)$, where $(1-x_1, y_1) \in C$. When $x_1 > 1$ and $y_1 = 0$, then $(1-x_1, 0) \in B$ is such that $(x_1, y_1) \sim (1-x_1, 0)$. If $x_1 > 1$ and $y_1 = 1$, then the vertex $(1-x_1, p^l - 1) \in A$ is such that $(x_1, y_1) \sim (1-x_1, p^l-1)$. We have $|A \cup B \cup C| = (\frac{p^l+1}{2} - 2)2^k + (\frac{2^k}{2})2 = 2^{k-1}(p^l - 1)$. Let $D$ be any independent set of cardinality $|A \cup B \cup C| + 1$. Then for every $y \in D$, $y = (a, b)$, where $a \in \mathbb{Z}_{2^k}$ and $b \in \mathbb{Z}_{p^l}$. If $b \in \{1, 2, ..., \frac{p^l-1}{2}\}$, then $|D| = \frac{p^l-1}{2} \times 2^k = 2^{k-1}(p^l-1)$, which is a contradiction. Otherwise, if $b \in \{0, p^l-1, ..., \frac{p^l+1}{2}\}$, then $|D| = (\frac{p^l+1}{2} - 2)2^k + (\frac{2^k}{2})2 = 2^{k-1}(p^l-1)$, which is again a contradiction. Hence, the independence number $\alpha(G_{Id}(\mathbb{Z}_{2^k p^l})) = 2^{k-1}(p^l-1)$, when $2^k < p^l$.

Case 2. If $p^l < 2^k$, we consider the set $A$=$\{(0,1), (1,1), ..., (p^l-1, 1), (0,2), (1,2), ..., (p^l-1, 2), ..., (0, 2^{k-1}-1), (1, 2^{k-1}-1), ..., (p^l-1, 2^{k-1}-1)\}$ and the set $B$=$\{(0, 2^{k-1}), (p^l-1, 2^{k-1}), (p^l-2, 2^{k-1}), ..., (\frac{p^l+1}{2}, 2^{k-1})\}$. We claim that $A \cup B$ is an independent subset of $V(G_{Id}(\mathbb{Z}_{p^l} \times \mathbb{Z}_{2^k}))$. Suppose $x = (a, b) \in A$, where $a \in \mathbb{Z}_{p^l}$, $b \in \mathbb{Z}_{2^k}$ and $1 \leq b < 2^{k-1}$. For any $y = (a_1, b_1) \in A$, $x$ is not adjacent to $y$ as $b + b_1 \not\cong 0$ $or$ $1$ $(mod\ 2^k)$. Also, no vertex of the set $A$ is adjacent to a vertex of $B$ because for any $x = (a, b) \in A$ and $y = (a_1, b_1) \in B$, $b + b_1 < 2^k$. Next, the set $B$ is independent as for any $x = (a, b) \in B$, $a$ belongs to the set of non adjacent vertices of the idempotent graph of $\mathbb{Z}_{p^l}$ and $b = 2^{k-1}$ which shows that $x$ is not adjacent to any element of $B$. So, $A \cup B$ is an independent set.

Now, we prove that $A \cup B$ is the maximum independent set. Suppose $(x_1, y_1) \in V(\mathbb{Z}_{p^l 2^k}) \setminus \{A \cup B\}$. If $(x_1, y_1)$ is an idempotent element, then $(x_1, y_1) = (0, 0)$ $or$ $(1, 0)$ and $(x_1, y_1) \sim (0, 1)$ in A. Now, suppose $(x_1, y_1)$ is a non idempotent element with at least one of $x_1$ or $y_1 > 1$. If $y_1 > 1$, then $1 < y_1 \leq 2^{k-1}$. When $1 < y_1 < 2^{k-1}$, $\exists$ a vertex $(1-x_1, 2^k - y_1)$ in $A$ such that $(x_1, y_1) \sim (1-x_1, 2^k - y_1)$. If $y_1 = \frac{2^k}{2}$, then $(x_1, y_1) \sim (1-x_1, y_1)$, where $(1-x_1, y_1) \in B$. When $x_1 > 1$ and $y_1 = 0$, $\exists$ $(1-x_1, 1) \in A$ is such that $(x_1, y_1) \sim (1-x_1, 1)$. Here, $|A \cup B| = (2^l - 1)p^l + (\frac{p^l+1}{2}) = \frac{p^l(2^k-1)+1}{2}$. Let $D$ be any independent set of cardinality $|A \cup B| + 1$. Then for every $y \in D$, $y = (a, b)$, where $a \in \mathbb{Z}_{p^l}$ and $b \in \mathbb{Z}_{2^k}$. If $b \in \{1, 2, ..., 2^{k-1}-1\}$, then $|D| = \frac{p^l(2^k-1)+1}{2}$, which is a contradiction. If $b \in \{0, 2^k-1, ..., 2^{k-1}+1\}$, then $|D| = (2^l-1)p^l + (\frac{p^l+1}{2}) = \frac{p^l(2^k-1)+1}{2}$, which is again a contradiction. Hence, the independence number $\alpha(G_{Id}(\mathbb{Z}_{p^l 2^k})) = \frac{p^l(2^k-1)+1}{2}$. $\square$

**Proposition 3.8.** *If $R \cong \mathbb{Z}_n$, where $n = \Pi_{i=1}^k p_i^{n_i}$, $i \geq 3$ and $p_1^{n_1} < p_2^{n_2} < ... < p_k^{n_k}$, then $\alpha(G_{Id}(\mathbb{Z}_n)) = \sum_{i=1}^{k+1} |S_i|$.*

**Proof.** Let $R \cong \mathbb{Z}_n$, where $n = \Pi_{i=1}^k p_i^{n_i}$, $i \geq 3$ $and$ $p_1^{n_1} < p_2^{n_2} < ... < p_k^{n_k}$ and $p_k = 2$.
Case 1. We consider the following independent sets consisting of the elements having entry $a_i \in \mathbb{Z}_{p_i}^{n_i}$ and entry $b_{k-i} = \frac{p_{k-i}^{n_{k-i}}+1}{2}$ $or$ $0$ for all $i$, $1 \leq i \leq k-1$:

$S_1 = \{(a_1, a_2, ..., a_{k-1}, 1), (a_1, a_2, ..., a_{k-1}, 2), ..., (a_1, a_2, ..., a_{k-1}, p_k^{n_k-1} - 1)\}$,

$S_2 = \{(a_1, a_2, ..., p_{k-1}^{n_{k-1}} - 1, p_k^{n_k-1}), (a_1, a_2, ..., p_{k-1}^{n_{k-1}} - 2, p_k^{n_k-1}), ..., (a_1, a_2, ..., \frac{p_{k-1}^{n_{k-1}}+1}{2} + 1, p_k^{n_k-1})\}$,

$S_3 = \{(a_1, a_2, ..., p_{k-2}^{n_{k-2}} - 1, b_{k-1}, p_k^{n_k-1}), (a_1, a_2, ..., p_{k-2}^{n_{k-2}} - 2, b_{k-1}, p_k^{n_k-1}), ..., (a_1, a_2, ..., \frac{p_{k-2}^{n_{k-2}}+1}{2} + 1, b_{k-1}, p_k^{n_k-1})\}$,

.

$S_l = \{(a_1, a_2, ..., p_{k-(l-1)}^{n_{k-(l-1)}} - 1, ..., b_{k-2}, b_{k-1}, p_k^{n_k-1}), (a_1, a_2, ..., p_{k-(l-1)}^{n_{k-(l-1)}} - 2, ..., b_{k-2}, b_{k-1}, p_k^{n_k-1}), ..., (a_1, a_2, ..., \frac{p_{k-(l-1)}^{n_{k-(l-1)}}+1}{2} + 1, ..., b_{k-2}, b_{k-1}, p_k^{n_k-1})\}$,

.

$S_k = \{(p_1^{n_1} - 1, b_2, ..., b_{k-3}, b_{k-2}, b_{k-1}, p_k^{n_k-1}), (p_1^{n_1} - 2, b_2, ..., b_{k-3}, b_{k-2}, b_{k-1}, p_k^{n_k-1}), ..., (\frac{p_1^{n_1}+1}{2} + 1, b_2, ..., b_{k-3}, b_{k-2}, b_{k-1}, p_k^{n_k-1})\}$,

$S_{k+1} = \{(b_1, b_2, ..., b_{k-3}, b_{k-2}, b_{k-1}, p_k^{n_k-1})\}$.

Now, we claim that $\bigsqcup_{i=1}^{k+1} S_i$ is an independent subset of $V(G_{Id}(\mathbb{Z}_n))$. Suppose $x = (x_1, x_2, ..., x_k)$ and $y = (y_1, y_2, ..., y_k)$ are any two distinct elements of $\bigsqcup_{i=1}^{k+1} S_i$. If $x_k$ or $y_k \in \{1, 2, ..., p_k^{n_k-1} - 1\}$, then $x$ is not adjacent to $y$ as these vertices are non adjacent in the idempotent graph $G_{Id}(\mathbb{Z}_{p_k^{n_k}})$. Otherwise, if $x_k = y_k = p_k^{n_k-1}$, then we check the preceding component. If $x_{k-1}$ or $y_{k-1} \in \{p_{k-1}^{n_k-1} - 1, p_{k-1}^{n_k-1} - 2, ..., \frac{p_{k-1}^{n_{k-1}}+1}{2} + 1\}$, then $x$ is not adjacent to $y$ as these vertices are not adjacent in the idempotent graph $G_{Id}(\mathbb{Z}_{p_{k-1}^{n_{k-1}}})$. Otherwise, if $x_{k-1}$ or $y_{k-1} \in \{\frac{p_{k-1}^{n_{k-1}}+1}{2}, 0\}$ and $x_{k-1} \neq y_{k-1}$, then $x$ is not adjacent to $y$ and if $x_{k-1} = y_{k-1}$, then we consider $x_{k-3}$. We repeat the process when the components are equal to the end vertices of their corresponding line graphs. Finally if $x_1$ or $y_1 \in \{p_1^{n_1} - 1, p_1^{n_1} - 2, ..., \frac{p_1^{n_1}+1}{2} + 1\}$, then $x$ is not adjacent to $y$ as these vertices are non adjacent in the idempotent graph $G_{Id}(\mathbb{Z}_{p_1^{n_1}})$. Otherwise if $x_1$ or $y_1 \in \{\frac{p_1^{n_1}+1}{2}, 0\}$ and $x_1 \neq y_1$, then $x$ is not adjacent to $y$ and if $x_1 = y_1$, then we get $x = y$, which is a contradiction. Therefore, $\bigsqcup_{i=1}^{k+1} S_i$ is an independent set and $|\bigsqcup_{i=1}^{k+1} S_i| = \sum_{i=1}^{k+1} |S_i|$, where $|S_1| = ((\frac{p_k^{n_k}}{2}) - 1)\Pi_{i=1}^{k-1} p_{k-i}^{n_{k-i}}$, $|S_i| = ((\frac{p_{k-(i-1)}^{n_{k-(i-1)}}+1}{2}) - 2)\Pi_{j=1}^{k-i} p_{k-j-(i-1)}^{n_{k-j-(i-1)}} \times 2^{i-2}$ for all $i,\ 2 \leq i \leq k$ and $|S_{k+1}| = 2^{k-1}$. Let $x = (x_1, x_2, ..., x_k) \in \mathbb{Z}_n \setminus \{\bigsqcup_{i=1}^{k+1} S_i\}$. If $x_k \in \{0, p_k^{n_k} - 1, p_k^{n_k} - 2, ..., p_k^{n_k-1} + 2\}$, then the vertex $(1 - x_1, 1 - x_2, ..., 1 - x_k) \in S_1$ is adjacent to $x$. If $x_k = p_k^{n_k-1} + 1$, then the vertex $(1 - x_1, 1 - x_2, ..., p_k^{n_k} - x_k) \in S_1$ is adjacent to $x$. If $x_k = p_k^{n_k-1}$, then the following cases arise;

Subcase 1: If $x_{k-1} \in \{1, 2, ..., \frac{p_{k-1}^{n_{k-1}}+1}{2} - 2\}$, then $x$ is adjacent to the vertex $(1 - x_1, 1 - x_2, ..., 1 - x_{k-1}, x_k) \in S_2$.

Subcase 2: If $x_{k-1} = \frac{p_{k-1}^{n_{k-1}}+1}{2} - 1$, then $x$ is adjacent to the vertex $(1 - x_1, 1 - x_2, ..., p_{k-1}^{n_{k-1}} - x_{k-1}, x_k) \in S_2$.

Subcase 3: If $x_{k-1} = \frac{p_{k-1}^{n_{k-1}}+1}{2}$ *or* 0, then there exist $(y_1, y_2, ..., y_{k-2}, x_{k-1}, x_k) \in S_l$ for some $l$, $2 < l \leq k+1$, where $y_{k-i}$ can be obtained by carrying out the same procedure as in the subcases 1, 2 and 3 for $x_{k-1}$, for all $i,\ 1 \leq i \leq k-2$ such that $(y_1, y_2, ..., y_{k-2}, x_{k-1}, x_k)$ will be adjacent to $x$.

Suppose $A$ is another independent set of cardinality $|\bigsqcup_{i=1}^{k+1} S_i| + 1$. Let $x = (x_1, x_2, ..., x_k)$ be a vertex in $A$. If $x_k \in \{1, 2, ..., p_k^{n_k-1} - 1, p_k^{n_k-1}\}$, then $A = \bigsqcup_{i=1}^{k+1} S_i$. So, let $x_k \in \{0, p_k^{n_k} - 1, p_k^{n_k} - 2, ..., p_k^{n_k-1} + 1\}$. By following the same process of construction of $\bigsqcup_{i=1}^{k+1} S_i$, we get that $|A| = |\bigsqcup_{i=1}^{k+1} S_i|$, which shows that $|A| \neq |\bigsqcup_{i=1}^{k+1} S_i| + 1$. Therefore, $|\bigsqcup_{i=1}^{k+1} S_i|$ is the independence number of the idempotent graph of $\mathbb{Z}_n$, where $n = \Pi_{i=1}^{k} p_i^{n_i},\ i \geq 3$, $p_1^{n_1} < p_2^{n_2} < ... < p_k^{n_k}$ and $p_k = 2$.

Case 2. In the case when all $p_i's$ are odd, by a similar construction as in case 1 we get $|S_1| = ((\frac{p_k^{n_k}+1}{2}) - 2)\Pi_{i=1}^{k-1} p_{k-i}^{n_{k-i}}$, $|S_i| = ((\frac{p_{k-(i-1)}^{n_{k-(i-1)}}+1}{2}) - 2)\Pi_{j=1}^{k-i} p_{k-j-(i-1)}^{n_{k-j-(i-1)}} \times 2^{i-1}$ for all $i,\ 2 \leq i \leq k$, here in this expression the term $2^{i-1}$ appear for the end vertices of the line graph $G_{Id}(\mathbb{Z}_{p_{k-j}^{n_{k-j}}})$ for all $j$, $0 \leq j < i-1$ and $|S_{k+1}| = 2^k$, since all $p_i's$ are odd we consider both the end vertices of the line graph $G_{Id}(\mathbb{Z}_{p_i^{n_i}})$.

Case 3. In the case when $p_{k-l} = 2$ for some $l \geq 1$, by a similar construction as in case 1 we get $|S_i| = ((\frac{p_{k+1-i}^{n_{k+1-i}}+1}{2}) - 2)\Pi_{j=1}^{k-i} p_{k+1-j-i}^{n_{k+1-j-i}} \times 2^{i-1}$ for all $i,\ 1 \leq i \leq l$, here in this expression the term $2^{i-1}$ appear for the end vertices of the line graph $G_{Id}(\mathbb{Z}_{p_{k-j}^{n_{k-j}}})$ for all $j$, $0 \leq j < i$, $|S_{k-l}| = ((\frac{p_{k-l}^{n_{k-l}}}{2}) - 1)\Pi_{j=1}^{k-l-1} p_{k-j-l}^{n_{k-j-l}} \times 2^l$, $|S_i| = ((\frac{p_{k-(i-1)}^{n_{k-(i-1)}}+1}{2}) - 2)\Pi_{j=1}^{k-i} p_{k-j-(i-1)}^{n_{k-j-(i-1)}} \times 2^{i-2}$

for all $i$, $l+2 \leq i \leq k$ and $|S_{k+1}| = 2^{k-1}$ as we consider only one end vertex of the line graph $G_{Id}(\mathbb{Z}_{p_{k-l}^{n_{k-l}}})$ and both the end vertices of the line graph $G_{Id}(\mathbb{Z}_{p_i}^{n_i})$ for odd $p_i's$. □

In the next three propositions, we exhibit the planar embedding and the Hamiltonian cycle of $G_{Id}(\mathbb{Z}_n)$.

**Proposition 3.9.** *$G_{Id}(\mathbb{Z}_n)$ is planar if and only if $k \leq 2$, where $k$ is the number of distinct prime divisors of $n$.*

**Proof.** Suppose $G_{Id}(\mathbb{Z}_n)$ is planar. If possible, let $k > 2$. Then $\delta \geqslant 7 > 5$, so there does not exist any vertex of degree less than 5. This contradicts our assumption. Hence, $k \leq 2$.
Conversely, let us assume that $k \leqslant 2$.
Case 1. If $\frac{n}{2}$ is an idempotent element, then $0 - 1 - (n-1) - 2 - (n-2) - ... - \frac{n}{2} - 0$ is a cycle. Therefore, 0 will be adjacent to $\frac{n}{2}$ and $\frac{n}{2}+1$ which are consecutive in the cycle and other vertices are adjacent to alternating pairs. The figure 1 shows the planar embedding of the idempotent graph of the ring $\mathbb{Z}_n$.

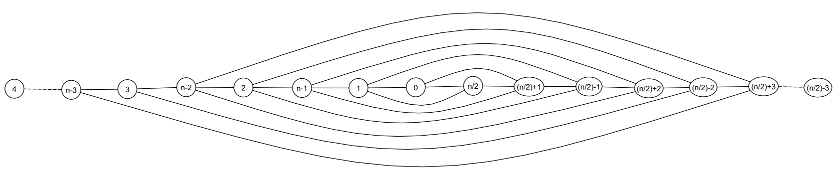


**Figure 1.** $G_{Id}(\mathbb{Z}_n)$

Case 2. If $\frac{n}{2}$ or $\frac{n+1}{2}$ is not an idempotent element, then we draw the line graph $0 - 1 - (n-1) - 2 - (n-2) - ... - \frac{n}{2}$ as directed by the arrow in the figure 2. The idempotents $e$ and $1-e$ are adjacent in the path and will be adjacent to 0. Next, the vertex 1 will be adjacent to a vertex next to $1-e$ and a vertex before $e$ in the path. We repeat this process of adding edges to the vertices in the middle row till a vertex degree reaches 3 (marked black in the figure). Now, all the vertices of the middle row will attain its maximum degree and new edges will be added to the outer vertices. If the end vertex is adjacent to $x$ and $y$, then $\frac{n}{2}+x = e$ and $\frac{n}{2}+y = 1-e$ which implies that $x+y = 1$. Hence, the end vertex of the above line graph is of degree 3 and will be adjacent to the consecutive vertex in the path $0 - 1 - (n-1) - 2 - (n-2) - ... - \frac{n}{2}$ as shown by the dotted edges in the figure 2. The second last vertex is adjacent to the vertex next to $y$ in the path. Repeating this process we get the planar embedding for $G_{Id}(\mathbb{Z}_n)$ as shown in the figure 2. Similarly, we can draw the planar embedding of $G_{Id}(\mathbb{Z}_n)$ for odd $n$. □

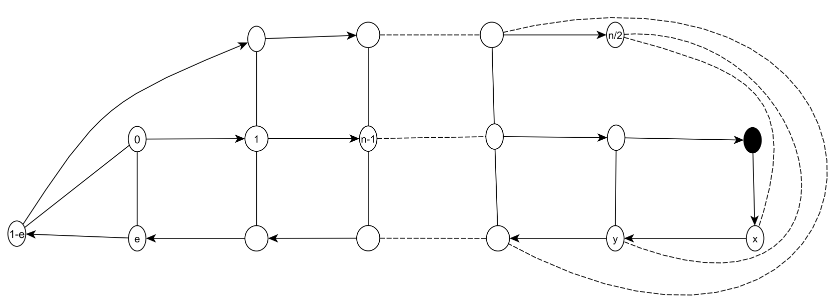


**Figure 2.** $G_{Id}(\mathbb{Z}_n)$

**Remark 3.10.** $G_{Id}(\mathbb{Z}_n)$ have at least four vertices of odd degree and hence cannot have an Eulerian trial.

**Proposition 3.11.** *If $n \cong 2 \ (mod\ 4)$, then $G_{Id}(\mathbb{Z}_n)$ is a Hamiltonian graph.*

**Proof.** Let $n = 4t + 2$ for some $t \in \mathbb{N}$. Then $2(2t+1) \mid (2t)(2t+1)$. Hence, $\frac{n}{2}$ is an idempotent element. Therefore, $0 - 1 - (n-1) - 2 - (n-2) ... - \frac{n}{2} - 0$ is a Hamiltonian cycle in $G_{Id}(\mathbb{Z}_n)$. $\square$

**Proposition 3.12.** *If $n \cong 0 \ (mod\, 4)$, then $G_{Id}(\mathbb{Z}_n)$ is a Hamiltonian graph.*

**Proof.** Let $n = 2^k m$, where $k \geq 2$. Then $\mathbb{Z}_n \cong \mathbb{Z}_{2^k} \times \mathbb{Z}_m$ since $gcd(2^k, m) = 1$. We know $G_{Id}(\mathbb{Z}_{2^k})$ is a line graph and $0 - 1 - (m-1) - 2 - (m-2) ... - \frac{m+1}{2}$ is a path in $G_{Id}(\mathbb{Z}_m)$.

$$
\begin{array}{cccccccc}
(\frac{m+1}{2},0)\longrightarrow & \swarrow(\frac{m+1}{2},1) & (\frac{m+1}{2},2^k-1)\longrightarrow & \swarrow(\frac{m+1}{2},2) & \dots & (\frac{m+1}{2},2^{k-1}+1)\searrow & \longleftarrow(\frac{m+1}{2},2^{k-1}) \\
(\frac{m+1}{2}-1,0)\searrow & \nwarrow(\frac{m+1}{2}-1,1) & (\frac{m+1}{2}-1,2^k-1)\searrow & \nwarrow(\frac{m+1}{2}-1,2) & \dots & (\frac{m+1}{2}-1,2^{k-1}+1)\nearrow & \swarrow(\frac{m+1}{2}-1,2^{k-1}) \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
(m-1,0)\nearrow & \swarrow(m-1,1) & (m-1,2^k-1)\searrow & \nwarrow(m-1,2) & \dots & (m-1,2^{k-1}+1)\searrow & \nwarrow(m-1,2^{k-1}) \\
(1,0)\downarrow & \nwarrow(1,1) & (1,2^k-1)\nearrow & (1,2)\searrow & \dots & (1,2^{k-1}+1)\nearrow & \downarrow(1,2^{k-1}) \\
(0,0)\longrightarrow & (0,\bar{1})\nearrow & \nwarrow(0,2^k-1) & \longleftarrow(0,2) & \dots & \nwarrow(0,2^{k-1}+1) & \longleftarrow(0,2^{k-1})
\end{array}
$$

$\square$

**Example:** We know that $\mathbb{Z}_{60} \cong \mathbb{Z}_{15} \times \mathbb{Z}_{2^2}$, $G_{Id}(\mathbb{Z}_{2^2})$ is a line graph and $0 - 1 - 14 - 2 - 13 - 3 - 12 - 4 - 11 - 5 - 10 - 6 - 9 - 7 - 8$ is a path in $G_{Id}(\mathbb{Z}_{15})$. So, the following diagram depicts Hamiltonion cycle in $G_{Id}(\mathbb{Z}_{60})$.

$$
\begin{array}{cccc}
(8,0)\longmapsto & \swarrow(8,1) & (8,3)\searrow & \longleftarrow(8,2) \\
(7,0)\searrow & \nwarrow(7,1) & (7,3)\nearrow & \swarrow(7,2) \\
(9,0)\nearrow & \swarrow(9,1) & (9,3)\searrow & \nwarrow(9,2) \\
(6,0)\searrow & \nwarrow(6,1) & (6,3)\nearrow & \swarrow(6,2) \\
(10,0)\nearrow & \swarrow(10,1) & (10,3)\searrow & \nwarrow(10,2) \\
(5,0)\searrow & \nwarrow(5,1) & (5,3)\nearrow & \swarrow(5,2) \\
(11,0)\nearrow & \swarrow(11,1) & (11,3)\searrow & \nwarrow(11,2) \\
(4,0)\searrow & \nwarrow(4,1) & (4,3)\nearrow & \swarrow(4,2) \\
(12,0)\nearrow & \swarrow(12,1) & (12,3)\searrow & \nwarrow(12,2) \\
(3,0)\searrow & \nwarrow(3,1) & (3,3)\nearrow & \swarrow(3,2) \\
(13,0)\nearrow & \swarrow(13,1) & (13,3)\searrow & \nwarrow(13,2) \\
(2,0)\searrow & \nwarrow(2,1) & (2,3)\nearrow & \swarrow(2,2) \\
(14,0)\nearrow & \swarrow(14,1) & (14,3)\searrow & \nwarrow(14,2) \\
(1,0)\downarrow & \nwarrow(1,1) & (1,3)\nearrow & \downarrow(1,2) \\
(0,0)\longrightarrow & (0,1)\nearrow & \nwarrow(0,3) & \longleftarrow(0,2)
\end{array}
$$

We end this article by proving that the clique number is equal to the chromatic number of $G_{Id}(\mathbb{Z}_n)$.

**Proposition 3.13.** *Let $R \cong \mathbb{Z}_n$, where $n = \Pi_{i=1}^{k} p_i^{n_i}$. Then $\chi(G_{Id}(\mathbb{Z}_n)) = \omega(G_{Id}(\mathbb{Z}_n)) = k+1$.*

**Proof.** Without loss of generality we can assume that $p_k^{n_k} > p_{k-1}^{n_{k-1}} > ... > p_1^{n_1}$. We begin by coloring the line graph of each component $G_{Id}(\mathbb{Z}_{p_i^{l_i}})$ in decreasing order of $p_i^{n_i}$. We assign two colors to $G_{Id}(\mathbb{Z}_{p_k^{l_k}})$ and color the next component by using one color from the preceding component and another new color. Repeating this process we can color all the components with $k+1$ distinct colors. Suppose $x = (a_1, a_2, ..., a_k)$ and $y = (b_1, b_2, ..., b_k)$ be any two adjacent elements. If $a_k$ is adjacent to $b_k$ in the component $G_{Id}(\mathbb{Z}_{p_k^{l_k}})$ we color the vertices by two colors used in the component $G_{Id}(\mathbb{Z}_{p_k^{l_k}})$. Otherwise, if $a_k = b_k$ we check the preceding coordinate. If $a_{k-1}$ is adjacent to $b_{k-1}$ in the component $G_{Id}(\mathbb{Z}_{p_{k-1}^{l_{k-1}}})$ we color the vertices by two colors used in the component $G_{Id}(\mathbb{Z}_{p_{k-1}^{l_{k-1}}})$. Otherwise, if $a_{k-1} = b_{k-1}$ we check the preceding coordinate to assign the color and if the preceding coordinates are equal we repeat till the last component $G_{Id}(\mathbb{Z}_{p_1^{l_1}})$. We see that this coloring is a proper $k+1$ coloring of $G_{Id}(\mathbb{Z}_n)$. $\square$

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Received: November 20th, 2021
Accepted: May 3rd, 2022