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# ON RICCI PSEUDO-SYMMETRIC SUPER QUASI-EINSTEIN NEARLY KAEHLER MANIFOLD

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**Abstract** In this paper, we have studied some curvature properties of projective curvature tensor in nearly Kaehler manifold. The present paper deals the study of a projective Ricci psedo-symmetric super quasi-Einstein nearly Kaehler manifold also found some properties of such a manifold.

## 1 Introduction

An even dimensional differentiable manifold $M^n$ is said to be a nearly Kaehler manifold, if a complex structure $F$ of type (1, 1) and a pseudo-Riemannian metric $g$ of the manifold satisfy

$$F^2 = -I, \tag{1.1}$$

$$g(FX, FY) = g(X, Y), \tag{1.2}$$

and

$$(\nabla_X F)Y + (\nabla_Y F)X = 0, \tag{1.3}$$

where $X, Y \in \chi(M)$ and $\chi(M)$ is Lie algebra of vector fields on the manifold.
A Riemannian manifold or pseudo-Riemannian manifold $(M^n, g)(n \geq 2)$ in which Ricci tensor be scalar multiple of Riemannian metric i.e.

$$S(X, Y) = \alpha g(X, Y), \tag{1.4}$$

is called Einsntein manifold. The concept of Einstein manifold was given by Albert Einstein in differential geometry and mathematical physics. In equation (1.4), $S$ represents the Ricci tensor of the manifold and $\alpha$ is a non-zero scalar. The concept of Einstein manifold plays very important role in the study of Riemannian geometry and general theory of relativity.
contracting equation (1.4), we get the expression for scalar curvature, which is given by

$$r = n\alpha, \tag{1.5}$$

where $r$ denotes scalar curvature and $n$ is dimension of the manifold. We note that in case of $n \geq 2$, $\alpha$ is constant and hence $r$ becomes constant.
A non-flat Riemannian manifold is said to be quasi-Einstein manifold if the non-zero Ricci tensor satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1.6}$$

where $\alpha$, $\beta$ are scalars such that $\beta \neq 0$ and $A(X)$ is non-zero 1-form associated with a unit vector field $\rho$ defined by $g(X, \rho) = A(X)$, for every vector field $X$, $\rho$ is called the generator of the manifold. The concept of quasi-Einstein manifold is given by M. C. Chaki and R. K. Maity [5] in 2000. An $n$-dimensional quasi-Einstein manifold is denoted by $(QE)_n$. Taking contraction of equation (1.6), we get the expression for scalar curvature in quasi-Einstein manifold, which is given by

$$r = n\alpha + \beta. \tag{1.7}$$

With the help of equations (1.2) and (1.6), we get

$$\begin{cases} S(X,\rho) = (\alpha+\beta)A(X), \\ S(\rho,\,\rho) = (\alpha+\beta), \\ g(F\rho,\,\rho) = 0 \ \text{ and} \\ S(F\rho,\,\rho) = 0. \end{cases} \tag{1.8}$$

The Walker-space time is an example of a quasi-Einstein manifold. The concept of quasi-Einstein manifold came out during the study of exact solutions of Einstein fields equations as well as considerations of a quasi-umblical hypersurfaces of semi-Euclidean space. Also a quasi-Einstein manifolds can be taken as a model of the perfect fluid space time in general theory of relativity [12]. Several authors [2], [14], [17], [18] studied quasi-Einstein manifolds in different ways.

The notion of generalised quasi-Einstein manifold was introduced and studied by M. C. Chaki [6] in 2001. Later on in 2004 U. C. De and G. C. Ghosh [19] cited an example of a generalised quasi-Einstein manifold and studied its geometrical properties.

If a non-zero Ricci tensor of type (0,2) of a Riemannian manifold $(M^n, g)(n \geq 2)$ satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma C(X)C(Y), \tag{1.9}$$

then manifold is called generalised quasi-Einstein manifold.

where $\alpha,\ \beta$ and $\gamma$ are scalars such that $\beta \neq 0$, $\gamma \neq 0$ and $A$, $C$ are non-vanishing 1-forms associated with two orthogonal unit vectors $\rho$ and $\mu$ by

$$\begin{cases} g(X,\rho) = A(X), \\ g(X,\mu) = C(X), \\ g(\rho,\,\rho) = g(\mu,\,\mu) = 1. \end{cases} \tag{1.10}$$

An $n$-dimensional generalised quasi-Einstein manifold is denoted by $G(QE)_n$.

After contraction of equation (1.9), we get

$$r = \alpha n + \beta + \gamma. \tag{1.11}$$

From the equations (1.2), (1.9) and (1.10), we can easily write

$$\begin{cases} S(X,\,\rho) = (\alpha+\beta)A(X), \\ S(X,\,\mu) = (\alpha+\gamma)C(X), \\ S(\mu,\,\mu) = \alpha+\gamma, \\ S(\rho,\,\rho) = \alpha+\beta, \\ g(F\rho,\,\rho) = g(F\mu,\mu) = 0 \ \text{ and} \\ S(F\mu,\,\mu) = S(F\rho,\,\rho) = 0. \end{cases} \tag{1.12}$$

The several authors generalised the concept of quasi-Einstein manifolds. In 2009, C. $\ddot{O}$zg$\ddot{u}$r [15] generalised the concept of quasi-Einstein manifold. N(k)-quasi-Einstein manifolds were developed by [3, 7, 8, 16]. In 2004, M. C. Chaki [20] developed the concept of super quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be super quasi-Einstein manifolds if its Ricci tensor $S$ of type (0,2) satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)C(Y) + C(X)A(Y)] + \delta D(X,Y), \tag{1.13}$$

where $\alpha,\ \beta$, $\gamma$ and $\delta$ are non-zero scalars and $A$, $C$ are non-vanishing 1-forms defined as (1.10) and $\rho$, $\mu$ are orthogonal unit vector fields, $D$ is a non-zero symmetric tensor of type (0, 2) which satisfies the condition

$$D(X,\rho) = 0,\ \forall X. \tag{1.14}$$

also $\alpha$, $\beta$, $\gamma$ and $\delta$ are called the associated scalars. $A$ and $C$ are associated 1-forms of the manifold and $D$ is called the structure tensor of the manifold.
After contraction of equation (1.13), we get

$$r = \alpha n + \beta + \gamma + \delta D(e_i, e_i). \tag{1.15}$$

From the equations (1.2), (1.10), (1.14) and (1.13) we can easily write

$$\begin{cases} S(X,\, \rho) = (\alpha + \beta)A(X) + \gamma C(X), \\ S(X,\, \mu) = \alpha C(X) + \gamma A(X), \\ S(\mu,\, \mu) = \alpha + \delta D(\mu, \mu), \\ S(\rho,\, \rho) = \alpha + \beta + \delta D(\rho, \rho), \\ g(F\rho,\, \rho) = g(F\mu, \mu) = 0, \\ S(F\mu,\, \mu) = \gamma A(F\mu) + \delta D(F\mu, \mu) \ \text{ and} \\ S(F\rho,\, \rho) = \gamma C(F\rho) + \delta D(F\rho, \rho). \end{cases} \tag{1.16}$$

In 2009, A. A. Shaikh [1] introduced the notion of pseudo quasi-Einstein manifold. A semi-Riemannian manifold $(M^n, g)(n \geq 2)$ is called a pseudo quasi-Einstein manifold if a non-zero Ricci tensor $S$ of type (0,2) satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \delta D(X, Y), \tag{1.17}$$

where $\alpha,\ \beta$, and $\delta$ are non-zero scalars and $A$ is non-vanishing 1-form defined as (1.10) and $D$ is a non-zero symmetric tensor with vanishing trace and satisfying $D(X, \rho) = 0$ for all vector fields $X$.
From the equations (1.2), (1.10), (1.14) and (1.17), we can easily write

$$\begin{cases} S(X,\, \rho) = (\alpha + \beta)A(X), \\ S(\rho,\, \rho) = \alpha + \beta, \\ g(F\rho,\, \rho) = 0 \ \text{ and} \\ S(F\rho,\, \rho) = \delta D(\rho, \rho). \end{cases} \tag{1.18}$$

In 2012, S. K. Hui and R. S. Lemence [10] studied generalised quasi-Einstein manifold and addmitting a $W_2$-curvature tensor and they have been shown that if a $W_2$-curvature tensor satisfies $W_2 \cdot S = 0$ then either the associated scalars $\beta$ and $\gamma$ are equal or the curvature tensor $R$ satisfies a definite condition. Some results on generalised quasi-Einstein manifolds was studied by D. G. Prakasha and H. Venkatesha [4]. During their studied they have proved that if a conharmonic curvature tensor $L \cdot S = 0$ in generalised quasi-Einstein manifold, then either $M$ is a nearly quasi-Einstein manifold $N(QE)_n$ or the curvature tensor $R$ satisfies a definite condition.
We have gone through all developments in quasi-Einstein manifold $(QE)_n$, generalised quasi-Einstein manifold $G(QE)_n$, super quasi-Einstein manifold and plan to study projective Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold.

## 2 Projective Ricci Pseudo-Symmetric Nearly Kaehler Manifold

A Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where $R$ is a Riemannian curvature tensor of $(M^n, g)$ and $\nabla$ is a Levi-Civita connection on $(M^n, g)$. Different differential geometers [30, 31] studied the concept of locally symmetric properties of the manifold through different aproaches. In 1982, Z. I. $Szab\grave{o}$ studied structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. In 1950, A. G. Walker [9] studied Ruse‘s spaces of recurrent curvature and a Riemannian space with recurrent conformal curvature tensor is studied by A. Adati and T. Miyazawa [11].
According to Z. I. $Szab\grave{o}$ [13], if the manifold $M$ satisfies the condition

$$(R(X, Y) \cdot R)(U, V)W = 0, \quad X, Y, U, V, W \in \chi(M) \tag{2.1}$$

for all vector fields $X$ and $Y$, then the manifold is called a semi-symmetric manifold.
For a $(0,k)$-tensor field $T$ on $M$, $k \geq 1$ and a symmetric (0,2)-tensor field $A$ on $M$, the $(0,k+2)$-tensor fields $R \cdot T$ and $Q(A,T)$ are defined by [28], [29]

$$\begin{aligned}(R \cdot T)(X_1, .....X_k; X, Y) = &-T(R(X,Y)X_1, X_2, .......X_k)\\ &- ....... - T(X_1, ..........X_{k-1}, R(X,Y)X_k),\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}Q(A,T)(X_1, .....X_k; X, Y) = &-T((X \wedge_A Y)X_1, X_2, .......X_k)\\ &- ....... - T(X_1, ..........X_{k-1}, (X \wedge_A Y)X_k),\end{aligned} \tag{2.3}$$

where $X \wedge_A Y$ is the endomorphism given by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{2.4}$$

**Definition 2.1.** ([27]) An $n$-dimensional Riemannian manifold $(M^n, g)$ is said to be Ricci pseudo-symmetric if and only if tensor $R \cdot S$ and $Q(g,S)$ are linearly dependent, i.e.

$$(R(X, Y) \cdot S)(Z,U) = L_S Q(g,S)(Z,U; X,Y), \tag{2.5}$$

holds on $U_S$ where $U_S = \left\{x \in M : S \neq \frac{r}{n} g \ at \ x\right\}$ and $L_S$ is a certain function on $U_S$.

In the continuation of above studied B. B. Chaturvedi and B. K. Gupta studied Bochner Ricci semi-symmetric Hermitian manifold [21], Ricci pseudo-symmetric mixed generalised quasi-Einstein Hermitian manifolds [22], Ricci pseudo-symmetric super quasi-Einstein Hermitian manifold [23], Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold [24] and Bochner Ricci pseudo-symmetric Hermitian manifolds [25].
The projective curvature tensor $W$ in nearly Kaehler manifold [26] is defined by

$$\begin{aligned}W(\bar{X}, \bar{Y}, Z, U) = &R(X,Y,Z,U) + g((\nabla_X F)Y, (\nabla_Z F)U)\\ &+ \frac{1}{(n-1)}\Big\{ 'F(X,U)S(Y,Z) - 'F(Y,U)S(X,Z)\Big\}.\end{aligned} \tag{2.6}$$

Where $g$ is the positive definite metric of the manifold, $S$ is a Ricci tensor of type (0,2) and $'F(X,Y) = g(FX,Y)$.
In a nearly Kaehler manifold, a projective curvature tensor satisfies the condition

$$W(\bar{X}, \bar{Y}, Z, U) = W(X, Y, \bar{Z}, \bar{U}). \tag{2.7}$$

## 3 Projective Ricci Pseudo-Symmetric Super Quasi-Einstein Nearly Kaehler Manifold

In this section, we will define and study projective Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold.
Now, we introduce the following:

**Definition 3.1.** A nearly kaehler manifold is said to be a super quasi-Einstein nearly Kaehler manifold if its Ricci tensor of type (0,2) satisfies the equation (1.13). Throughout this paper, we denote the super quasi-Einstein nearly Kaehler manifold by $S(QENK)_n$.

**Definition 3.2.** A super quasi-Einstein nearly Kaehler manifold is said to be a projective Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold if and only if tensor $W \cdot S$ and $Q(g,S)$ are linearly dependent i.e.

$$(W(X, Y) \cdot S)(Z,U) = L_S Q(g,S)(Z,U; X,Y), \tag{3.1}$$

holds on $U_S$ where $U_S = \left\{x \in M : S \neq \frac{r}{n} g \ at \ x\right\}$ and $L_S$ is a certain function on $U_S$.

If we take projective Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold, then from equation (3.1) and (1.13), we have

$$\begin{aligned} &S(W(X,Y)Z,U)+S(Z,W(X,Y)U)\\ &=L_S[g(Y,Z)S(X,U)-g(X,Z)S(Y,U)+g(Y,U)S(X,Z)-g(X,U)S(Y,Z)]. \end{aligned} \tag{3.2}$$

Using equation (1.13) in equation (3.2), we get

$$\begin{aligned} &\alpha[W(X,Y,Z,U)+W(X,Y,U,Z)]+\beta[A(W(X,Y)Z)A(U)\\ &+A(Z)A(W(X,Y)U)]+\gamma[A(W(X,Y)Z)C(U)+A(U)C(W(X,Y)Z)\\ &+A(Z)C(W(X,Y)U)+A(W(X,Y)U)C(Z)]+\delta[D(W(X,Y)Z,U)\\ &+D(Z,W(X,Y)U)]=L_S\{\beta[g(Y,Z)A(X)A(U)-g(X,Z)A(Y)A(U)\\ &+g(Y,U)A(X)A(Z)-g(X,U)A(Y)A(Z)]+\gamma[g(Y,Z)A(X)C(U)\\ &+g(Y,Z)A(U)C(X)-g(X,Z)A(Y)C(U)-g(X,Z)A(U)C(Y)\\ &+g(Y,U)A(X)C(Z)+g(Y,U)A(Z)C(X)-g(X,U)A(Y)C(Z)\\ &-g(X,U)A(Z)C(Y)]+\delta[g(Y,Z)D(X,U)-g(X,Z)D(Y,U)\\ &+g(Y,U)D(X,Z)-g(X,U)D(Y,Z)]\}, \end{aligned} \tag{3.3}$$

where $g(W(X,Y)U,Z)=W(X,Y,U,Z)$.
Replacing $\bar{X}$, $\bar{Y}$ in place of $X$, $Y$ respectively in equation (3.3), we get

$$\begin{aligned} &\alpha[W(\bar{X},\bar{Y},Z,U)+W(\bar{X},\bar{Y},U,Z)]+\beta[A(W(\bar{X},\bar{Y})Z)A(U)\\ &+A(Z)A(W(\bar{X},\bar{Y})U)]+\gamma[A(W(\bar{X},\bar{Y})Z)C(U)+A(U)C(W(\bar{X},\bar{Y})Z)\\ &+A(Z)C(W(\bar{X},\bar{Y})U)+A(W(\bar{X},\bar{Y})U)C(Z)]+\delta[D(W(\bar{X},\bar{Y})Z,U)\\ &+D(Z,W(\bar{X},\bar{Y})U)]=L_S\{\beta[g(\bar{Y},Z)A(\bar{X})A(U)-g(\bar{X},Z)A(\bar{Y})A(U)\\ &+g(\bar{Y},U)A(\bar{X})A(Z)-g(\bar{X},U)A(\bar{Y})A(Z)]+\gamma[g(\bar{Y},Z)A(\bar{X})C(U)\\ &+g(\bar{Y},Z)A(U)C(\bar{X})-g(\bar{X},Z)A(\bar{Y})C(U)-g(\bar{X},Z)A(U)C(\bar{Y})\\ &+g(\bar{Y},U)A(\bar{X})C(Z)+g(\bar{Y},U)A(Z)C(\bar{X})-g(\bar{X},U)A(\bar{Y})C(Z)\\ &-g(\bar{X},U)A(Z)C(\bar{Y})]+\delta[g(\bar{Y},Z)D(\bar{X},U)-g(\bar{X},Z)D(\bar{Y},U)\\ &+g(\bar{Y},U)D(\bar{X},Z)-g(\bar{X},U)D(\bar{Y},Z)]\}. \end{aligned} \tag{3.4}$$

Putting $\bar{Z}$, $\bar{U}$ in place of $Z$, $U$ respectively, in equation (3.3), we obtain

$$\begin{aligned} &\alpha[W(X,Y,\bar{Z},\bar{U})+W(X,Y,\bar{U},\bar{Z})]+\beta[A(W(X,Y)\bar{Z})A(\bar{U})\\ &+A(\bar{Z})A(W(X,Y)\bar{U})]+\gamma[A(W(X,Y)\bar{Z})C(\bar{U})+A(\bar{U})C(W(X,Y)\bar{Z})\\ &+A(\bar{Z})C(W(X,Y)\bar{U})+A(W(X,Y)\bar{U})C(\bar{Z})]+\delta[D(W(X,Y)\bar{Z},\bar{U})\\ &+D(\bar{Z},W(X,Y)\bar{U})]=L_S\{\beta[g(Y,\bar{Z})A(X)A(\bar{U})-g(X,\bar{Z})A(Y)A(\bar{U})\\ &+g(Y,\bar{U})A(X)A(\bar{Z})-g(X,\bar{U})A(Y)A(\bar{Z})]+\gamma[g(Y,\bar{Z})A(X)C(\bar{U})\\ &+g(Y,\bar{Z})A(\bar{U})C(X)-g(X,\bar{Z})A(Y)C(\bar{U})-g(X,\bar{Z})A(\bar{U})C(Y)\\ &+g(Y,\bar{U})A(X)C(\bar{Z})+g(Y,\bar{U})A(\bar{Z})C(X)-g(X,\bar{U})A(Y)C(\bar{Z})\\ &-g(X,\bar{U})A(\bar{Z})C(Y)]+\delta[g(Y,\bar{Z})D(X,\bar{U})-g(X,\bar{Z})D(Y,\bar{U})\\ &+g(Y,\bar{U})D(X,\bar{Z})-g(X,\bar{U})D(Y,\bar{Z})]\}. \end{aligned} \tag{3.5}$$

Subtracting equation (3.4) from (3.5) and using (2.7), we have

$$\begin{aligned}
&\beta[A(W(\bar{X},\bar{Y})Z)A(U) + A(Z)A(W(\bar{X},\bar{Y})U) - A(W(X,Y)\bar{Z})A(\bar{U})\\
&- A(\bar{Z})A(W(X,Y)\bar{U})] + \gamma[A(W(\bar{X},\bar{Y})Z)C(U) + A(U)C(W(\bar{X},\bar{Y})Z)\\
&+ A(Z)C(W(\bar{X},\bar{Y})U) + A(W(\bar{X},\bar{Y})U)C(Z) - A(W(X,Y)\bar{Z})C(\bar{U})\\
&- A(\bar{U})C(W(X,Y)\bar{Z}) - A(\bar{Z})C(W(X,Y)\bar{U}) - A(W(X,Y)\bar{U})C(\bar{Z})]\\
&+ \delta[D(W(\bar{X},\bar{Y})Z,U) + D(Z,W(\bar{X},\bar{Y})U) - D(W(X,Y)\bar{Z},\bar{U})\\
&- D(\bar{Z},W(X,Y)\bar{U})] = L_S\{\beta[g(\bar{Y},Z)A(\bar{X})A(U) - g(\bar{X},Z)A(\bar{Y})A(U)\\
&+ g(\bar{Y},U)A(\bar{X})A(Z) - g(\bar{X},U)A(\bar{Y})A(Z)] - g(Y,\bar{Z})A(X)A(\bar{U})\\
&+ g(X,\bar{Z})A(Y)A(\bar{U}) - g(Y,\bar{U})A(X)A(\bar{Z}) + g(X,\bar{U})A(Y)A(\bar{Z})]\\
&+ \gamma[g(\bar{Y},Z)A(\bar{X})C(U) + g(\bar{Y},Z)A(U)C(\bar{X}) - g(\bar{X},Z)A(\bar{Y})C(U)\\
&- g(\bar{X},Z)A(U)C(\bar{Y}) + g(\bar{Y},U)A(\bar{X})C(Z) + g(\bar{Y},U)A(Z)C(\bar{X})\\
&- g(\bar{X},U)A(\bar{Y})C(Z) - g(\bar{X},U)A(Z)C(\bar{Y}) - g(Y,\bar{Z})A(X)C(\bar{U})\\
&- g(Y,\bar{Z})A(\bar{U})C(X) + g(X,\bar{Z})A(Y)C(\bar{U}) + g(X,\bar{Z})A(\bar{U})C(Y)\\
&- g(Y,\bar{U})A(X)C(\bar{Z}) - g(Y,\bar{U})A(\bar{Z})C(X) + g(X,\bar{U})A(Y)C(\bar{Z})\\
&+ g(X,\bar{U})A(\bar{Z})C(Y)] + \delta[g(\bar{Y},Z)D(\bar{X},U) - g(\bar{X},Z)D(\bar{Y},U)\\
&+ g(\bar{Y},U)D(\bar{X},Z) - g(\bar{X},U)D(\bar{Y},Z) - g(Y,\bar{Z})D(X,\bar{U})\\
&+ g(X,\bar{Z})D(Y,\bar{U}) - g(Y,\bar{U})D(X,\bar{Z}) + g(X,\bar{U})D(Y,\bar{Z})]\}.
\end{aligned} \tag{3.6}$$

Putting $Z = U = \rho$ in equation (3.6) and using (1.10), (1.14) and (1.16), we can write

$$2\{\beta W(\bar{X},\bar{Y},\rho,\rho) + \gamma W(\bar{X},\bar{Y},\rho,\mu) - \gamma L_S[C(\bar{X})A(\bar{Y}) - C(\bar{Y})A(\bar{X})]\} = 0. \tag{3.7}$$

Taking $\bar{X}$, $\bar{Y}$ in place of $X$ and $Y$ respectively, we have

$$\beta W(X,Y,\rho,\rho) + \gamma\{W(X,Y,\rho,\mu) - L_S[C(X)A(Y) - C(Y)A(X)]\} = 0. \tag{3.8}$$

If we take $\beta = 0$ in equation (3.8), we get

$$\begin{cases} \text{either } \gamma = 0 \text{ or} \\ W(X,Y,\rho,\mu) = L_S[C(X)A(Y) - C(Y)A(X)]. \end{cases} \tag{3.9}$$

Now, if $\gamma = 0$ then from equation (1.13), we have

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \delta D(X,Y). \tag{3.10}$$

Therefore the manifold become a pseudo quasi-Einstein manifold.
Thus we conclude:

**Theorem 3.3.** *In a projective Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold if $\beta = 0$, then the manifold is either a projective Ricci pseudo-symmetric pseudo quasi-Einstein nearly Kaehler manifold or $W(X,Y,\rho,\mu) = L_S[C(X)A(Y) - C(Y)A(X)]$.*

From equation (3.8), we can also conclude:

**Corollary 3.4.** *In a projective Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold if $\beta = 0$ and $\gamma \neq 0$ then $W(X,Y,\rho,\mu) = 0$ iff vector fields $\rho$ and $\mu$ corresponding to 1-forms $A$ and $C$ respectively are codirectional.*

## 4 Projectively flat Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold with $(W(X,\ Y)\cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y)$

If we take a projectively flat then from equation (2.6), we have

$$R(X,Y,Z,U) = -g((\nabla_X F)Y, (\nabla_Z F)U) - \frac{1}{(n-1)}\Big\{ {}'F(X,U)S(Y,Z) - {}'F(Y,U)S(X,Z)\Big\}. \tag{4.1}$$

In nearly Kaehler manifold the curvature tensor satisfied the following relation [26] $R(\bar{X},\bar{Y},Z,U) - R(X,Y,Z,U) = g((\nabla_X F)Y, (\nabla_Z F)U)$ and ${}'F(X,Y) = g(\bar{X},Y)$, now using equation (2.5) and (4.1), we can write

$$- \frac{1}{(n-1)}\Big\{g(\bar{X},U)S(QY,Z) - g(\bar{Y},U)S(QX,Z)\Big\} = L_S[g(\bar{Y},Z)S(\bar{X},U) - g(\bar{X},Z)S(\bar{Y},U) + g(\bar{Y},U)S(\bar{X},Z) - g(\bar{X},U)S(\bar{Y},Z)]. \tag{4.2}$$

If we take $\lambda$ is an eigen value of $Q$ corresponding to eigen vector $X$, then $QX = \lambda X$ i.e. $S(X,Z) = \lambda g(X,Z)$ (where the manifold is not Einstein) and hence $S(QX,Z) = \lambda S(X,Z)$. Then from equation (4.2), we have

$$L_S[g(\bar{Y},Z)S(\bar{X},U) - g(\bar{X},Z)S(\bar{Y},U) + g(\bar{Y},U)S(\bar{X},Z) - g(\bar{X},U)S(\bar{Y},Z)] + \frac{\lambda}{(n-1)}\Big\{g(\bar{X},U)S(Y,Z) - g(\bar{Y},U)S(X,Z)\Big\} = 0. \tag{4.3}$$

Putting $Z = \rho$ and $U = \rho$ in equation (4.3), we have

$$\gamma\{L_S[A(\bar{Y})C(\bar{X}) - A(\bar{X})C(\bar{Y})] + \frac{\lambda}{2(n-1)}[A(\bar{X})C(Y) - A(\bar{Y})C(X)]\} + \frac{\lambda(\alpha+\beta)}{2(n-1)}[A(\bar{X})A(Y) - A(X)A(\bar{Y})] = 0. \tag{4.4}$$

From equation (4.4), let $\gamma = 0$ and $\lambda \neq 0$ then either $\alpha + \beta = 0$ or $A(\bar{X})A(Y) = A(X)A(\bar{Y})$ Thus we conclude:

**Theorem 4.1.** *In a projective flat Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold if $\gamma = 0$ and $\lambda$ be non-zero eigen value of the Ricci operator Q then either $\alpha + \beta = 0$ or $A(\bar{X})A(Y) = A(X)A(\bar{Y})$.*

Now if we take $\lambda = 0$ and $\gamma \neq 0$ then, equation (4.4) implies that $L_S[C(\bar{X})A(\bar{Y}) - C(\bar{Y})A(\bar{X})] = 0$.
Again we conclude:

**Theorem 4.2.** *In a projective flat Ricci pseudo-symmetric super quasi-Einstein nearly Kaehler manifold if $\lambda = 0$ is an eigen value of the Ricci operator Q and $\gamma \neq 0$ then $L_S[C(\bar{X})A(\bar{Y}) - C(\bar{Y})A(\bar{X})] = 0$.*

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