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1-ABSORBING PRIME AVOIDANCE THEOREM IN MULTIPLICATIVE HYPERRINGS

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Abstract Let R be a multiplicative hyperring. The main purpose of this paper is to state the 1-absorbing Prime Avoidance Theorem for multiplicative hyperrings. Some properties of 1-absorbing prime hyperideals in multiplicative hyperrings are studied. Also, 1-absorbing prime hyperideals of valuation hyperdomains, Prüfer hyperdomains and idealization of hypermodules are characterized.

1 Introduction

Throughout this paper R is a commutative multiplicative hyperring with scalar identity 1. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Hyperstructures have many applications to several sectors of both pure and applied mathematics (see [5, 7]). The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [12], at the 8th Congress of Scandinavian Mathematicians. Contrary to classical algebra, in hyperstructure theory, there are various kinds of hyperrings and studied by many authors. The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication is an operation [11]. One important class of hyperrings was introduced by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation, which is called multiplicative hyperrings [14]. Also, hypermodules over a hyperring is a generalization of the classical modules over a ring. In 2007, Badawi [3] introduced the concept of 2-absorbing ideals of commutative rings with identity, which is a generalization of prime ideals, and investigated some properties of them. After that in [1, 2, 4, 9, 10], the authors extened the notion of 2-absorbing ideals. In this paper, we introduce and study the concept of 1-absorbing prime hyperideals in a multiplicative hyperring which is also a generalization of prime hyperideals and obtain their basic properties. For example, we show that if R is a 1-absorbing prime hyperideal that is not a prime hyperideal, then R is a quasilocal hyperring. Second, we state the 1-Absorbing Prime Avoidance Theorem for 1-absorbing prime hyperideals in multiplicative hyperrings and get some results concerning it.

In the following, we give some definitions and results of hyperstructures which we need to develop our paper. We refer to [6, 7, 8] for these basic properties and information on hyperstructures.

Definition 1.1. [7] Let H be a non-empty set. By $P^*(H)$, we mean the set of all non-empty subsets of H. A hyperoperation on non-empty set H is a map $\circ : H \times H \to P^*(H)$. $(H, \circ)$ is called a *hypergroupoid*. A *hypergroup* is a hypergroupoid $(H, \circ)$ which satisfies the associative and the reproductive law, i.e.,
(1) $x \circ (y \circ z) = (x \circ y) \circ z,\ \forall x, y, z \in H$ (associative law),
(2) $x \circ H = H \circ x = H,\ \forall x \in H$ (reproductive law).

Let $A \subset H$. Then A is called a *subhypergroup* of H if $0 \in H$ and $(A, \circ)$ is itself a hypergroup.

Definition 1.2. [7] A triple $(R, +, \circ)$ is called a *multiplicative hyperring* if
(1) $(R, +)$ is an abelian group;

(2) $(R, \circ)$ is semihypergroup;
(3) For all $a, b, c \in R$, we have $a \circ (b + c) \subseteq a \circ b + a \circ c$ and $(b + c) \circ a \subseteq b \circ a + c \circ a$;
(4) For all $a, b \in R$, we have $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

If in (2) the equality holds, then we say that the multiplicative hyperring is *strongly distributive*. We assume throughout this paper that all multiplicative hyperrings are strongly distributive.
For any two non-empty subsets A and B of R and $x \in R$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b; \quad A \circ x = A \circ \{x\}$$

Also, $x^n = x \circ x \circ \cdots \circ x$ (n times).

Definition 1.3. (a) A non-zero element a of a multiplicative hyperring R is said to be *unit*, if $1 \in a \circ x$ and $1 \in x \circ a$ for some $x \in R$. The set of all unit elements of R is denoted by $U(R)$.
(b) A subset S of a multiplicative hyperring R is said to be a *subhyperring* of R if $(S, +, \circ)$ is itself a multiplicative hyperring.
(c) A commutative hyperring R with identity 1 is called *hyperdomain*, if for every $a, b \in R$, $0 \in a \circ b$, then $a = 0$ or $b = 0$ [13].
(d) A commutative hyperring R with identity 1 is called *hyperfield* if every non-zero element of R is unit.
(e) A non-empty subset I of a multiplicative hyperring R is a *hyperideal* of R if
(1) $a, b \in I$, then $a - b \in I$,
(2) $a \in I$ and $r \in R$, then $r \circ a \subseteq I$.
(f) A hyperideal I of a commutative multiplicative hyperring R with identity 1 is *finitely generated* if $I = \langle r_1, \ldots, r_n \rangle$ for some $r_1, \ldots, r_n \in R$, i.e., for any $x \in I$, there exist $x_1, \ldots, x_n \in R$ such that $x \in r_1 \circ x_1 + \cdots + r_n \circ x_n$.
(g) A hyperideal I of R is called *principal* if $I = \langle x \rangle$ for some $x \in R$. Also, R is called *principal hyperideal hyperdomain*, if every hyperideal of R is principal [6].
(h) Let R and S be hyperrings. A mapping $\phi : R \to S$ is said to be a *hyperring homomorphism*, if for all $a, b \in R$;
(1) $\phi(a + b) = \phi(a) + \phi(b)$.
(2) $\phi(a \circ b) = \phi(a) \circ \phi(b)$ ([7]).

Let I be a hyperideal of a multiplicative hyperring R and let $R/I = \{r + I \mid r \in R\}$. Define the operation $+$ and the hyperoperation $\circ$ on R/I by $(a + I) + (b + I) = a + b + I$ and $(a + I) \circ (b + I) = \cup\{c + I \mid c \in a \circ b\}$. Then $(R/I, +, \circ)$ is called a *quotient hyperring* [7]. Let I, J be two hyperideals of R. We define $(I :_R J) = \{a \in R \mid a \circ J \subseteq I\}$. It is clear that $(I :_R J)$ is a hyperideal of R. Let C be the class of all finite products of elements of R i.e., $C = \{r_1 \circ r_2 \circ \cdots \circ r_n \mid r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A hyperideal I of R is said to be a *C-hyperideal* of R, if whenever $A \cap I \neq \emptyset$ for any $A \in C$, then $A \subseteq I$ [6]. Prime and primary hyperideals in multiplicative hyperrings has been introduced and studied by U. Dasgupta in [6]. A proper hyperideal P of a multiplicative hyperring R is said to be *prime (primary)*, if $a \circ b \subseteq P$, where $a, b \in R$, then $a \in P$ or $b \in P$ ($a \in P$ or $b^n \subseteq P$ for some $n \in \mathbb{N}$). The intersection of all prime hyperideals of R containing I is called the *radical* of I and denoted by $rad(I)$. If the multiplicative hyperring R does not have any prime hyperideal containing I, we define $rad(I) = R$. We refer to the prime hyperideal $P = rad(Q)$ as the associated prime hyperideal of Q and on the other hand Q is referred to as a P-primary hyperideal of R. Let I be a hyperideal of a multiplicative hyperring R. Then $D(I) \subseteq rad(I)$ where $D(I) = \{r \in R \mid r^n \subseteq I \text{ for some } n \in \mathbb{N}\}$. The equality holds when I is a C-hyperideal of R [6, Proposition 3.2]. A proper hyperideal I of a hyperring R is said to be *maximal*, if $I \subseteq J \subseteq R$ for some hyperideal J of R, then $I = J$ or $J = R$. A hyperring R is called *quasilocal*, if it has a unique hyperideal M (see [6]). P. Ghiasvand in [10] has introduced and studied the concept of 2-absorbing hyperideals of a multiplicative hyperring as a generalization of prime hyperideals. Also, M. Anbarloei has studied 2-absorbing and 2-absorbing primary hyperideals of a multiplicative hyperring in [1]. It is clear that every prime hyperideal is a 2-absorbing hyperideal. But the converse is not true, in general. A proper hyperideal I of a multiplicative hyperring R is said to be a 2-*absorbing hyperideal* of R if $x \circ y \circ z \subseteq I$ for $x, y, z \in R$, then $x \circ y \subseteq I$ or $x \circ z \subseteq I$ or $y \circ z \subseteq I$.

2 Characterizations of 1-absorbing Prime Hyperideals

In this section, basic properties of 1-absorbing prime hyperideals are studied. Moreover, 1-absorbing prime hyperideals of valuation hyperdomains, Prüfer hyperdomains and idealization of hypermodules are characterized.

Definition 2.1. Let R be a multiplicative hyperring. A proper hyperideal I of R is called 1-absorbing prime if for all non-unit elements $a, b, c \in R$ such that $a \circ b \circ c \subseteq I$, then either $a \circ b \subseteq I$ or $c \in I$.

Remark 2.2. If $a \circ b \circ c \subseteq I$ for some $a, b, c \in R$ and a is unit, then we have $1 \in a \circ x$ for some $x \in R$. Thus $b \circ c \subseteq a \circ b \circ c \circ x \subseteq I$. Hence in the definition of 1-absorbing prime hyperideals we can assume that a, b, c are non-unit elements.

Every prime hyperideal is a 1-absorbing prime hyperideal and every 1-absorbing prime hyperideal is a 2-absorbing hyperideal.

Example 2.3. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers. We define the hyperoperation $a \circ b = \{2ab, 4ab\}$ for all $a, b \in \mathbb{Z}$. Then $R = (\mathbb{Z}, +, \circ)$ is a multiplicative hyperring. Consider the hyperideal $I = 15\mathbb{Z}$. Then I is a 2-absorbing hyperideal of R that is not prime.

Example 2.4. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers and $\mathbb{Z}[x]$ be the ring of polynomials in indeterminate x. Suppose that $R = \mathbb{Z} + 3x\mathbb{Z}[x]$. Define the hyperoperation $a \circ b = \{2ab, 4ab\}$ for all $a, b \in \mathbb{Z}$. It is easy to see that $P = 3x\mathbb{Z}[x]$ is a prime hyperideal of R, and so P^2 is a 2-absorbing hyperideal of R, by [1, **?**]. But P^2 is not 1-absorbing prime, since $3 \circ 3 \circ x^2 \subseteq P^2$, but $3 \circ 3 = \{18, 36\} \nsubseteq P^2$ and $x^2 \notin P^2$.

Theorem 2.5. *Let I be a 1-absorbing prime hyperideal of a multiplicative hyperring R. Then $D(I)$ is a prime hyperideal of R. Moreover, $(I : c) = \{x \in R \mid c \circ x \subseteq I\}$ is a prime hyperideal of R for every non-unit element $c \in R \setminus I$.*

Proof. Let I be a 1-absorbing prime hyperideal of R and let $x \circ y \subseteq D(I)$ for some $x, y \in R$. If x or y is unit, then there is nothing to prove, so assume that x, y are non-unit elements of R. Then there exists a positive integer n such that $(x \circ y)^n \subseteq I$, and hence $x^m \circ x^{n-m} \circ y^n \subseteq I$ for some positive integer $m < n$. Since I is a 1-absorbing prime hyperideal of R, we conclude that $x^n = x^m \circ x^{n-m} \subseteq I$ or $y^n \subseteq I$ and hence $x \in D(I)$ or $y \in D(I)$. Thus $D(I)$ is a prime hyperideal of R. Now suppose that $a \circ b \subseteq (I : c)$ for some elements $a, b \in R$ and non-unit element $c \in R \setminus I$ such that $a \notin (I : c)$. Assume that a, b are non-unit elements of R. Then $a \circ c \nsubseteq I$, so $b \in I \subseteq (I : c)$ since I is a 1-absorbing prime hyperideal of R and $a \circ b \circ c \subseteq I$. Thus $(I : c)$ is a prime hyperideal of R. □

Lemma 2.6. *Let R be a multiplicative hyperring. Suppose that for every non-unit element x of R and for every unit element u of R, we have $x + u$ is a unit element of R. Then R is a quasilocal hyperring.*

Proof. Suppose that R has at least two maximal hyperideals, say M_1, M_2. We have $M_1 \subset M_1 + M_2 \subseteq R$, hence $M_1 + M_2 = R$. Then $m_1 + m_2 = 1$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Thus $1 - m_1 = m_2$ is a unit element of R by hypothesis, which is impossible. Thus R is a quasilocal hyperring. □

Theorem 2.7. *Let R be a multiplicative hyperring and I be a hyperideal of R. If I is a 1-absorbing prime hyperideal of R that is not a prime hyperideal, then R is a quasilocal hyperring.*

Proof. If I is a 1-absorbing prime hyperideal that is not a prime hyperideal of R, then there are non-unit elements $a, b \in R$ such that $a \circ b \subseteq I$ and $a, b \notin I$. Suppose that x is a non-unit element and y is a unit element of R. We show that $x + y$ is a unit element of R, so the proof follows from Lemma 2.6. Suppose that $x + y$ is a non-unit element of R. Since I is a 1-absorbing prime hyperideal, $x \circ a \circ b \subseteq I$ and $b \notin I$, then $x \circ a \subseteq I$. But $(x + y) \circ a \circ b \subseteq I$, we have $(x + y) \circ a \subseteq I$ and since $x \circ a \subseteq I$ we conclude $y \circ a \subseteq I$, which follows that $a \in I$ because y is unit, which is a contradiction. Hence $x + y$ is a unit element and the proof is complete. □

Corollary 2.8. *Let $R = R_1 \times R_2$ be a decomposable hyperring where R_1 and R_2 are multiplicative hyperrings with identity 1 and J be a proper hyperideal of R. Then the following statements are equivalent:*

(i) J is a 1-absorbing prime hyperideal of R.

(ii) J is a prime hyperideal of R.

(iii) $J = I \times R_2$ or $J = R_1 \times K$, where I and K are prime hyperideals of R_1 and R_2, respectively.

Next, it is proved that a proper hyperideal I of R is 1-absorbing prime if and only if the inclusion $I_1I_2I_3 \subseteq I$ for some proper hyperideals I_1, I_2, I_3 of R implies that $I_1I_2 \subseteq I$ or $I_3 \subseteq I$. First, we need the following lemma.

Lemma 2.9. *Let I be a 1-absorbing prime hyperideal of a multiplicative hyperring R. If $a \circ b \circ J \subseteq I$ for proper hyperideal J of R and non-unit elements $a, b \in R$, then $a \circ b \subseteq I$ or $J \subseteq I$.*

Proof. Suppose that $a \circ b \circ J \subseteq I$ for some proper hyperideal J of R and non-unit elements $a, b \in R$ such that $a \circ b \nsubseteq I$ and $J \nsubseteq I$. Then there exists an element $c \in J \setminus I$. But $a \circ b \circ c \subseteq I$ and $a \circ b \nsubseteq I$ and $c \notin I$, which is a contradiction. □

Theorem 2.10. *Suppose that I is a proper hyperideal of a multiplicative hyperring R. Then the following statements are equivalent:*

(i) I is a 1-absorbing prime hyperideal of R.

(ii) If $I_1I_2I_3 \subseteq I$ for some proper hyperideals I_1, I_2, I_3 of R, then $I_1I_2 \subseteq I$ or $I_3 \subseteq I$.

Proof. $(i) \Rightarrow (ii)$ Suppose that I is a 1-absorbing prime hyperideal of R and $I_1I_2I_3 \subseteq I$ for some proper hyperideals I_1, I_2, I_3 of R such that $I_1I_2 \nsubseteq I$. Then there are non-unit elements $a \in I_1$ and $b \in I_2$ such that $a \circ b \nsubseteq I$. Since $a \circ b \circ I_3 \subseteq I$, $a \circ b \nsubseteq I$, it follows from Lemma 2.9 that $I_3 \subseteq I$.

$(ii) \Rightarrow (i)$ Suppose that $a \circ b \circ c \subseteq I$ for some non-unit elements $a, b, c \in R$ and $a \circ b \nsubseteq I$. Suppose also that $I_1 = aR$, $I_2 = bR$, and $I_3 = cR$. Then $I_1I_2I_3 \subseteq I$ and $I_1I_2 \nsubseteq I$. Hence $I_3 = cR \subseteq I$, thus $c \in I$. □

A hyperring R is said to be divided if for every prime hyperideal P of R, we have $P \subseteq Rx$ for every $x \in R \setminus P$. It is known that the prime hyperideals of a divided hyperring are linearly ordered; i.e., if P_1, P_2 are prime hyperideals of R, then $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ [13].

Lemma 2.11. *Let R be a divided hyperring and I be a C-hyperideal of R. If I is a 1-absorbing prime hyperideal of R with $rad(I) = P$, then I is a primary hyperideal of R such that $P^2 \subseteq I$.*

Proof. By Theorem 2.5, $D(I) = rad(I) = P$ is a prime hyperideal of R. Suppose that I is a 1-absorbing prime hyperideal of R. First we show that $P^2 \subseteq I$. Let $x, y \in P = rad(I)$. Then $x^{n-2} \circ x \circ x \subseteq I$ for some positive integer n, so $x^2 \subseteq I$, similarly, $y^2 \subseteq I$. Thus $x \circ (x+y) \circ y \subseteq I$. Since I is a 1-absorbing prime hyperideal of R, either $x \circ (x+y) = x^2 + xy \subseteq I$ or $y \in I$. Hence $x \circ y \subseteq I$ and thus $P^2 \subseteq I$. Now we show that I is a primary hyperideal of R. Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $y \notin P$. Then $x \in P$ and since P is a divided hyperideal of R, we conclude that $x \in y \circ w$ for some $w \in R$. Hence $x \circ y \subseteq y^2 \circ w$, so $\emptyset \neq x \circ y = x \circ y \cap I \subseteq y^2 \circ w \cap I$. Therefore $y^2 \circ w \subseteq I$ because I is a C- hyperideal of R. But $y^2 \nsubseteq I$ and I is a 1-absorbing prime hyperideal of R, so $w \in I$. Therefore $x \in I$ and thus I is a primary hyperideal of R such that $P^2 \subseteq I$. □

A valuation hyperring is a hyperdomain R with the property that if I and J are hyperideals of R then either $I \subseteq J$ or $J \subseteq I$.

Theorem 2.12. *Let P be a prime hyperideal of a valuation hyperring R. Then the following hold:*

(i) If Q is a P-primary C-hyperideal of R and $x \in R \setminus P$, then $Q = Q\langle x \rangle$.

(ii) The product of P-primary C-hyperideals of R is a P-primary hyperideal. If $P \neq P^2$, then the only P-primary hyperideals are powers of P.

Proof. (i) Since $x \notin P$, we have $Q \subset \langle x \rangle$ because R is a valuation hyperring. Let K be the quotient hyperfield of R and let $A = \{y \mid y \in K \; and \; y \circ x \subseteq Q\}$. Since $Q \subset \langle x \rangle$, A is a subset of R. Furthermore, it is easy to check that A is a hyperideal of R and $Q = A \langle x \rangle$. Moreover, since Q is P-primary and $\langle x \rangle \nsubseteq P$, we have $A \subseteq Q$. Thus $Q = A$ and $Q = Q \langle x \rangle$, as claimed.

(ii) Let Q_1, Q_2 be P-primary hyperideals of R. Clearly, $rad(Q_1 Q_2) = P$. Let x, y be elements of R with $x \circ y \subseteq Q_1 Q_2$ and $x \notin P$. By (i), $Q_1 = Q_1 \langle x \rangle$. Hence $x \circ y \subseteq \langle x \rangle Q_1 Q_2$. Since $x \circ y \neq \emptyset$, then there exist $z \in x \circ y$ and so $z \in \langle x \rangle Q_1 Q_2$. Hence $z \in x \circ q$ for some $q \in q_1 \circ q_2 \subseteq Q_1 Q_2$ ($q_1 \in Q_1$ and $q_2 \in Q_2$). Therefore $0 = z - z \in x \circ y - x \circ q = x \circ (y - q)$, so $y - q = 0$ because R is a hyperdomain, this implies that $y \in Q_1 Q_2$. Thus $Q_1 Q_2$ is P-primary. Now suppose that $P \neq P^2$ and let Q be a P-primary hyperideal of R. Hence Q contains a power of P^2 and so contains a power of P. Thus there is a positive integer m such that $P^m \subseteq Q$ but $P^{m-1} \nsubseteq Q$. Let $x \in P^{m-1}$ and $x \notin Q$, then $Q \subseteq \langle x \rangle$. If we define A as in the proof of (i), then $Q = A \langle x \rangle$. Since Q is P-primary and $x \notin Q$, $A \subseteq P$. Therefore, $Q = A \langle x \rangle \subseteq P \langle x \rangle \subseteq P^m$, so we conclude that $Q = P^m$. □

Theorem 2.13. *Let R be a valuation hyperdomain and I be a non-zero proper C-hyperideal of R such that $P = rad(I)$. If I is a 1-absorbing prime hyperideal of R, then $I = P$ or $I = P^2$ where $P = rad(I)$ is a prime hyperideal of R.*

Proof. Let R be a valuation hyperdomain and I be a non-zero proper C-hyperideal of R such that $P = rad(I)$. Since every valuation hyperdomain is a divided hyperdomain, it follows from Lemma 2.11 that I is a primary hyperideal of R such that $P^2 \subseteq I$. Since R is a valuation hyperdomain, we conclude that either $I = P$ or $I = P^2$ where $P = rad(I)$ is a prime hyperideal of R by Theorem 2.12. □

Let R be a hyperdomain with quotient hyperfield K. A proper hyperideal I of R is called invertible if $II^{-1} = R$, where $I^{-1} = \{r \in K : r \circ I \subseteq R\}$. A hyperdomain R is called a Prüfer hyperdomain if every non-zero finitely generated hyperideal of R is invertible. A hyperdomain is called a Dedekind hyperdomain if every nonzero proper hyperideal of R is invertible. In the following results, 1-absorbing prime hyperideals of Dedekind hyperdomains and Prüfer hyperdomains are completely described.

Lemma 2.14. *Let R be a Prüfer hyperdomain and Q be a P-primary C-hyperideal of R such that $P = rad(I)$. Then if $P \neq P^2$, then $Q = P^m$ for some positive integer m.*

Proof. The proof holds by Theorem 2.13. □

Theorem 2.15. *Let R be a Prüfer hyprdomain and I be a non-zero proper C-hyperideal of R such that $P = rad(I)$ where P is an invertible hyperideal. If I is a 1-absorbing prime hyperideal of R, then $I = P$ or $I = P^2$ where $P = rad(I)$ is a prime hyperideal of R.*

Proof. Suppose that R is a Prüfer hyperdomain and I is a non-zero proper hyperideal of R such that $P = rad(I)$. If R is quasilocal with maximal hyperideal M, then it is known that R is a valuation hyperdomain since R is a Prüfer hyperdomain, thus the claim follows from Theorem 2.13. So suppose that R is not a quasilocal hyperring. Then it follows from Theorem 2.7 that I is a prime hyperideal of R and hence I is a P-primary hyperideal of R such that $P^2 \subseteq I$. Thus $I = P$ or $I = P^2$ where $P = rad(I)$ is a prime hyperideal of R by Lemma 2.14. □

Theorem 2.16. *Let R be a Noetherian hyperdomain that is not a hyperfield and I be a hyperideal of R. Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.*

(i) R is a Dedekind hyperdomain;

(ii) If I is a 1-absorbing prime hyperideal of R, then $I = M$ or $I = M^2$ where M is a maximal hyperideal of R;

(iii) If I is a 1-absorbing prime hyperideal of R, then $I = P$ or $I = P^2$ where $P = rad(I)$ is a prime hyperideal of R.

Proof. $(i) \Rightarrow (ii)$ Suppose that R is a Noetherian hyperdomain that is not a hyperfield and I is a 1-absorbing prime hyperideal of R such that $P = rad(I)$. Since R is a Dedekind hyperdomain, we conclude that every non-zero prime hyperideal of R is a maximal hyperideal of R. Hence P is a maximal hyperideal of R. This means that I is a primary hyperideal of R such that $P^2 \subseteq I$. Therefore, by Theorem 2.15, $I = M$ or $I = M^2$ where M is a maximal hyperideal of R.

$(ii) \Rightarrow (iii)$ is obvious. □

In view of Theorem 2.16, we have the following result.

Corollary 2.17. *Let R be a principal hyperideal hyperdomain that is not a hyperfield and I be a non-zero proper hyperideal of R. If I is a 1-absorbing prime hyperideal of R, then $I = pR$ or $I = p^2R$ for some non-zero prime element p of R.*

Theorem 2.18. *Let I be a P-primary hyperideal of a hyperring R. If $(P^2 : x) \subseteq I$ for every $x \in P \setminus I$, then I is a 1-absorbing prime hyperideal of R.*

Proof. Let I be a P-primary hyperideal of a hyperring R. Suppose that $(P^2 : x) \subseteq I$ for every $x \in P \setminus I$ and $x \circ y \circ z \subseteq I$ for some non-unit elements $x, y, z \in R$. Assume that $x \circ y \nsubseteq I$ and $z \notin I$. Since I is a P-primary hyperideal of R, we conclude that $z \in P \setminus I$, so $x \circ y \subseteq (P^2 : z) \subseteq I$, which is a contradiction. Hence either $x \circ y \subseteq I$ or $z \in I$, so I is a 1-absorbing prime hyperideal of R. □

Theorem 2.19. *Let R and S be multiplicative hyperrings and $f : R \to S$ be a hyperring homomorphism such that $f(1) = 1$ and $f(a)$ is non-unit in S for every non-unit element a in R. Then the following statements hold:*

(i) If J is a 1-absorbing prime hyperideal of S, then $f^{-1}(J)$ is a 1-absorbing prime hyperideal of S.

(ii) If f is onto and I is a 1-absorbing prime hyperideal of R with $Ker(f) \subseteq I$, then $f(I)$ is a 1-absorbing prime hyperideal of S.

Proof. (i) Suppose that J is a 1-absorbing prime hyperideal of S and $a \circ b \circ c \subseteq f^{-1}(J)$ for some non-unit elements $a, b, c \in R$. Then $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c) \subseteq J$, which means that $f(a) \circ f(b) \subseteq J$ or $f(c) \in J$. It follows $a \circ b \subseteq f^{-1}(J)$ or $c \in f^{-1}(J)$. Hence $f^{-1}(J)$ is a 1-absorbing prime hyperideal of R.

(ii) Suppose that f is onto and I is a 1-absorbing prime hyperideal of R with $Ker(f) \subseteq I$ and $x \circ y \circ z \subseteq f(I)$ for some non-unit elements $x, y, z \in S$. Since f is onto, there exist non-unit elements $a, b, c \in R$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$. Therefore $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c) = x \circ y \circ z \subseteq f(I)$. Since $Ker(f) \subseteq I$, we conclude that $a \circ b \circ c \subseteq I$. Thus $a \circ b \subseteq I$ or $c \in I$, so $x \circ y \subseteq f(I)$ or $z \in f(I)$. Hence $f(I)$ is a 1-absorbing prime hyperideal of S. □

Corollary 2.20. *Let I and J be proper hyperideals of a multiplicative hyperring R with $I \subseteq J$ and $U(R/I) = \{a + I \mid a \in U(R)\}$. Then J is a 1-absorbing prime hyperideal of R if and only if J/I is a 1-absorbing prime hyperideal of R/I.*

Proof. Suppose that I and J are proper hyperideals of R with $I \subseteq J$ and let $f : R \to R/I$ such that $f(a) = a + I$. Then f is a hyperring homomorphism from R onto R/I, $f(1) = 1$ and $f(a) \in R/I$ is non-unit for every non-unit a in R. But $Ker(f) = I \subseteq J$ and f is onto, hence $f(J) = J/I$ is a 1-absorbing prime hyperideal of R/I by Theorem 2.19 (ii). Assume that J/I is a 1-absorbing prime hyperideal of R/I. Then $f^{-1}(J/I) = J$ is a 1-absorbing prime hyperideal of R by Theorem 2.19 (i). □

Let $(R, +, \circ)$ be a multiplicative hyperring with identity 1. An *R-(left) hypermodule* M is an abelian group $(M, +)$ together with a map $\cdot : R \times M \longrightarrow M$ defined by

$$(a, m) \mapsto a \cdot m = am \in M$$

such that for all $r_1, r_2 \in R$ and $m_1, m_2, m \in M$ we have

(1) $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_2 \cdot m_2$;

(2) $(r_1 + r_2) \cdot m = (r_1 \cdot m) + (r_2 \cdot m)$;
(3) $(r_1 \circ r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$;
(4) $1m = m$;
(5) $r0_M = 0_R m = 0_M$.

A non-empty subset N of an R-hypermodule M is called a *subhypermodule* if N is an R-hypermodule with the operations of M [15].

Let R be a commutative multiplicative hyperring with identity 1 and M be an R-hypermodule. Let $R(+)M = \{(r, m) : r \in R, m \in M\}$. Consider addition and multiplication as follows: For each $a, b \in R$ and $m, n \in M$; $(a, n) + (b, m) = (a + b, m + n)$ and $(a, m) \circ (b, n) = \{(c, an + bm) \mid c \in a \circ b\}$. It is easy to see that $R(+)M$ is a commutative multiplicative hyperring with identity $(1, 0)$ and we call it the idealization of M . Suppose that I is a hyperideal of R and N is a subhypermodule of M. Then $I(+)N$ is a hyperideal of $R(+)M$ if and only if $IM \subseteq N$. In this case, $I(+)N$ is called a homogeneous hyperideal of $R(+)M$.

Theorem 2.21. *Let M be an R-hypermodule and $I(+)N$ be a homogeneous hyperideal of the hyperring $R(+)M$. If $I(+)N$ is a 1-absorbing prime hyperideal of $R(+)M$, then I is a 1-absorbing prime hyperideal of R.*

Proof. Let $a \circ b \circ c \subseteq I$ for some non-unit elements $a, b, c \in R$. Then $(a, 0) \circ (b, 0) \circ (c, 0) = \{(t, 0) \mid t \in a \circ b \circ c\} \subseteq I(+)N$, and so $a \circ b \circ c \subseteq I$. Since $I(+)N$ is a 1-absorbing prime hyperideal of $R(+)M$, either $(a, 0) \circ (b, 0) \subseteq I(+)N$ or $(c, 0) \in I(+)N$. Hence either $a \circ b \subseteq I$ or $c \in I$ and thus I is a 1-absorbing prime hyperideal of R. □

3 1-Absorbing Prime Avoidance Theorem

In this section, we state the 1-Absorbing Prime Avoidance Theorem for 1-absorbing prime hyperideals of R.

Let $I, I_1, I_2, \ldots, I_n$ be hyperideals of R. A covering $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ is said to be efficient precisely when I is not contained in the union of any $n-1$ of the hyperideals $I_1, I_2, \ldots, I_n$. We shall say that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ is an efficient union if none of the I_k, $1 \leq k \leq n$, may be excluded.

Lemma 3.1. *Let $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ ($n \geq 2$) be an efficient covering. If $I_i \nsubseteq (I_j :_R x)$ for every $x \in R \setminus I_j$ and $i \neq j$, then no I_j is 1-absorbing prime for every $j \in \{1, \ldots, n\}$.*

Proof. Suppose to the contrary, I_j is a 1-absorbing prime hyperideal of R for some $j \in \{1, \ldots, n\}$. It is easy to see that $I = (I \cap I_1) \cup (I \cap I_2) \cup \cdots \cup (I \cap I_n)$ is an efficient union. So there exists an element $x_j \in I \setminus I_j$ for every $j \in \{1, \ldots, n\}$. Since $I = (I \cap I_1) \cup (I \cap I_2) \cup \cdots \cup (I \cap I_n)$ is an efficient union, we conclude that $(\cap_{i \neq j} I_i) \cap I \subseteq I_j \cap I$. By hypothesis, $I_i \nsubseteq (I_j : x_j)$ such that $x_j \in I \setminus I_j$ and $i \neq j$. Hence there exists $y_i \in I_i \setminus (I_j : x_j)$ for every $i \neq j$. Let $Y = y_1 \circ \cdots y_{j-1} \circ y_{j+1} \circ \cdots \circ y_n$. Then $Y \circ x_j \subseteq \cap_{i \neq j}(I_i \bigcap I)$ but $Y \circ x_j \nsubseteq I_j \cap I$. Since for otherwise, assume that $Y \circ x_j \subseteq I_j \cap I$. Since I_j is 1-absorbing prime and $(I_j : x_j)$ is a prime hyperideal of R by Theorem 2.5, we have $y_i \in (I_j : x_j)$ for some $i \neq j$, which is impossible. Therefore, $Y \circ x_j \nsubseteq I \cap I_j$ and this contradicts the fact that $(\bigcap_{i \neq j} I_i) \cap I \subseteq I_j \cap I$. The proof is complete. □

In following theorem we state 1-Absorbing Prime Avoidance Theorem in multiplicative hyperrings.

Theorem 3.2. *Let $I, I_1, I_2, \ldots, I_n$ be hyperideals of R and at most two of $I_1, I_2, \ldots I_n$ are not 1-absorbing prime. Suppose that I is a hyperideal of R such that $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ and $I_i \nsubseteq (I_j : x)$ for every $x \in R \setminus I_j$ and $i \neq j$. Then $I \subseteq I_j$ for some $j \in \{1, \ldots, n\}$.*

Proof. Let $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ be a covering such that at least $n-2$ of the hyperideals $I_1, I_2, \ldots I_n$ are 1-absorbing prime. Without loss of generality, one may reduce the covering to an efficient covering. If $n = 2$, then it is obvious. Suppose that $n > 2$. Since the covering is efficient and $I_i \nsubseteq (I_j : x)$ for every $x \in R \setminus I_j$ and $i \neq j$, by Lemma 3.1, $n < 2$. Hence $n = 1$

and $I \subseteq I_j$ for some $j \in \{1, \dots, n\}$. □

Corollary 3.3. *Let $I = \langle r_1, \dots, r_n \rangle$ be a finitely generated hyperideal of R for some $r_1, \dots, r_n \in R$. Let $I_1, I_2, \dots, I_n$ be 1-absorbing prime hyperideals of R, $I \nsubseteq I_i$ for every $i \in \{1, \dots, n\}$ and $I_i \nsubseteq (I_j : x)$ for every $x \in R \setminus I_j$ and $i \neq j$. Then there exist $b_2, \dots, b_s \in R$ such that $A = r_1 + b_2 \circ r_2 + \cdots + b_s \circ r_s \nsubseteq \bigcup_{i=1}^n I_i$.*

Proof. We prove the corollary by induction on n. If $n = 1$, then the result is clear. So suppose that $n \geq 1$ and the result has been proved for smaller values than n. Then there exist $a_2, \dots, a_s \in R$ such that $B = r_1 + a_2 \circ r_2 + \cdots + a_s \circ r_s \nsubseteq \bigcup_{i=1}^{n-1} I_i$. If $B \nsubseteq I_n$, so $B \nsubseteq \bigcup_{i=1}^n I_i$, because if $B \subseteq \bigcup_{i=1}^n I_i$, then $B \subseteq (\bigcup_{i=1}^{n-1} I_i) \bigcup I_n$, hence $B \subseteq \bigcup_{i=1}^{n-1} I_i$ or $B \subseteq I_n$, a contradiction and so there is nothing to prove. Hence suppose that $B \subseteq I_n$. If $r_2, \dots, r_s \in I_n$, then $r_1 \in I_n$, a contradiction, as $I \nsubseteq I_n$. Thus we assume $r_i \notin I_n$ for some i. Without loss of generality, suppose that $r_2 \notin I_n$. By the hypothesis, $I_i \nsubseteq (I_j : x)$ for every $x \in R \setminus I_j$ and $i \neq j$. Hence, there exists $y_i \in I_i \setminus (I_n : r_2)$ for every $i \neq n$. Let $Y = y_1 \circ y_2 \circ \cdots y_{n-1}$. Then $Y \subseteq I_i$ for every $i \neq n$ but $Y \nsubseteq (I_n : r_2)$. Therefore, $Y \subseteq I_i \setminus (I_n : r_2)$ for every $i \neq n$. Let $A = r_1 + (a_2 + Y) \circ r_2 + \cdots + a_s \circ r_s$. We consider two cases. Case one: Suppose that $I \subseteq I_1 \bigcup I_2 \cup \cdots \bigcup I_n$. By Theorem 3.2, $I \subseteq I_j$ for some $j \in \{1, \dots, n\}$, which is a contradiction. Case two: Suppose that $I \nsubseteq I_1 \bigcup I_2 \bigcup \cdots \bigcup I_n$. Then by a similar argument as above, we assume $r_2 \notin I_n$. Hence $A = B + Y \circ r_2 \nsubseteq \bigcup_{i=1}^n I_i$ and so the proof is complete. □

Corollary 3.4. *Let $I_1, I_2, \dots, I_n$ be 1-absorbing prime hyperideals of R, I be a hyperideal of R and $I_i \nsubseteq (I_j : x)$ for every $x \in R \setminus I_j$ and $i \neq j$. If $r \in R$ and $Rr + I \nsubseteq \bigcup_{i=1}^n I_i$, then there exists $x \in I$ such that $r + x \notin \bigcup_{i=1}^n I_i$.*

Proof. Suppose that $r \in \bigcap_{i=1}^k I_i$ but $r \notin \bigcup_{i=k+1}^n I_i$. If $k = 0$, then $r = r + 0 \notin \bigcup_{i=1}^n I_i$ and so we are done. Thus assume that $1 \leq k$. By the hypothesis, $I_i \nsubseteq (I_j : x)$ for every $x \in R \setminus I_j$ and $i \neq j$, so Theorem 3.2 implies that $I \nsubseteq \bigcup_{i=1}^k I_i$. Hence there exists $a \in I \setminus \bigcup_{i=1}^k I_i$. We show that $\bigcap_{i=1}^k I_i \nsubseteq \bigcup_{i=1}^k (I_j : x)$ for every $x \in R \setminus I_j$. Suppose that $\bigcap_{i=1}^k I_i \subseteq \bigcup_{i=1}^k (I_j : a)$ for $a \in I \setminus I_j$. By Theorem 3.2, we get $\bigcap_{i=1}^k I_i \nsubseteq (I_j : a)$ for some $j \in \{1, \dots, k\}$. This implies that $\bigcap_{i=1}^k I_i \nsubseteq (I_j : a)$ for some $j \in \{1, \dots, k\}$ and $a \in I \setminus I_j$. Since $(I_j : a)$ is prime, we conclude that $I_i \subseteq (I_j : a)$ where $i \in \{k+1, \dots, n\}$ and $j \in \{1, \dots, k\}$, which contradicts the hypothesis. Thus there exists $b \in \bigcap_{i=k+1}^n I_i \setminus \bigcup_{j=1}^k (I_j : a)$. Let $x \in a \circ b$. Then $x \in I$. We also have $x \in \bigcap_{i=k+1}^n I_i$, but $x \notin \bigcup_{i=1}^k I_i$, because otherwise $x \in a \circ b \subseteq I_i$ for some $i \in \{1, \dots, k\}$ and hence $b \in (I_i : a)$ for some $i \in \{1, \dots, k\}$, a contradiction. Thus $x \in \bigcap_{i=k+1}^n I_i \setminus \bigcup_{i=1}^k I_i$. Now $r \in \bigcap_{i=1}^k I_i \setminus \bigcup_{i=k+1}^n I_i$ shows that $r + x \notin \bigcup_{i=1}^n I_i$. □

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