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SOME FIXED POINT THEOREMS IN 2-INNER PRODUCT SPACES

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Abstract This paper aims to establish some results on the structure of fixed point sets for mappings in the 2-inner product spaces. To this end, we employ some well-known techniques of 2-inner product spaces.

1 Introduction and Preliminaries

The concept of 2-metric spaces, linear 2-normed spaces, and 2-inner product spaces, introduced by Gähler [4]. After that, several authors like White [11], Lewandowska [8, 9], Freese [3], and Diminnie [2], worked on possible applications of Metric Geometry, Functional Analysis, and Topology in these settings. Some other related results are also discussed in [1, 5, 7, 10].

Let $\mathscr{X}$ be a linear space of dimension greater than 1 over the field $K = \mathbb{R}$ of real numbers, or the field $K = \mathbb{C}$ of complex numbers. Suppose that $\langle \cdot, \cdot | \cdot \rangle$ is a K-valued function defined on $\mathscr{X} \times \mathscr{X} \times \mathscr{X}$ satisfying the following conditions:

(I1) $\langle x, x|z \rangle \geq 0$ and $\langle x, x|z \rangle = 0$ if and only if x and z are linearly dependent;

(I2) $\langle x, x|z \rangle = \langle z, z|x \rangle$;

(I3) $\langle y, x|z \rangle = \overline{\langle x, y|z \rangle}$;

(I4) $\langle \alpha x, y|z \rangle = \alpha \langle x, y|z \rangle$ for any scalar $\alpha \in K$;

(I5) $\langle x + x', y|z \rangle = \langle x, y|z \rangle + \langle x', y|z \rangle$.

$\langle \cdot, \cdot | \cdot \rangle$ is called a 2-inner product on $\mathscr{X}$ and $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ is called a 2-inner product space (or 2-pre-Hilbert space). Some basic properties of 2-inner product $\langle \cdot, \cdot | \cdot \rangle$ can be immediately obtained as follows:

(P1) $\langle 0, y|z \rangle = \langle x, 0|z \rangle = \langle x, y|0 \rangle = 0$;

(P2) $\langle x, \alpha y|z \rangle = \overline{\alpha} \langle x, y|z \rangle$;

(P3) $\langle x, y|\alpha z \rangle = |\alpha|^2 \langle x, y|z \rangle$, for all $x, y, z \in \mathscr{X}$ and $\alpha \in K$.

By the above properties, we can prove the Cauchy-Schwarz inequality

$$|\langle x, y|z \rangle|^2 \leq \langle x, x|z \rangle \langle y, y|z \rangle .$$

The most standard example for a linear 2-inner product $\langle \cdot, \cdot | \cdot \rangle$ is defined on $\mathscr{X}$ by

$$\langle x, y|z \rangle := \det \begin{bmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{bmatrix}$$

for all $x, y, z \in \mathscr{X}$. In [2], it is shown that, in any given 2-inner product space $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$, we can define a function

$$\|x, z\| = \sqrt{\langle x, x|z \rangle} \tag{1.1}$$

for all $x, z \in \mathscr{X}$. It is easy to see that this function satisfies the following conditions:

(N1) $\|x, y\| = 0$ if and only if x and y are linearly dependent;

(N2) $\|x, y\| = \|y, x\|$;

(N3) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for any real number α;

(N4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

Any function $\|\cdot, \cdot\|$ defined on $\mathscr{X} \times \mathscr{X}$ and satisfying the above conditions is called a 2-norm on $\mathscr{X}$ and $(\mathscr{X}, \|\cdot, \cdot\|)$ is called linear 2-normed space. Some of the fundamental properties of 2-norms are that they are non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in \mathscr{X}$ and all $\alpha \in \mathbb{R}$. Whenever a 2-inner product space $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ is given, we consider it as a linear 2-normed sapce $(\mathscr{X}, \|\cdot, \cdot\|)$ with the 2-norm defined by (1.1).

An operator $A \in \mathcal{B}(\mathscr{X})$ is said to be bounded, if there exists $M > 0$ such that

$$\|Ax, y\| \leq M \, \|x, y\| ,$$

for every $x \in \mathscr{X}$.

The norm of the b-operator is defined by

$$\|A\|_2 = \sup \{\|Ax, b\| : \ \|x, b\| = 1\} . \tag{1.2}$$

where b is fix element in $\mathscr{X}$. It is easy to check that (1.2) is equivalent with

$$\sup \{|(Ax, y|b)| : \ \|x, b\| = \|y, b\| = 1\} .$$

Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space, and $z \in \mathscr{X}$. A sequence $\{x_n, z\}$ in $\mathscr{X}$ is a z-Cauchy sequence if

$$\forall \varepsilon > 0 \, \exists N > 0, \text{ s.t} \, \forall \, m, n \geq N \; 0 < \|x_m - x_n, z\| < \varepsilon.$$

Meanwhile, $\mathscr{X}$ is called z-Hilbert if every z-Cauchy sequence is converges in the semi normed $(\mathscr{X}, \|\cdot, z\|)$.

2 Main Results

Let C be a non empty closed convex subset of a 2-inner product space. A mapping $A : C \to C$ is called non spreading if

$$2\|Ax - Ay, z\|^2 \leq \|Ax - y, z\|^2 + \|Ay - x, z\|^2$$

for all $x, y \in C$.

We say $A : C \to C$ is an asymptotic non-spreading mapping if there exists two functions $\alpha : C \to [0, 2)$ and $\beta : C \to [0, k]$, $k < 2$, such that

(a) $2\|Ax - Ay, z\|^2 \leq \alpha(x) \|Ax - y, z\|^2 + \beta(x) \|Ay - x, z\|^2$, for all $x, y, z \in C$.

(b) $0 < \alpha(x) + \beta(x) \leq 2$, for all $x \in C$.

It is necessary to remark that

(a) If $\alpha(x) = \beta(x) = 1$, for all $x \in C$, then A is a non-spreading mapping.

(b) If $\alpha(x) = \frac{4}{3}$ and $\beta(x) = \frac{2}{3}$ for all $x \in C$, then A is a AJ-2 mapping.

Let $\mathscr{X}$ be a real 2-inner product space and let C be a non empty subset of $\mathscr{X}$. A mapping $A : C \to \mathscr{X}$ is called symmetric generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\|Ax - Ay, z\|^2 + \beta \left(\|x - Ay, z\|^2 + \|Ax - y, z\|^2\right) + \gamma\|x - y, z\|^2$$
$$+ \delta \left(\|x - Ax, z\|^2 + \|y - Ay, z\|^2\right) \leq 0$$

for all $x, y, z \in \mathscr{X}$. Such mapping A is also called $(\alpha, \beta, \gamma, \delta)$- symmetric generalized hybrid.

Theorem 2.1. *Let C be a non-empty closed convex subset of a 2-inner product space $\mathscr{X}\times\mathscr{X}$. Let α,β be the same as in the above. Then $A:C\to C$ is an asymptotic non spreading mapping if*

$$\begin{aligned}&\|Ax-Ay,z\|^2\\&\leq\frac{\alpha(x)-\beta(x)}{2-\beta(x)}\|Ax-x,z\|^2\\&\quad+\frac{\alpha(x)\|x-y,z\|}{2-\beta(x)}\frac{2\langle Ax-x,\alpha(x)(x-y)+\beta(x)(Ay-x)|z\rangle}{2-\beta(x)}.\end{aligned}$$

Proof. We have that for $x,y,z\in C$

$$\begin{aligned}2\|Ax-Ay,z\|^2&\leq\alpha(x)\|Ax-y,z\|^2+\beta(x)\|Ay-x,z\|^2\\&=\alpha(x)\|Ax-x,z\|^2+2\alpha(x)\langle Ax-x,x-y|z\rangle\\&\quad+\alpha(x)\|x-y,z\|^2+\beta(x)\|Ay-Ax,z\|^2\\&\quad+2\beta(x)\langle Ay-Ax,Ax-x|z\rangle+\beta(x)\|Ax-x,z\|^2\\&=(\alpha(x)+\beta(x))\|Ax-x,z\|^2+\beta(x)\|Ay-Ax,z\|^2\\&\quad+\alpha(x)\|x-y,z\|^2+2\alpha(x)\langle Ax-x,x-y|z\rangle\\&\quad+2\beta(x)\langle Ay-x+x-Ax,Ax-x|z\rangle\\&=(\alpha(x)-\beta(x))\|Ax-x,z\|^2+\beta(x)\|Ay-Ax,z\|^2\\&\quad+\alpha(x)\|x-y,z\|^2+\langle Ax-x,2\alpha(x)(x-y)\\&\qquad+2\beta(x)Ay-x|z\rangle,\end{aligned}$$

and this implies that desired result. □

Theorem 2.2. *Let $\mathscr{X}\times\mathscr{X}$ be a real 2-inner product space, let C be a nonempty closed convex subset of $\mathscr{X}$ and let A be an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions*

(i) $\alpha+2\beta+\gamma\geq 0$

(ii) $\alpha+\beta+\delta>0$

(iii) $\delta\geq 0$

holds. Then A has a fixed point if and only if there exists $y\in C$ such that $\{A^ny:\ n\in\{0,1,\ldots\}\}$ is bounded. In particular, a fixed point of A is unique in the case of $\alpha+2\beta+\gamma>0$ on the condition.

Proof. Assume that A has a fixed point y. Then $\{A^ny:\ n\in\{0,1,\ldots\}\}=\{y\}$ and hence $\{A^ny:\ n\in\{0,1,\ldots\}\}$ is bounded. Conversely, suppose that there exists $y\in C$ such that $\{A^ny:\ n\in\{0,1,\ldots\}\}$ is bounded. Since A is an $(\alpha,\beta,\gamma,\delta)$-symmetric generalized hybrid mapping of C into itself, we have that

$$\begin{aligned}&\alpha\left\|Ax-A^{n+1}y,z\right\|^2+\beta\left(\left\|x-A^{n+1}y,z\right\|^2+\|Ax-A^ny,z\|^2\right)\\&\quad+\gamma\|x-A^ny,z\|^2+\delta\left(\|x-Ax,z\|^2\right.\\&\quad\left.+\left\|A^ny-A^{n+1}y,z\right\|^2\right)\leq 0\end{aligned}$$

for all $n\in\mathbb{N}\cup\{0\}$ and $x\in C$. Since $\{A^ny\}$ is bounded, we can apply Banach limit μ to both sides of the inequality. Since $\mu_n\|Ax-A^ny,z\|^2=\mu_n\left\|Ax-A^{n+1}y,z\right\|^2$ and $\mu_n\left\|x-A^{n+1}y,z\right\|^2=\mu_n\|x-A^ny,z\|^2$, we have that

$$\begin{aligned}&(\alpha+\beta)\mu_n\|Ax-A^ny,z\|^2+(\beta+\gamma)\mu_n\|x-A^ny,z\|^2\\&\quad+\delta\left(\|x-Ax,z\|^2+\mu_n\left\|A^ny-A^{n+1}y,z\right\|^2\right)\leq 0.\end{aligned}$$

Furthermore, since

$$\mu_n\|Ax - A^n y, z\|^2 = \|Ax - x, z\|^2 + 2\mu_n\left(Ax - x, x - A^n, z\right) \\ + \mu_n\|x - A^n x, z\|^2$$

we have that

$$(\alpha+\beta+\delta)\|Ax - x, z\|^2 + 2(\alpha+\beta)\mu_n\left(Ax - x, x - A^n|z\right) \\ + (\alpha+2\beta+\gamma)\mu_n\|x - A^n y, z\|^2 + \delta\mu_n\left\|A^n x - A^{n+1}x, z\right\|^2 \leq 0.$$

From (i) and (iii) we have

$$(\alpha+\beta+\delta)\|Ax - x, z\|^2 + 2(\alpha+\beta)\mu_n\left(Ax - x, x - A^n, z\right) \leq 0. \tag{2.1}$$

Since there exists $p \in \mathscr{X}$ such that

$$\mu_n\left(w, A^n y, z\right) = (w, p, z)$$

for all $w \in \mathscr{X}$. We have from (2.1) that

$$(\alpha+\beta+\delta)\|Ax - x, z\|^2 + 2(\alpha+\beta)\mu_n\left(Ax - x, x - p, z\right) \leq 0. \tag{2.2}$$

Since C is closed and convex, we have that

$$p \in \overline{co}\left\{A^n x : \ n \in \mathbb{N}\right\} \subset C.$$

Putting $x = p$ we obtain from (2.2) that

$$(\alpha+\beta+\delta)\|Ap - p, z\|^2 \leq 0. \tag{2.3}$$

We have from (ii) that $\|Ap - p, b\|^2 \leq 0$. This implies that p is a fixed point in A.

New suppose that $\alpha + 2\beta + \gamma > 0$. Let p_1 and p_2 be fixed points of A. Then we have that

$$\alpha\|Ap_1 - Ap_2, z\|^2 + \beta\left(\|p_1 - Ap_2, z\|^2 + \|Ap_1 - p_2, z\|^2\right) \\ + \gamma\|p_1 - p_2, z\|^2 + \delta\left(\|p_1 - Ap_1, z\|^2 \right. \\ \left. + \|p_2 - Ap_2, z\|^2\right) \leq 0$$

and hence $(\alpha + 2\beta + \gamma)\|p_1 - p_2, z\|^2 \leq 0$. We have from $\alpha+2\beta+\gamma > 0$ that $p_1 = p_2$. Therefore a fixed point of A is unique. This completes the proof. □

Corollary 2.3. *Let $\mathscr{X} \times \mathscr{X}$ be a real 2-inner product space, let C be a nonempty closed convex subset of $\mathscr{X}$ and let A be an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping from C into itself such that the conditions*

(i) $\alpha + 2\beta + \gamma \geq 0$

(ii) $\alpha + \beta + \delta > 0$

(iii) $\delta \geq 0$

holds. Then A has a fixed. In particular, a fixed point of A is unique in the case of $\alpha+2\beta+\gamma > 0$ on the condition.

Following theorem is the generalization of the Banach contraction principle in the 2-inner product space, involving four rational square terms in the inequality.

Theorem 2.4. *Let $A : \mathscr{X} \to \mathscr{X}$ be a self mapping satisfying the following condition*

$$\begin{aligned}\|Ax - Ay, z\|^2 \leq a_1 &\frac{\|y - Ay, z\|^2 \left(1 + \|x - Ax, z\|^2\right)}{1 + \|x - y, z\|^2} \\ &+ a_2 \frac{\|x - Ax, z\|^2 \left(1 + \|y - Ay, z\|^2\right)}{1 + \|x - y, z\|^2} \\ &+ a_3 \frac{\|x - Ay, z\|^2 \left(1 + \|y - Ax, z\|^2\right)}{1 + \|x - y, z\|^2} \\ &+ a_4 \frac{\|y - Ax, z\|^2 \left(1 + \|x - Ay, z\|^2\right)}{1 + \|x - y, z\|^2} \\ &+ a_5 \|x - y, z\|^2\end{aligned}$$

for all $x, y, z \in \mathscr{X}$ and $x \neq y$, where a_1, a_2, a_3, a_4, a_5 are non negative reals with $a_1 + a_2 + a_3 + 4a_4 + a_5 < 1$. Therefore, A has a unique fixed point in $\mathscr{X}$.

Proof. For some $x_0 \in \mathscr{X}$, we define a sequence $\{x_n\}$ of iterates of A as follows

$$x_1 = Ax_0, z,\ x_2 = Ax_1, z,\ x_3 = Ax_2, z, \ldots,\ x_{n+1} = Ax_n, z$$

for $n \in \{0, 1, \ldots\}$.

Now, we show that $\{x_n, z\}$ is a z-Cauchy sequence in $\mathscr{X} \times \mathscr{X}$. For this, consider

$$\|x_{n+1} - x_n, z\|^2 = \|Ax_n - Ax_{n-1}, z\|^2.$$

Then by using the hypothesis, we have

$$\begin{aligned}\|x_{n+1} - x_n, z\|^2 \leq a_1 &\frac{\|x_{n-1} - Ax_{n-1}, z\|^2 \left(1 + \|x_n - Ax_n, z\|^2\right)}{1 + \|x_n - x_{n-1}, z\|^2} \\ &+ a_2 \frac{\|x_n - Ax_n, z\|^2 \left(1 + \|x_{n-1} - Ax_{n-1}, z\|^2\right)}{1 + \|x_n - x_{n-1}, z\|^2} \\ &+ a_3 \frac{\|x_n - Ax_{n-1}, z\|^2 \left(1 + \|x_{n-1} - Ax_n, z\|^2\right)}{1 + \|x_n - x_{n-1}, z\|^2} \\ &+ a_4 \frac{\|x_{n-1} - Ax_n, z\|^2 \left(1 + \|x_n - Ax_{n-1}, z\|^2\right)}{1 + \|x_n - x_{n-1}, z\|^2} \\ &+ a_5 \|x_n - x_{n-1}, z\|^2.\end{aligned}$$

Which implies that

$$\begin{aligned}&(1 - a_2 - 2a_4) \|x_{n+1} - x_n, z\|^2 \\ &\quad + (1 - a_1 - a_2) \|x_{n+1} - x_n, z\|^2 \|x_n - x_{n-1}, z\|^2 \\ &\leq \left((a_1 + 2a_4 + a_5) + a_5 \|x_n - x_{n-1}, z\|^2\right) \|x_n - x_{n-1}, z\|^2.\end{aligned}$$

Resulting in

$$\|x_{n+1} - x_n, z\|^2 \leq p(n) \|x_n - x_{n-1}, b\|^2$$

where

$$p(n) = \frac{(a_1 + 2a_4 + a_5) + a_5 \|x_n - x_{n-1}, z\|^2}{(1 - a_2 - 2a_4) + (1 - a_1 - a_2) \|x_n - x_{n-1}, z\|^2}$$

for $n \in \{0, 1, \ldots\}$. Clearly $p(n) < 1$, for all n as $a_1 + a_2 + a_3 + 4a_4 + a_5 < 1$. Repeating the same argument, we find some $S < 1$, such that

$$\|x_{n+1} - x_n, z\|^2 \leq \lambda^n \|x_1 - x_0, z\|^2$$

where $\lambda = S^2$. Letting $n \to \infty$, we obtain $\|x_{n+1} - x_n, z\| \to 0$. It follows that $\{x_n, z\}$ is a z-Cauchy sequence in $\mathscr{X}$. So by the completeness of $\mathscr{X}$ there exists a point $\mu \in \mathscr{X}$ such that $x_n \to \mu$ as $n \to \infty$. Also $\{x_{n+1}, z\} = \{Ax_n, z\}$ is sub sequence of $\{x_n, z\}$ converges to the same limit μ. Since A is continuous, we obtain

$$A(\mu) = A\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_{n+1} = \mu.$$

Hence μ is a fixed point of $\mathscr{X}$. New, we show the uniqueness of μ. If A has another fixed point γ and $\mu \neq \gamma$, then

$$\begin{aligned}
\|p - p', z\|^2 &= \|Ap - Ap', z\|^2 \\
&\leq a_1 \frac{\|p' - Ap', z\|^2 \left(1 + \|p - Ap, z\|^2\right)}{1 + \|p - p', b\|^2} \\
&\quad + a_2 \frac{\|p - Ap, z\|^2 \left(1 + \|p' - Ap', z\|^2\right)}{1 + \|p - p', z\|^2} \\
&\quad + a_3 \frac{\|p - Ap', z\|^2 \left(1 + \|p' - Ap, z\|^2\right)}{1 + \|p - p', z\|^2} \\
&\quad + a_4 \frac{\|p' - Ap, z\|^2 \left(1 + \|p - Ap', z\|^2\right)}{1 + \|p - p', z\|^2} \\
&\quad + a_5 \|p - p', z\|^2
\end{aligned}$$

which in turn, implies that

$$\|p - p', z\|^2 \leq (a_3 + a_4 + a_5) \|p - p', z\|^2.$$

This gives a contradiction, for $a_3 + a_4 + a_5 < 1$. Thus p is a unique fixed point of A in $\mathscr{X}$. □

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