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On Statistically Internal Chain Transitive Sets in a Discrete Dynamical System

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Abstract Following the concept of statistical convergence, the notions of statistical ω-limit sets and statistical ω-cluster sets have been studied in a discrete dynamical system X in [1]. In this paper we further introduce the concept of statistically internal chain transitive sets and give a characterization of the sets for a continuous function in X.

1 Introduction

The recurrence property of a dynamical system can be described by by the notion of chain recurrence introduced by Conley [2]. This concept is being applied increasingly in economics [3], game theory [4], epidemiology [5], numerical analysis [6] and many other areas. Chain recurrence is an inherent property of ω limit sets (see [7],[8],[9],[10],[11] etc.) which helps us to gain an idea about the behaviour of the dynamical system. For the present paper we are concerned about the discrete the dynamical system.
In a compact metric metric space (X, γ), let $f : X \longrightarrow X$ be a map. The forward orbit of some point $x \in X$ corresponding to the map f is given by

$$Orb^+(x) = \{f^i(x) : i \in \{0, 1, 2, 3, ...\}\}$$

and f^0 is the identity map on X. The backward orbit of x is given by, $Orb^-(x) = \{x_{-i}\}_{i\geq 0} \subset X$, where $f(x_{-i}) = x_{-i+1}$, for $i > 0$.
The ω-limit set for a sequence of points $\{x_n\}$ is the collection of all accumulation points of $\{x_n\}$. It is given by

$$\omega(\{x_n\}) = \bigcap_{n=0}^{\infty} \overline{\{x_k : k \geq n\}}$$

For a map $f : X \to X$, if $\{x_n\}$ is the orbit of a point $x \in X$ then the ω-limit set can be described as,

$$\omega(x, f) = \bigcap_{n=0}^{\infty} \overline{\{f^k(x) : k \geq n\}}.$$

In [1] the author studied the concept of omega limit sets in a more subtle way using the idea of statistical convergence (see [12-22]) where ω-limit sets consists of points which are not limit points in ordinary sense and introduced the notion of statistical omega limit set. The concept of invariant set and weak incompressibility has been studied in connection to statistical ω-limit sets. The concept of invariant sets [10] are central to control theory and to validation of systems, such as programs, physical systems or hybrid systems whereas the property of weak incompressibility is an inherent property of ω-limit sets. Originally the weak incompressibility was stated as a property of invariant sets, but Barwell [8] modified the definition slightly to remove the necessity of invariance.

H. Fast [12] and Steinhaus [13] introduced the concept of statistical convergence incorporating the idea of natural density [14] and later it was developed by Salat [15], Millar [16], Fridy [20] and others.

The natural density or asymptotic density of a subset S of the set of natural numbers $\mathbb{N}$ is given by $\delta(S)$, if $\lim_{n\to\infty} \frac{|S(n)|}{n} = \delta(S)$, where $S(n) = \{k < n : k \in S\}$ and $|S|$ denotes the cardinality of the set S. A sequence $\{x_n\}$ in X is said to be statistically convergent to some element $\xi \in X$ if for every $\epsilon > 0$, $\delta(\{k \in \mathbb{N} : \gamma(x_k, \xi) \geq \epsilon\}) = 0$.
The concept of statistical limit points and statistical cluster points was introduced by Fridy [21]. A subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ is said to be of density zero or thin subsequence if $\delta(\{n_k : k \in \mathbb{N}\}) = 0$ and $\{x_{n_k}\}$ is said to be non-thin if either $\delta(\{n_k : k \in \mathbb{N}\}) > 0$ or density of the set $\{n_k : k \in \mathbb{N}\}$ does not exist.
A number λ is said to be a statistical limit point of a sequence $\{x_n\}$ if there is a non-thin subsequence of $\{x_n\}$ converging to λ and a number μ is called a statistical cluster point of $\{x_n\}$ if for every $\epsilon > 0$, $\{k \in \mathbb{N} : \gamma(x_k, \mu) \geq \epsilon\}$ does not have density zero. Fridy established that the set of statistical cluster points of a bounded sequence is non-empty and if this set contains only one point then the sequence is statistically convergence to that point.
In the present paper we have studied the concept of internally chain transitive set using the concept of statistical convergence and introduced the notion of statistically internal chain transitive set and established some properties of this set in connection to invariant set and find its relation with statistical ω-cluster sets.

2 Definitions and notations

In this section we provide some well known definitions and state some already established results related to dynamical system, statistical ω- limit sets and statistical ω-cluster sets.

Lemma 2.1. *If a map $f : I \to I$ is continuous in a compact interval I, then f has a fixed point on I.*

Definition 2.2. [10] A set $A \subset X$ is said to be invariant under a map $f : X \to X$ if $f(A) \subset A$ and is strongly invariant (i.e., s-invariant) if $f(A) = A$.

Property 2.3. For any point $x_0 \in X$ and $f : X \to X$, the set $\omega(x_0, f)$ is closed, non-empty and s-invariant.

Definition 2.4. [10] For $\epsilon > 0$, a (finite or infinite) sequence of points $\{x_0, x_1, x_2, ...\} \subset X$ is an ϵ-pseudo orbit if $\gamma(f(x_i), x_{i+1}) < \epsilon$ for every $i \geq 0$.

Definition 2.5. [10] A set $A \subset X$ is said to be Chain Transitive if for any $\epsilon > 0$ and for any pair of points $x, y \in A$ there is an ϵ-pseudo orbit $\{x = x_0, x_1, x_2, ..., x_n = y\}$.

Definition 2.6. [10] A set $A \subset X$ is said to be Internally Chain Transitive if for any $\epsilon > 0$ and for any pair of points $x, y \in A$ there is an ϵ-pseudo orbit $\{x = x_0, x_1, x_2, ..., x_n = y\} \subset A$ for $n \geq 1$.

Definition 2.7. [10] A sequence of points $\{x_n\}$ is said to be an asymptotic pseudo orbit of f if $\lim_{n\to\infty} \gamma(f(x_n), x_{n+1}) = 0$.

Definition 2.8. [1] If $\{x_n\}$ is the orbit of a point $x \in X$ for a map $f : X \to X$, then the collection of all statistical limit points of $\{x_n\}$ is said to be the statistical ω-limit set of $\{x_n\}$. Accordingly, the collection of all statistical cluster points of $\{x_n\}$ is the statistical ω-cluster set of $\{x_n\}$.
We denote the statistical ω-limit set and statistical ω-cluster set of $\{x_n\}$ by $\omega_l(x, f)$ and $\omega_c(x, f)$ respectively. Also the set of all ordinary limit points of $\{x_n\}$ is given by the ω-limit set $\omega(x, f)$.

Lemma 2.9. [1] *If $\{x_n\}$ is the orbit of a point $x \in X$ for a map $f : X \to X$, then*
(a) $\omega_l(x, f) \subset \omega(x, f)$
(b) $\omega_c(x, f) \subset \omega(x, f)$
(c) $\omega_l(x, f) \subset \omega_c(x, f)$
(d) $\omega_l(x, f)$ *is an F_σ set.*
(e) $\omega_c(x, f)$ *is a closed point set and hence compact in X.*
(f) *If $\{x_n\}$ and $\{y_n\}$ are orbits of two pints x and y such that $x_k = y_k$ for almost all $k \in \mathbb{N}$, then*

$$\omega_l(x, f) = \omega_l(y, f) \text{ and } \omega_c(x, f) = \omega_c(y, f)$$

(g) For the orbit $\{x_n\}$ of any point $x \in X$ there exists an orbit $\{y_n\}$ of some point $y \in X$ such that
(i) $\omega(y, f) = \omega_c(x, f)$ and $x_k = y_k$ for almost all k and
(ii) $\{y_n : n \in \mathbb{N}\} \subset Orb^+(x)$.
(h) If the orbit of some point $x \in X$ for a function $f : X \to X$ is bounded, then $\omega_c(x, f) \neq \phi$.

Lemma 2.10. $[1]$ *For a point $x_0 \in X$ and a function $f : X \to X$, $\omega_c(x, f)$ is s-invariant.*

Lemma 2.11. $[1]$ *For some point $x_0 \in X$ and for a function $f : X \to X$ if $y_0 \in \omega_c(x_0, f)$, then $\omega_c(y_0, f) \subset \omega_c(x_0, f)$.*

Definition 2.12. $[8]$ A set $A \subset X$ is said to have weak incompressibility if for any proper non-empty open subset U in A, $\overline{f(U)} \cap \overline{(A - U)} \neq \phi$. Equivalently, if for any proper non-empty closed subset D in A we have $D \cap \overline{f(A - D)} \neq \phi$.

Lemma 2.13. $[1]$ *For $x_0 \in X$, $\omega_c(x_0, f)$ has weak incompressibility.*

3 Statistically Chain Transitive Set

In this section we introduce the notion of statistically internal chain transitive sets using the concept of statistical convergence and study some properties of the sets.

Definition 3.1. For $\epsilon > 0$, a sequence of points $\{x_0, x_1, x_2, ...\} \subset X$ is said to be $st - \epsilon$-pseudo orbit if $\delta\{i \in \mathbb{N} : \gamma(f(x_i), x_{i+1}) \geq \epsilon\} = 0$.

We prove the following important lemma which in ordinary sense is used to deduce the behaviour of maps near pseudo-orbits using uniform continuity.

Lemma 3.2. *Let (X, f) be a dynamical system. For any $\epsilon > 0$ and $n \in \mathbb{N}$ there exists $\sigma > 0$ depending on ϵ and n such that if $\{x_0, x_1, x_2, ...\}$ is a $st - \sigma$-pseudo orbit and $y \in X$ is such that $\gamma(y, x_0) < \sigma$, then $\delta\{k \in \mathbb{N} : \gamma(f^k(y), x_k) \geq \epsilon\} = 0$.*

Proof. Since f is continuous on X, there exists a positive real number $\sigma < \frac{\epsilon}{2}$ such that $\gamma(f^i(x), f^i(y)) < \epsilon/2$, for all $i = 1, 2, 3, ...$ whenever $\gamma(x, y) < \sigma$.
Then for every $st - \sigma$-pseudo orbit $\{x_0, x_1, x_2, ...\}$ of f we have $\delta(B) = 0$, where $B = \{j \in \mathbb{N} : \gamma(f^j(x_0), x_j) \geq \sigma\}$. Let $A = \{j \in \mathbb{N} : \gamma(f^j(x_0), x_j) \geq \epsilon/2\}$. If $k \in B^c$ (complement set of B), then $\gamma(f^k(x_0), x_k) < \sigma < \epsilon/2$. This implies that $k \in A^c$. Thus, $A \subset B$ and so $\delta(A) = 0$.
Let $y \in X$ be such that $\gamma(y, x_0) < \sigma$. Then $\gamma(f^j(x_0), f^j(y)) < \epsilon/2$ for all $j = 1, 2, 3, ...$
Again, consider $C = \{j \in \mathbb{N} : \gamma(f^j(y), x_j) \geq \epsilon\}$. Here $k \in A^c$ implies that $\gamma(f^k(x_0), x_k) < \epsilon/2$. So,

$$\gamma(f^k(y), x_k) \leq \gamma(f^k(y), f^k(x_0)) + \gamma(f^k(x_0), x_k) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Clearly, $k \in C^c$ and $C \subset A$, which implies that $\gamma(C) = 0$. This proves the lemma. □

Definition 3.3. A set $A \subset X$ is said to be Statistically Chain Transitive if for any $\epsilon > 0$ and for any pair of points $x, y \in A$ there is a $st - \epsilon$-pseudo orbit $\{x_0, x_1, x_2, ..., \}$ such that $x = x_0$ and $st - \lim x_n = y$.

Definition 3.4. A set $A \subset X$ is said to be Statistically Internal Chain Transitive (SICT) if for any $\epsilon > 0$ and for any pair of points $x, y \in A$ there is a $st - \epsilon$-pseudo orbit $\{x_0, x_1, x_2, ...\} \subset A$ such that $x = x_0$ and $st - \lim x_n = y$.

The following result provides the connection between SICT and invariant set.

Theorem 3.5. *If a closed subset of X is SICT then it is strongly invariant.*

Proof. Let A be a closed subset of X and SICT. We are to prove that $f(A) = A$. Let $x \in A$.
Case I: If x is a fixed point then the result follows trivially.
Case II: Let $f(x) \neq x$. For every $\epsilon > 0$, there exists a $st - \epsilon$-pseudo orbit of points in A from x to x via some distinct points. Then we can find points y_n and z_n in A such that $\{n \in \mathbb{N} :$

$\gamma(f(y_n), x) \geq \epsilon/2\} = 0$ and $\{n \in \mathbb{N} : \gamma(f(x), z_n) \geq \epsilon/2\} = 0$.
Since A is compact, there exist two points y and z in A such that $st-\lim y_n = y$ and $st-\lim z_n = z$. By the continuity of f, $st - \lim f(y_n) = f(y)$ and consequently $x = f(y)$. So, $A \subset f(A)$
Let $P = \{n \in \mathbb{N} : \gamma(f(x), z_n) \geq \epsilon/2\}$ and $Q = \{n \in \mathbb{N} : \gamma(z_n, z) \geq \epsilon/2\}$. Then $\delta(P) = \delta(Q) = 0$. If $k \in P^c \cap Q^c$, then $\gamma(f(x), z_k) < \epsilon/2$ and $\gamma(z_k, z) < \epsilon/2$. This follows that

$$\gamma(f(x), z) \leq \gamma(f(x), z_k) + \gamma(z_k, z) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since ϵ is arbitrary, $f(x) = z$ and clearly $f(A) \subset A$ which follows $f(A) = A$. □

We establish the relation between SICT and statistical ω-cluster set in the following.

Theorem 3.6. *In the dynamical system (X, f), the set $S = \omega_c(x_0, f)$ is SICT for any $x_0 \in X$.*

Proof. Since X is compact, f is uniformly continuous on X. For $\epsilon > 0$ there exists an $\eta > 0$ with $\eta < \epsilon/3$ such that whenever $x, y \in X$ and $\gamma(x, y) < \eta$, we have $\gamma(f(x), f(y)) < \epsilon/3$. Also for $T = \{n \in \mathbb{N} : f^n(x_0) \in N_\eta(S)\}$, $\delta(T) \neq 0$ where $N_\eta(S)$ is an open neighbourhood of S.

Let $a, b \in S$. If $A = \{n \in \mathbb{N} : \gamma(f^{n-1}(x_0), a) < \eta\}$ and $B = \{n \in \mathbb{N} : \gamma(f^n(x_0), b) < \epsilon/3\}$ then $\delta(A) \neq 0$ and $\delta(B) \neq 0$.
Consider $C = \{n \in \mathbb{N} : \gamma(f^n(x_0), f(a)) < \epsilon/3\}$. If $k \in A$, $\gamma(f^{k-1}(x_0), a) < \eta$. By uniform continuity of f, $\gamma(f^k(x_0), f(a)) < \epsilon/3$ and so $k \in C$. Thus $A \subset C$ and $\delta(C) \neq 0$. Clearly, $\delta(T - B) = 0$ and $\delta(T - C) = 0$, i.e., $\delta(T \cap B) \neq 0$ and $\delta(T \cap C) \neq 0$.

Set, $B \cap T = \{p_k : k \in \mathbb{N}\}$ and $C \cap T = \{q_k : k \in \mathbb{N}\}$ and define $Y = \{y_n\}, n = 0, 1, 2, ...$ where,

$$\begin{aligned} y_n &= a; \, when \; n = 0 \\ &= f^{q_n}(x_0); \, if \; n = k^2, k = 1, 2, ... \\ &= f^{p_n}(x_0); \, if \; n \neq k^2, k = 1, 2, ... \end{aligned}$$

Here $Y \subset N_\eta(S)$ forms a $st - \epsilon/3$-pseudo orbit and $st - \lim y_n = b$. So, for each $y_i \in Y$ there is a $z_i \in S$ such that $\gamma(y_i, z_i) < \eta < \epsilon/3$ for all $i = 1, 2, 3, ...$ and $st - \lim z_n = b$.
Again, we consider $U = \{n \in \mathbb{N} : \gamma(f(z_i), z_{i+1}) \geq \epsilon\}$. Since $\gamma(y_i, z_i) < \eta$, by uniform continuity of f we have $\gamma(f(y_i), f(z_i)) < \epsilon/3$ for all i. Also Y forms a $st - \epsilon/3$-pseudo orbit and this implies that $\delta(V) = 0$ where $V = \{n \in \mathbb{N} : \gamma(f(y_n), y_{n+1}) \geq \epsilon/3\}$. If $i \in V^c$, $\gamma(f(y_i), y_{i+1}) < \epsilon/3$. Then,

$$\gamma(f(z_i), z_{i+1}) \leq \gamma(f(z_i), f(y_i)) + \gamma(f(y_i), y_{i+1}) + \gamma(y_{i+1}, z_{i+1}) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Thus $i \in U^c$ and $U \subset V$ which implies that $\delta(U) = 0$ and consequently $\{a = z_0, z_1, z_2, ...\}$ forms a $st - \epsilon$-psuedo orbit in S and it is SICT. □

Definition 3.7. A sequence of points $\{x_n\}$ is said to be a statistically asymptotic pseudo orbit of f if $st - \lim \gamma(f(x_n), x_{n+1}) = 0$.

Statistical ω-cluster set of a statistically asymptotic pseudo orbit may be characterised by SICT.

Theorem 3.8. *In the dynamical system (X, f), a closed set $A \subset X$ is SICT if A is the statistical ω-cluster set of some statistically asymptotic pseudo orbit of f in A.*

Proof. Suppose A is the statistical ω-cluster set of some statistically asymptotic pseudo orbit $\{x_n\}$. We will prove that A is SICT.
Let $\epsilon > 0$ be arbitrary. Since f is uniformly continuous on x, there is a $\eta > 0$ with $\eta < \epsilon/3$ such that whenever $x, y \in X$ and $\gamma(x, y) < \eta$, we have $\gamma(f(x), f(y)) < \epsilon/3$. Again it is to note that for $S = \{n \in \mathbb{N} : f^n(x_0) \in N_\eta(A)\}$, $\delta(S) \neq 0$ and for $T = \{n \in \mathbb{N} : \gamma(f(x_n), x_{n+1}) \geq \epsilon/3\}$, $\delta(T) \neq 0$.

Let $a, b \in A$. Then for $A = \{n \in \mathbb{N} : \gamma(x_{n-1}, a) < \eta\}$ and $B = \{n \in \mathbb{N} : \gamma(x_n, b) < \epsilon/3\}$ then $\delta(A) \neq 0$ and $\delta(B) \neq 0$. Consider $C = \{n \in \mathbb{N} : \gamma(x_n, f(a)) < \epsilon/3\}$. Following the same process of the previous theorem we can show that $A \subset C$ and thus $\delta(C) \neq 0$.

Set, $B \cap T = \{p_k : k \in \mathbb{N}\}$ and $C \cap T = \{q_k : k \in \mathbb{N}\}$ and define $Y = \{y_n\}, n = 0, 1, 2, ...$ where,

$$\begin{aligned} y_n &= a; when\, n = 0 \\ &= x_{q_n}; if\ n = k^2, k = 1, 2, ... \\ &= x_{p_n}; if\ n \neq k^2, k = 1, 2, ... \end{aligned}$$

Y forms a $st - \epsilon/3$-pseudo orbit and $st - \lim y_n = b$. Here, $Y \subset N_\eta(A)$, so for each $y_i \in Y$ there is a $z_i \in A$ such that $\gamma(y_i, z_i) < \eta < \epsilon/3$ for all $i = 1, 2, 3, ...$ and $st - \lim z_n = b$.

Again we consider $U = \{n \in \mathbb{N} : \gamma(f(z_i), z_{i+1}) \geq \epsilon\}$. Since $\gamma(y_i, z_i) < \delta$, by uniform continuity of f we have $\gamma(f(y_i), f(z_i)) < \epsilon/3$ for all i. Also Y forms a $st - \epsilon/3$-pseudo orbit and this implies that $\delta(V) = 0$ where $V = \{n \in \mathbb{N} : \gamma(f(y_n), y_{n+1}) \geq \epsilon/3\}$. If $i \in V^c$, $\gamma(f(y_i), y_{i+1}) < \epsilon/3$. Then,

$$\gamma(f(z_i), z_{i+1}) \leq \gamma(f(z_i), f(y_i)) + \gamma(f(y_i), y_{i+1}) + \gamma(y_{i+1}, z_{i+1}) < \epsilon$$

Thus $i \in U^c$ and $U \subset V$ which implies that $\delta(U) = 0$ and consequently $\{a = z_0, z_1, z_2, ...\}$ forms a $st - \epsilon$-pseudo orbit in A and it is SICT. □

Theorem 3.9. *In the dynamical system (X, f), a closed set $A \subset X$ is the statistical omega cluster set of some statistically asymptotic pseudo orbit of f in A if A is SICT.*

Proof. Let us assume that A is SICT and $x \in A$. Since A is closed in X, then A is compact. Choose $\epsilon > 0$. There are finite number of points $x = x_0, x_1, ..., x_m, x_{m+1} = x$ in A such that $A \subset \bigcup_{i=0}^m N_\epsilon(x_i)$.
Since A is SICT, for each $i = 1, 2, ..., n$, there exists a $st - \epsilon$-pseudo orbit $\{y_n^i\}_n$ in A joining x_i to x_{i+1}. i.e., $\delta\{Y_i\} = 0$ where $Y_i = \{n \in \mathbb{N} : \gamma(f(y_n^i), y_{n+1}^i) \geq \epsilon\}$ and $st - \lim y_n^i = x$ for all $i = 1, 2, ..., n$.
For each i, let $Y_i = \{p_{i,n}\}_n$ and consider a sequence $U_\epsilon = \{t_n\}_{n=0}$ in A as follows:

$$\begin{aligned} t_n &= a; when\, n = 0 \\ &= x_{p_{1,n}}; if\ n = k^2, k = 1, 2, ... \\ &= x_{p_{2,n}}; if\ n = k^3, k = 1, 2, ... \\ &= x_{p_{3,n}}; if\ n = k^4, k = 1, 2, ... \\ .. &\quad .... \\ &= x_{p_{m,n}}; if\ n = k^{m+1}, k = 1, 2, ... \\ &= x_{p_{m+1,n}}; otherwise \end{aligned}$$

Clearly, $st - \lim t_n = x$ and $U_\epsilon \subset A$ forms a $st - \epsilon$-pseudo orbit connecting x to x and $A \subset \bigcup_{y \in U_\epsilon} N_\epsilon(y)$. This is true for all $\epsilon = 1/k$ for all $k = 1, 2, ...,$. Then $U = \bigcup_{k \in \mathbb{N}} U_{1/k}$ forms a statistically asymptotic pseudo orbit in A.
It is now sufficient to show that $\omega_c(U) = A$. If $\alpha \in \omega_c(U)$ then there is a sub sequence $\{x_{n_k}\}$ in U with non-zero density converging to α. Since $\{x_{n_k}\} \subset U \subset A$ and A is closed, then $\alpha \in A$. i.e., $\omega_c(U) \subset A$.
Conversely, if $\beta \in A \subset \bigcup_{k \in \mathbb{N}} U_{1/k}$, for every k we have some $z_k \in U$ such that $\delta\{k \in \mathbb{N} : \beta \in N_{1/k}(z_k)\} > 0$. Thus $y \in \omega_c(U)$ which implies $A \subset \omega_c(U)$ and hence $\omega_c(U) = A$. □

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