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DERIVATION OF THE ANGLE COSINE FORMULAS VIA LAGRANGE'S THEOREM (FOR POLAR SPHERICAL TRIANGLES)

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Abstract In the context of Spherical Trigonometry, this study presents an alternative proof for the Angle Cosine Formulas, using the Langrange's Theorem as support. This text is a continuation of the article [1].

1 Introduction

Spherical Geometry is a very well developed area, studied for centuries. There are many widely known theorems and formulas with classical demonstrations, but it still constitutes an interesting field to look for new approaches.

In article [1], we presented a proof for the Half-Side Tangent Formulas, using the Langrange's Theorem as support. In this work, we intend to do the same for the Angle Cosine Formulas.

2 Preliminary Concepts and Classical Proof for The Angle Cosine Formulas

We consider this text as a continuation of the work published in the article [1]. Therefore, we will not repeat in this document the basic definitions of the subject addressed. To check the definitions of a spherical triangle and its elements, the reader must read the work [1].

Our purpose in this section is to display the classical proof for the Angle Cosine Formulas. Also, in this section, we will state the Langrange's Theorem, which will support our main result in this article.

The Cosine Formulas in spherical triangles are classified into two forms: The Side Cosine Formulas and The Angle Cosine Formulas. The first of them follows below:

Theorem 2.1. *(The Side Cosine Formulas) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively, as in the Figure 1. So:*

$$\cos(a) = \cos(b).\cos(c) + \sin(b).\sin(c).\cos(\widehat{A})$$

$$\cos(b) = \cos(a).\cos(c) + \sin(a).\sin(c).\cos(\widehat{B})$$

$$\cos(c) = \cos(a).\cos(b) + \sin(a).\sin(b).\cos(\widehat{C})$$

The readers can find proof for Theorem 2.1 in [2, 3, 4, 5].

To display the classical proof for the Angle Cosine Formulas, we first should establish the following result:

Lemma 2.2. *Let be $x,\ y,\ z \in \mathbb{R}$ such that $x^2 < 1$, $y^2 < 1$, $z^2 < 1$ and $2xyz - x^2 - y^2 - z^2 + 1 > 0$. If $m = \dfrac{x - yz}{\sqrt{1-y^2}\sqrt{1-z^2}}$, $n = \dfrac{y - xz}{\sqrt{1-x^2}\sqrt{1-z^2}}$ and $p = \dfrac{z - xy}{\sqrt{1-x^2}\sqrt{1-y^2}}$ then $x = \dfrac{m + np}{\sqrt{1-n^2}\sqrt{1-p^2}}$, $y = \dfrac{n + mp}{\sqrt{1-m^2}\sqrt{1-p^2}}$ and $z = \dfrac{p + mn}{\sqrt{1-m^2}\sqrt{1-n^2}}$.*

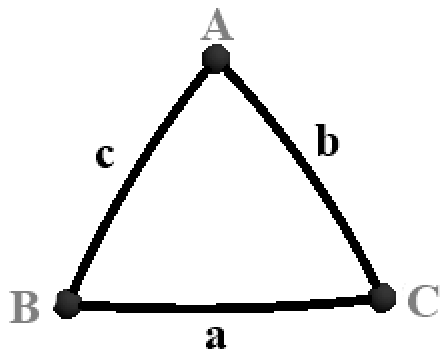

Figure 1. Oblique triangle ABC.

Proof. Denoting

$$L = 2xyz - x^2 - y^2 - z^2 + 1$$

follows

$$1 - m^2 = 1 - \left(\frac{x - yz}{\sqrt{1-y^2}\sqrt{1-z^2}}\right)^2 = 1 - \frac{(x-yz)^2}{(1-y^2)(1-z^2)} =$$

$$= \frac{1 - z^2 - y^2 + y^2z^2 + 2xyz - x^2 - y^2z^2}{(1-y^2)(1-z^2)} = \frac{2xyz - x^2 - y^2 - z^2 + 1}{(1-y^2)(1-z^2)}$$

That is

$$1 - m^2 = \frac{L}{(1-y^2)(1-z^2)} \tag{2.1}$$

Also

$$1 - n^2 = 1 - \left(\frac{y - xz}{\sqrt{1-x^2}\sqrt{1-z^2}}\right)^2 = 1 - \frac{(y-xz)^2}{(1-x^2)(1-z^2)} =$$

$$= \frac{1 - z^2 - x^2 + x^2z^2 + 2xyz - y^2 - x^2z^2}{(1-x^2)(1-z^2)} = \frac{2xyz - x^2 - y^2 - z^2 + 1}{(1-x^2)(1-z^2)}$$

That is

$$1 - n^2 = \frac{L}{(1-x^2)(1-z^2)} \tag{2.2}$$

And

$$1 - p^2 = 1 - \left(\frac{z - xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right)^2 = 1 - \frac{(z-xy)^2}{(1-x^2)(1-y^2)} =$$

$$= \frac{1 - y^2 - x^2 + x^2y^2 + 2xyz - z^2 - x^2y^2}{(1-x^2)(1-y^2)} = \frac{2xyz - x^2 - y^2 - z^2 + 1}{(1-x^2)(1-y^2)}$$

That is

$$1 - p^2 = \frac{L}{(1-x^2)(1-y^2)} \tag{2.3}$$

Moreover

$$m + np =$$

$$= \left(\frac{x-yz}{\sqrt{1-y^2}\sqrt{1-z^2}}\right) + \left(\frac{y-xz}{\sqrt{1-x^2}\sqrt{1-z^2}}\right)\left(\frac{z-xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right) =$$
$$= \left(\frac{x-yz}{\sqrt{1-y^2}\sqrt{1-z^2}}\right) + \left(\frac{yz-xy^2-xz^2+x^2yz}{(1-x^2).\sqrt{1-y^2}\sqrt{1-z^2}}\right) =$$
$$= \frac{(1-x^2)(x-yz)+(yz-xy^2-xz^2+x^2yz)}{(1-x^2).\sqrt{1-y^2}\sqrt{1-z^2}} =$$
$$= \frac{x-yz-x^3+x^2yz+yz-xy^2-xz^2+x^2yz}{(1-x^2).\sqrt{1-y^2}\sqrt{1-z^2}} =$$
$$= \frac{x(2xyz-x^2-y^2-z^2+1)}{(1-x^2).\sqrt{1-y^2}\sqrt{1-z^2}}$$

That is

$$m+np = \frac{x.L}{(1-x^2).\sqrt{1-y^2}\sqrt{1-z^2}} \tag{2.4}$$

Also

$$n+mp =$$
$$= \left(\frac{y-xz}{\sqrt{1-x^2}\sqrt{1-z^2}}\right) + \left(\frac{x-yz}{\sqrt{1-y^2}\sqrt{1-z^2}}\right)\left(\frac{z-xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right) =$$
$$= \left(\frac{y-xz}{\sqrt{1-x^2}\sqrt{1-z^2}}\right) + \left(\frac{xz-x^2y-yz^2+xy^2z}{(1-y^2).\sqrt{1-x^2}\sqrt{1-z^2}}\right) =$$
$$= \frac{(1-y^2)(y-xz)+(xz-x^2y-yz^2+xy^2z)}{(1-y^2).\sqrt{1-x^2}\sqrt{1-z^2}} =$$
$$= \frac{y-xz-y^3+xy^2z+xz-x^2y-yz^2+xy^2z}{(1-y^2).\sqrt{1-x^2}\sqrt{1-z^2}} =$$
$$= \frac{y(2xyz-x^2-y^2-z^2+1)}{(1-y^2).\sqrt{1-x^2}\sqrt{1-z^2}}$$

That is

$$n+mp = \frac{y.L}{(1-y^2).\sqrt{1-x^2}\sqrt{1-z^2}} \tag{2.5}$$

And

$$p+mn =$$
$$= \left(\frac{z-xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right) + \left(\frac{x-yz}{\sqrt{1-y^2}\sqrt{1-z^2}}\right)\left(\frac{y-xz}{\sqrt{1-x^2}\sqrt{1-z^2}}\right) =$$
$$= \left(\frac{z-xy}{\sqrt{1-x^2}\sqrt{1-y^2}}\right) + \left(\frac{xy-x^2z-y^2z+xyz^2}{(1-z^2).\sqrt{1-x^2}\sqrt{1-y^2}}\right) =$$
$$= \frac{(1-z^2)(z-xy)+(xy-x^2z-y^2z+xyz^2)}{(1-z^2).\sqrt{1-x^2}\sqrt{1-y^2}} =$$
$$= \frac{z-xy-z^3+xyz^2+xy-x^2z-y^2z+xyz^2}{(1-z^2).\sqrt{1-x^2}\sqrt{1-y^2}} =$$
$$= \frac{z(2xyz-x^2-y^2-z^2+1)}{(1-z^2).\sqrt{1-x^2}\sqrt{1-y^2}}$$

That is

$$p + mn = \frac{z.L}{(1-z^2).\sqrt{1-x^2}\sqrt{1-y^2}} \tag{2.6}$$

Now operating as follows:
Multiplying the Equation (2.2) by (2.3), we have

$$(1-n^2)(1-p^2) = \left(\frac{L}{(1-x^2)(1-z^2)}\right)\left(\frac{L}{(1-x^2)(1-y^2)}\right)$$
$$= \left(\frac{L^2}{(1-x^2)^2(1-y^2)(1-z^2)}\right)$$

Multiplying the Equation (2.1) by (2.3), we have

$$(1-m^2)(1-p^2) = \left(\frac{L}{(1-y^2)(1-z^2)}\right)\left(\frac{L}{(1-x^2)(1-y^2)}\right)$$
$$= \left(\frac{L^2}{(1-y^2)^2(1-x^2)(1-z^2)}\right)$$

Multiplying the Equation (2.1) by (2.2), we have

$$(1-m^2)(1-n^2) = \left(\frac{L}{(1-y^2)(1-z^2)}\right)\left(\frac{L}{(1-x^2)(1-z^2)}\right)$$
$$= \left(\frac{L^2}{(1-z^2)^2(1-x^2)(1-y^2)}\right)$$

Therefore:

$$\frac{m+np}{\sqrt{1-n^2}\sqrt{1-p^2}} =$$
$$= \frac{\dfrac{x.L}{(1-x^2).\sqrt{1-y^2}\sqrt{1-z^2}}}{\sqrt{\dfrac{L^2}{(1-x^2)^2(1-y^2)(1-z^2)}}} = \frac{\dfrac{x.L}{(1-x^2).\sqrt{1-y^2}\sqrt{1-z^2}}}{\dfrac{L}{(1-x^2)\sqrt{(1-y^2)}\sqrt{(1-z^2)}}} =$$
$$= \frac{x.L}{(1-x^2)\sqrt{(1-y^2)}\sqrt{(1-z^2)}}.\frac{(1-x^2)\sqrt{(1-y^2)}\sqrt{(1-z^2)}}{L} = x$$

Also

$$\frac{n+mp}{\sqrt{1-m^2}\sqrt{1-p^2}} =$$
$$= \frac{\dfrac{y.L}{(1-y^2).\sqrt{1-x^2}\sqrt{1-z^2}}}{\sqrt{\dfrac{L^2}{(1-y^2)^2(1-x^2)(1-z^2)}}} = \frac{\dfrac{y.L}{(1-y^2).\sqrt{1-x^2}\sqrt{1-z^2}}}{\dfrac{L}{(1-y^2)\sqrt{(1-x^2)}\sqrt{(1-z^2)}}} =$$
$$= \frac{y.L}{(1-y^2)\sqrt{(1-x^2)}\sqrt{(1-z^2)}}.\frac{(1-y^2)\sqrt{(1-x^2)}\sqrt{(1-z^2)}}{L} = y$$

Also

$$\frac{p+mn}{\sqrt{1-m^2}\sqrt{1-n^2}} =$$
$$= \frac{\dfrac{z.L}{(1-z^2).\sqrt{1-x^2}\sqrt{1-y^2}}}{\sqrt{\dfrac{L^2}{(1-z^2)^2(1-x^2)(1-y^2)}}} = \frac{\dfrac{z.L}{(1-z^2).\sqrt{1-x^2}\sqrt{1-y^2}}}{\dfrac{L}{(1-z^2)\sqrt{(1-x^2)}\sqrt{(1-y^2)}}} =$$

$$= \frac{z.L}{(1-z^2)\sqrt{(1-x^2)}\sqrt{(1-y^2)}} . \frac{(1-z^2)\sqrt{(1-x^2)}\sqrt{(1-y^2)}}{L} = z$$

Accordingly the lemma is proven. □

We are now in a position to present the classical proof for The Angle Cosine Formulas:

Theorem 2.3. *(The Angle Cosine Formulas) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively, as in the Figure 1. So:*

$$\cos(\widehat{A}) = -\cos(\widehat{B}).\cos(\widehat{C}) + \sin(\widehat{B}).\sin(\widehat{C}).\cos(a)$$

$$\cos(\widehat{B}) = -\cos(\widehat{A}).\cos(\widehat{C}) + \sin(\widehat{A}).\sin(\widehat{C}).\cos(b)$$

$$\cos(\widehat{C}) = -\cos(\widehat{A}).\cos(\widehat{B}) + \sin(\widehat{A}).\sin(\widehat{B}).\cos(c)$$

Proof. From Theorem 2.1, follows

$$\cos(\widehat{A}) = \frac{\cos(a) - \cos(b).\cos(c)}{\sin(b).\sin(c)} \tag{2.7}$$

$$\cos(\widehat{B}) = \frac{\cos(b) - \cos(a).\cos(c)}{\sin(a).\sin(c)} \tag{2.8}$$

$$\cos(\widehat{C}) = \frac{\cos(c) - \cos(a).\cos(b)}{\sin(a).\sin(b)} \tag{2.9}$$

On the other hand, there is also $\cos^2(a) < 1$, $\cos^2(b) < 1$ and $\cos^2(c) < 1$. Additionally, it is notorious that $\sin(a) = \sqrt{1-\cos^2(a)}$, $\sin(b) = \sqrt{1-\cos^2(b)}$ and $\sin(c) = \sqrt{1-\cos^2(c)}$.

Denoting $x = \cos(a)$, $y = \cos(b)$, $z = \cos(c)$, $m = \cos(\widehat{A})$, $n = \cos(\widehat{B})$ and $p = \cos(\widehat{C})$ follows from Equations (2.7), (2.8) and (2.9) that

$$m = \frac{x - yz}{\sqrt{1-y^2}\sqrt{1-z^2}}$$

$$n = \frac{y - xz}{\sqrt{1-x^2}\sqrt{1-z^2}}$$

$$p = \frac{z - xy}{\sqrt{1-x^2}\sqrt{1-y^2}}$$

Then, the hypotheses of the Lemma 2.2 are satisfied, which implies

$$x = \frac{m + np}{\sqrt{1-n^2}\sqrt{1-p^2}}$$

$$y = \frac{n + mp}{\sqrt{1-m^2}\sqrt{1-p^2}}$$

$$z = \frac{p + mn}{\sqrt{1-m^2}\sqrt{1-n^2}}$$

That is

$$\cos(a) = \frac{\cos(\widehat{A}) + \cos(\widehat{B})\cos(\widehat{C})}{\sin(\widehat{B})\sin(\widehat{C})}$$

$$\cos(b) = \frac{\cos(\widehat{B}) + \cos(\widehat{A})\cos(\widehat{C})}{\sin(\widehat{A})\sin(\widehat{C})}$$

$$\cos(c) = \frac{\cos(\widehat{C}) + \cos(\widehat{A})\cos(\widehat{B})}{\sin(\widehat{A})\sin(\widehat{B})}$$

because $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are also less than $180°$, causing $\sin(\widehat{A}) = \sqrt{1-\cos^2(\widehat{A})}$, $\sin(\widehat{B}) = \sqrt{1-\cos^2(\widehat{B})}$ and $\sin(\widehat{C}) = \sqrt{1-\cos^2(\widehat{C})}$.

Thus:

$$\cos(\widehat{A}) = -\cos(\widehat{B}).\cos(\widehat{C}) + \sin(\widehat{B}).\sin(\widehat{C}).\cos(a)$$

$$\cos(\widehat{B}) = -\cos(\widehat{A}).\cos(\widehat{C}) + \sin(\widehat{A}).\sin(\widehat{C}).\cos(b)$$

$$\cos(\widehat{C}) = -\cos(\widehat{A}).\cos(\widehat{B}) + \sin(\widehat{A}).\sin(\widehat{B}).\cos(c)$$

This ends the proof. □

To close this section, the Lagrange's Theorem will be presented, the proof for this theorem we already presented in article [1].

Theorem 2.4. *(The Lagrange's Theorem) - In two polar triangles, each angle of one is measured by the supplement of the corresponding side of the other. That is, in polar triangles arranged as in Figure 2, it is true that:*

$$\widehat{A} = 180° - a'$$
$$\widehat{B} = 180° - b'$$
$$\widehat{C} = 180° - c'$$

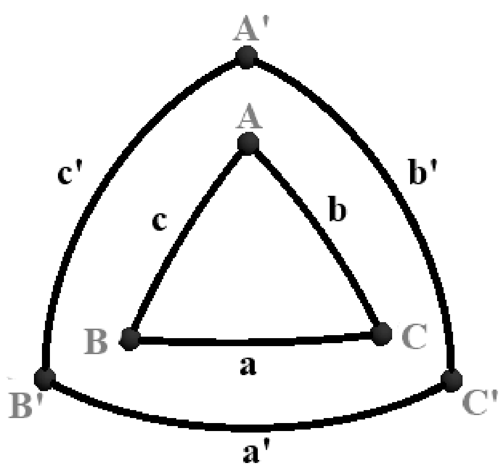

Figure 2. Figure support for Theorem 2.4.

3 The Angle Cosine Formulas via Lagrange's Theorem

Now comes the part where we will present the main result of this study. Below, we will present an alternative proof for Theorem 2.3, different from the classical presented in section 2, via Lagrange's Theorem 2.4.

Theorem 3.1. *(The Angle Cosine Formulas) - Let be a spherical triangle of vertices A, B and C, with internal angles measuring $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ whose opposite sides measure a, b and c, respectively, as in the Figure 1. So:*

$$\cos(\widehat{A}) = -\cos(\widehat{B}).\cos(\widehat{C}) + \sin(\widehat{B}).\sin(\widehat{C}).\cos(a)$$

$$\cos(\widehat{B}) = -\cos(\widehat{A}).\cos(\widehat{C}) + \sin(\widehat{A}).\sin(\widehat{C}).\cos(b)$$

$$\cos(\widehat{C}) = -\cos(\widehat{A}).\cos(\widehat{B}) + \sin(\widehat{A}).\sin(\widehat{B}).\cos(c)$$

Proof. Let be A'B'C 'and ABC polar triangles (See Figure 2). Consider Theorem 2.1 applied to the polar triangle A'B'C ':

$$\cos(a') = \cos(b').\cos(c') + \sin(b').\sin(c').\cos(\widehat{A'}) \tag{3.1}$$

$$\cos(b') = \cos(a').\cos(c') + \sin(a').\sin(c').\cos(\widehat{B'}) \tag{3.2}$$

$$\cos(c') = \cos(a').\cos(b') + \sin(a').\sin(b').\cos(\widehat{C'}) \tag{3.3}$$

From Theorem 2.4 can be stated:

$$\begin{aligned} a' &= 180^\circ - \widehat{A} \\ b' &= 180^\circ - \widehat{B} \\ c' &= 180^\circ - \widehat{C} \\ \widehat{A'} &= 180^\circ - a \\ \widehat{B'} &= 180^\circ - b \\ \widehat{C'} &= 180^\circ - c \end{aligned}$$

Replacing the previous considerations in Equations (3.1), (3.2) and (3.3):

$$\begin{aligned} \cos(180^\circ - \widehat{A}) = {} & \cos(180^\circ - \widehat{B}).\cos(180^\circ - \widehat{C}) + \\ & + \sin(180^\circ - \widehat{B}).\sin(180^\circ - \widehat{C}).\cos(180^\circ - a) \end{aligned}$$

$$\begin{aligned} \cos(180^\circ - \widehat{B}) = {} & \cos(180^\circ - \widehat{A}).\cos(180^\circ - \widehat{C}) + \\ & + \sin(180^\circ - \widehat{A}).\sin(180^\circ - \widehat{C}).\cos(180^\circ - b) \end{aligned}$$

$$\begin{aligned} \cos(180^\circ - \widehat{C}) = {} & \cos(180^\circ - \widehat{A}).\cos(180^\circ - \widehat{B}) + \\ & + \sin(180^\circ - \widehat{A}).\sin(180^\circ - \widehat{B}).\cos(180^\circ - c) \end{aligned}$$

$$-\cos(\widehat{A}) = -\cos(\widehat{B}).(-\cos(\widehat{C})) + \sin(\widehat{B}).\sin(\widehat{C}).(-\cos(a))$$

$$-\cos(\widehat{B}) = -\cos(\widehat{A}).(-\cos(\widehat{C})) + \sin(\widehat{A}).\sin(\widehat{C}).(-\cos(b))$$

$$-\cos(\widehat{C}) = -\cos(\widehat{A}).(-\cos(\widehat{B})) + \sin(\widehat{A}).\sin(\widehat{B}).(-\cos(c))$$

Consequently

$$\cos(\widehat{A}) = -\cos(\widehat{B}).\cos(\widehat{C}) + \sin(\widehat{B}).\sin(\widehat{C}).\cos(a)$$

$$\cos(\widehat{B}) = -\cos(\widehat{A}).\cos(\widehat{C}) + \sin(\widehat{A}).\sin(\widehat{C}).\cos(b)$$

$$\cos(\widehat{C}) = -\cos(\widehat{A}).\cos(\widehat{B}) + \sin(\widehat{A}).\sin(\widehat{B}).\cos(c)$$

The theorem is proved. □

4 Final Remarks

Following our studies presented in article [1], we continued to deepen Spherical Geometry and Trigonometry. Once again, we presented a proposal of proof for a well-known theorem using the Lagrange's Theorem as a support. In this time, our object of study was a theorem that presents formulas commonly known as the Angle Cosine Formulas.

In this text, we presented the classical proof and then we presented our proposal of proof using the Lagrange's Theorem as a support.

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