Palestine Journal of Mathematics

Vol. 12(1)(2023) , 930–936

A NEW PROOF OF SOMOS'S IDENTITIES OF LEVEL 6

D. Anu Radha, B. R. Srivatsa Kumar and N. V. Sayinath Udupa*

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11B65, 11F20; Secondary 33C05.

Keywords and phrases: Theta-functions, modular equations, q-identity.

Abstract M. Somos discovered around 6200 theta function identities of various levels by using PARI/GP script. He has not offered any proof for these identities. In this paper, we prove Somos's theta-function identities of level 6 by using modular equation of degree 3.

1 Introduction

Ramanujan documented many theta functions which involve quotients of the function $f(-q)$ at different arguments. For example, if [7, p. 204]

$$P := \frac{f^2(-q)}{q^{1/6}f^2(-q^3)} \qquad \text{and} \qquad Q := \frac{f^2(-q^2)}{q^{1/3}f^2(-q^6)}$$

then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

B. C. Berndt [6] proved similar type of identities and used it to evaluate various continued fractions, weber class invariants, theta functions and many more. After the publication of [6, 7], many mathematicians discovered similar identities in the spirit of Ramanujan. For the wonderful work, one can see [1, 2, 4, 5, 15, 16]. Motivated by the above work, M. Somos [9] used a computer to discover around 6277 new elegant Dedekind eta-function identities of various levels without offering the proof. He runs PARI/GP scripts and it works as a sophisticated programmable calculator. Many authors [3, 10, 11, 12, 13, 14, 17, 18] have given the proof of Somos's identities of various levels and found the applications of these in colored partitions. Ramanujan's theta function $\mathfrak{f}(x,y)$ is defined as

$$\mathfrak{f}(x,y) := \sum_{n=-\infty}^{\infty} x^{n(n+1)/2}y^{n(n-1)/2} \qquad |xy| < 1.$$

The function $f(x,y)$ enjoys the well-known Jacobi's triple-product identity [6, p. 35] given by

$$\mathfrak{f}(x,y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty$$

where here and throughout the paper, we assume $|q| < 1$ and employ the standard notation

$$(x;q)_\infty := \prod_{n=0}^{\infty}(1-xq^n).$$

The important special cases of $\mathfrak{f}(x,y)$ [6, p. 36] are as follows:

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$\varphi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty,$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

After Ramanujan, we define

$$\chi(q) := (-q;q^2)_\infty.$$

A theta function identity which relates $f(-q), f(-q^2), f(-q^n)$ and $f(-q^{2n})$ is called a theta function identity of level $2n$. After expressing theta-function identities which we are proving in terms of $f(-q^n)$, we obtain the arguments in $f(-q), f(-q^2), f(-q^3)$ and $f(-q^6)$, namely $-q, -q^2, -q^3$ and $-q^6$ all have exponents dividing 6, which is thus equal to the 'level' of the identity 6. For convenience in future we write $f(-q^n) = f_n$. Motivated by this, in the present work we prove some Somos's theta-function identities of level 6 by using modular equation of degree 3 in Section 2 and these identities are also proved by [11].
Before that we define a modular equation as given in the literature. A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(p,q;r;x) := \sum_{n=0}^{\infty}\frac{(p)_n(q)_n}{(r)_n n!}x^n \qquad |x|<1,$$

denotes an ordinary hypergeometric function with

$$(p)_n := p(p+1)(p+2)...(p+n-1).$$

Then, we say that β is of degree n over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

2 Main results

Theorem 2.1. *We have*

$$\frac{\psi^3(q)}{\psi(q^3)} - \frac{\varphi^3(-q^3)}{\varphi(-q)} = q\frac{\psi^3(q^3)}{\psi(q)}.$$

Proof. If $y = \pi\ {}_2F_1(1-x)/{}_2F_1(x)$ and $z =_2 F_1(x)$, then from Entry 10(i) and 12(v) [6, pp. 122-124] for $q = e^{-y}$, we have

$$\varphi(q) = \sqrt{z} \tag{2.1}$$

and

$$\chi(q) = 2^{\frac{1}{6}}\left\{x(1-x)q\right\}^{-1/24}. \tag{2.2}$$

Ramanujan in his notebook[8, p.230] and from Entry 5 [6, pp. 230-238] recorded the following modular equations of degree 3. If

$$P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \text{ and } Q := \left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/4}$$

then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0, \tag{2.3}$$

$$m = \frac{1-2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{1-2\left(\alpha\beta\right)^{1/4}} \tag{2.4}$$

and

$$\frac{3}{m}=\frac{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)-1}{1-2\left(\alpha\beta\right)^{1/4}}, \tag{2.5}$$

where β has degree 3 over α and $m=\frac{z_1}{z_3}$ the multiplier. From (2.4) and (2.5), we have

$$\frac{m^2}{3}=\frac{1-2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}-1}. \tag{2.6}$$

Now, On transforming (2.3) and (2.6) using (2.1) and (2.2), we obtain

$$\left(\frac{x}{y}\right)^6+\left(\frac{y}{x}\right)^6=(xy)^3-\frac{8}{(xy)^3} \tag{2.7}$$

and

$$\frac{\varphi^4(q)}{3\varphi^4(q^3)}=\frac{1-4\frac{x^3}{y^9}}{4\frac{y^3}{x^9}-1} \tag{2.8}$$

respectively, where $x:=x(q)=q^{-1/24}\chi(q)$ and $y:=y(q)=q^{-1/8}\chi(q^3)$. On Multiplying (2.7) by $(xy)^{-9}$, we obtain

$$\frac{x^3}{y^9}+\frac{8}{x^6y^6}-1+\frac{y^3}{x^9}=0.$$

which is equivalent to

$$3\frac{y^3}{x^9}\left(1-4\frac{x^3}{y^9}\right)+\left(\frac{x^3}{y^9}-1\right)\left(4\frac{y^3}{x^9}-1\right)=0. \tag{2.9}$$

On employing (2.8) and (2.9), we obtain

$$\frac{y^3}{x^9}\frac{\varphi^4(q)}{\varphi^4(q^3)}-1+\frac{x^3}{y^9}=0. \tag{2.10}$$

By using q-identities, one can easily deduce the following.

$$\varphi(q)=\frac{f_2^5}{f_1^2f_4^2},\qquad \varphi(-q)=\frac{f_1^2}{f_2},\qquad \psi(q)=\frac{f_2^2}{f_1}\qquad\text{and}\qquad \chi(-q)=\frac{f_1}{f_2}. \tag{2.11}$$

From (2.11) we observe that

$$\frac{\varphi(q)}{\varphi(q^3)}=\frac{x^2}{y^2}\frac{f_2}{f_6}. \tag{2.12}$$

Using (2.12) in (2.10), we obtain

$$\frac{q^{-2/3}}{xy^5}\left(\frac{f_2}{f_6}\right)^4-1+\frac{x^3}{y^9}=0.$$

On replacing $q\to -q$ in the above, rewriting $x(-q)$ and $y(-q)$ in terms of f_n by using (2.11) and then multiplying throughout by $f_1f_2^3f_3^9$, we obtain

$$f_2^8f_3^4f_6-f_1f_2^3f_3^9-qf_1^4f_6^9=0.$$

Finally, on dividing by $f_1^3f_2^2f_3^3f_6^3$ and after arrangement of terms, we obtain the required result. □

Theorem 2.2. *We have*

$$\frac{\varphi^3(-q)}{\varphi(-q^3)} = \frac{\psi^3(q)}{\psi(q^3)} - 9q\frac{\psi^3(q^3)}{\psi(q)}.$$

Proof. On multiplying (2.7) throughout by y^{-12} we obtain

$$8\frac{x^3}{y^9} - \frac{x^9}{y^3} + 1 + \frac{x^{12}}{y^{12}} = 0,$$

which is equivalent to

$$\left(\frac{x^9}{y^3} - 1\right)\left(1 - 4\frac{x^3}{y^9}\right) - 3\frac{x^{12}}{y^{12}}\left(4\frac{y^3}{x^9} - 1\right) = 0.$$

Using (2.8) in the above, we obtain

$$\frac{x^9}{y^3} - 1 - 9\frac{x^{12}}{y^{12}}\frac{\varphi^4(q^3)}{\varphi^4(q)} = 0.$$

Using (2.12) in the above, we obtain

$$\frac{x^9}{y^3} - 1 - 9q^{2/3}\frac{x^4}{y^4}\left(\frac{f_6}{f_2}\right)^4 = 0.$$

On replacing $q \to -q$ in the above and rewriting $x(-q)$ and $y(-q)$ in terms of f_n by using (2.11) and then multiplying throughout by $f_2^9 f_3^4$, we obtain

$$f_1^9 f_3 f_6^3 - f_2^9 f_3^4 + 9q f_1^4 f_2 f_6^8 = 0.$$

Finally, on dividing by $f_1^3 f_2^3 f_3^3 f_6^2$ and after rearrangement of terms, we obtain the required result. □

Theorem 2.3. *We have*

$$\frac{\varphi^3(-q)}{\varphi(-q^3)} = 9\frac{\varphi^3(-q^3)}{\varphi(-q)} - 8\frac{\psi^3(q)}{\psi(q^3)}.$$

Proof. On multiplying (2.7) throughout by $4(xy)^{-9}$, we obtain

$$4\frac{x^3}{y^9} + \frac{32}{x^6y^6} - 4 + 4\frac{y^3}{x^9} = 0, \tag{2.13}$$

which is equivalent to

$$3\left(1 + 8\frac{y^3}{x^9}\right)\left(1 - 4\frac{x^3}{y^9}\right) - 9\left(4\frac{y^3}{x^9} - 1\right) = 0.$$

Using (2.8) in the above, we obtain

$$\left(1 + 8\frac{y^3}{x^9}\right)\frac{\varphi^4(q)}{\varphi^4(q^3)} - 9 = 0.$$

Using (2.12) in the above, we obtain

$$\left(\frac{x^8}{y^8} + \frac{8}{xy^5}\right)\left(\frac{f_2}{f_6}\right)^4 - 9q^{2/3} = 0.$$

On replacing $q \to -q$ in the above, rewriting $x(-q)$ and $y(-q)$ in terms of f_n by using (2.11) and then multiplying throughout by $f_1 f_2^4 f_3^8$, we obtain

$$f_1^9 f_6^4 - 9 f_1 f_2^4 f_3^8 + 8 f_2^9 f_3^3 f_6 = 0.$$

Finally, on dividing by $f_1^3 f_2^2 f_3^2 f_6^3$ and after arrangement of terms, we obtain the required result. □

Theorem 2.4. *We have*

$$\frac{\varphi^3(-q)}{\varphi(-q^3)} = \frac{\varphi^3(-q^3)}{\varphi(-q)} - 8q\frac{\psi^3(q^3)}{\psi(q)}.$$

Proof. On multiplying (2.7) throughout by $4(xy)^{-9}$, we obtain

$$3\left(1-4\frac{x^3}{y^9}\right) - \left(1+8\frac{x^3}{y^9}\right)\left(4\frac{y^3}{x^9}-1\right) = 0.$$

Using (2.8) in the above, we obtain

$$\frac{\varphi^4(q)}{\varphi^4(q^3)} - 1 - 8\frac{x^3}{y^9} = 0.$$

Using (2.12) in the above, we obtain

$$\frac{x^8}{y^8}\left(\frac{f_2}{f_6}\right)^4 - q^{2/3} - 8q^{2/3}\frac{x^3}{y^9} = 0.$$

On replacing $q \to -q$ in the above, rewriting $x(-q)$ and $y(-q)$ in terms of f_n by using (2.11) and then multiplying throughout by $f_2^4 f_3^9$, we obtain

$$f_1^8 f_3 f_6^4 - f_2^4 f_3^9 + 8q f_1^3 f_2 f_6^9 = 0.$$

Finally, on dividing by $f_1^2 f_2^3 f_3^3 f_6^3$ and after arrangement of terms, we obtain the required result. □

Theorem 2.5. *We have*

$$\frac{\varphi^4(-q) - 9\varphi^4(-q^3)}{\varphi^4(-q) - \varphi^4(-q^3)} = \frac{\psi^4(q)}{q\psi(q^4)}.$$

Proof. On multiplying (2.7) throughout by $4(x^4-y^4)\left((x^2+y^2)^2 - x^2y^2\right)\left(x^4 - x^2y^2 + y^4\right)$, we obtain

$$4\frac{y^3}{x^9} + \frac{32}{x^6y^6} - 4 + 4\frac{x^{12}}{y^{12}} - 32\frac{x^6}{y^{18}} - 4\frac{x^{15}}{y^{21}} = 0,$$

which is equivalent to

$$\left(1-\frac{x^{12}}{y^{12}}\right)\left(1-4\frac{x^3}{y^9}\right)\left(4\frac{y^3}{x^9}-1\right) + 3\frac{x^{12}}{y^{12}}\left(4\frac{y^3}{x^9}-1\right)^2 - 3\left(1-4\frac{x^3}{y^9}\right)^2 = 0.$$

Using (2.8) in the above, we obtain

$$1 - \frac{x^{12}}{y^{12}} + 9\frac{x^{12}}{y^{12}}\frac{\phi^4(q^3)}{\phi^4(q)} - \frac{\phi^4(q)}{\phi^4(q^3)} = 0.$$

Using (2.12) in the above, we obtain

$$1 - \frac{x^{12}}{y^{12}} + 9q^{2/3}\frac{x^4}{y^4}\left(\frac{f_6}{f_2}\right)^4 - q^{-2/3}\frac{x^8}{y^8}\left(\frac{f_2}{f_6}\right)^4 = 0.$$

On replacing $q \to -q$ in the above, rewriting $x(-q)$ and $y(-q)$ in terms of f_n by using (2.11) and then multiplying throughout by $f_2^{12} f_3^{12}$, we obtain

$$q f_1^{12} f_6^{12} - 9q f_1^4 f_2^4 f_3^8 f_6^8 - f_1^8 f_2^8 f_3^4 f_6^4 + f_2^{12} f_3^{12} = 0.$$

Finally, on dividing by $f_1^4 f_2^4 f_3^4 f_6^4$ and after arrangement of terms, we obtain the required result. □

Remark:

As an application of Somos's theta function identities, one can see the application of these in colored partitions.

Acknowledgment

The authors thank the referee for the valuable comments and suggestions which helped us to improve the presentation of the paper.

References

[1] C. Adiga, N. A. S. Bulkhali, D. Ranganatha and H. M. Srivastava, Some new modular relations for the Rogers-Ramanujan type functions of order eleven with applications to partitions, *J. of Number Theory* **158**, 281–297 (2016).

[2] C. Adiga, N. A. S. Bulkhali, Y. Simsek and H. M. Srivastava, A Continued fraction of Ramanujan and some Ramanujan-Weber class invariants, *Filomat* **31**(13), 3975–3997 (2017).

[3] D. Anu Radha, B. R. Srivatsa Kumar and Sayinath Udupa, Two theta-function identities of level 10, *Advances in Mathematics : Scientific Journal*, **9**(7), 4929–4936 (2020).

[4] N. D. Baruah, Modular equations for Ramanujan's cubic continued fraction, *J. Math. Anal. Appl.*,, **268**(1), 244–255 (2002).

[5] N. D. Baruah, On some of Ramanujan's Schläfli-type "mixed" modular equations, *J. Number Theory,* **100**(2), 270–294 (2003).

[6] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York (1991).

[7] B. C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer-Verlag, New York (1996).

[8] S. Ramanujan, *The Lost Notebook and other unpublished papers*, Narosa, New Delhi (1988).

[9] M. Somos, Personal communication. **11**(1), 1–9 (2018).

[10] B. R. Srivatsa Kumar and R. G. Veeresha , Partition identities arising from Somos's theta-function identities, *Annali Dell 'Universita' Di Ferrara*, **63**, 303–313 (2017).

[11] B. R. Srivatsa Kumar and Anusha Kumari, Some theta-function identities of level six and its applications to partitions, *Advanced Studies in Contemporary Mathematics*, **28**(1), 57–71 (2018).

[12] B. R. Srivatsa Kumar and D. Anu Radha, Somos's theta-function identities of level 10, *Turkish Journal of Mathematics*, **42**, 763–773 (2018).

[13] B. R. Srivatsa Kumar, K. R. Rajanna, R. Narendra, New Theta-Function Identities of Level 6 in the Spirit of Ramanujan, *Mathematical Notes*, **106**(6), 922–929 (2019).

[14] B. R. Srivatsa Kumar, K. R. Rajanna, R. Narendra, Theta-function identities of level 8 and its application to partition, *Afrika Matematika*, **20**, 257–267 (2019).

[15] K. R. Vasuki and T. G. Sreeramamurthy, A note on P-Q modular equations, *Tamsui Oxf. J. Math. Sci.*, **21**(2), 109-120 (2005).

[16] K. R. Vasuki, On some of Ramanujan's P-Q modular equations, *J. Indian Math. Soc.*, **73**(3-4), 131-143 (2006).

[17] K. R. Vasuki and R. G. Veeresha, On Somos's theta-function identities of level 14, *Ramanujan J.*, **42**, 131–144 (2017).

[18] B. Yuttanan, New modular equations in the spirit of Ramanujan, *Ramanujan J.*, **29**, 257–272 (2012).

Author information

D. Anu Radha, Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal – 576104, India.
E-mail: `anu.radha@manipal.edu`

B. R. Srivatsa Kumar, Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal – 576104, India.
E-mail: `srivatsa.kumar@manipal.edu`

N. V. Sayinath Udupa*, Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal – 576104, India.
E-mail: sayinath.udupa@manipal.edu, *Corresponding Author

Received: December 12th, 2021
Accepted: May 18th, 2022