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Solving some partial q-differential equations using transformation methods

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Abstract In this paper, we introduce the concepts of transformed functions and q-difference inverse transforms. Then, we apply them to solve some partial q-difference equations. Finally, we provide four examples to show how we can benefit from the obtained results.

1 Introduction

The subject of q-calculus started in the nineteenth century in intense works by Jackson [20], Carmichael [11], Mason [23], Trjitzinsky [29], Adams [1], Mason [23], and Trjitzinsky [29]. Recently, q-calculus has been a goal of some recent studies such as [4, 7, 9].

The q-difference has a lot of applications in different fields of mathematics such as number theory, statistical physics [24], fractal geometry [14],[15], quantum mechanics, orthogonal polynomials [19] and other sciences including theory of relativity [5], mechanics and quantum theory.

In the last years, partial q-difference equations are generally studied by various methods of separation of variables, by the techniques of Lie symmetry, using q-integral transforms [3, 16, 17, 10, 22].

In this paper, we explain the solution of the partial q-difference equations using reduced transformation method.

First, let us recall some basic concepts of q-calculus mentioned in ([5, 6, 26, 12, 13, 18, 20, 21, 28]).

The shifted factorial $(r)_n$ is defined by:

$$\begin{aligned} (r;q)_0 &= 1, \\ (r;q)_n &= (1-r)(1-qr)(1-q^2r)(1-q^3r)...(1-q^{n-1}r) \\ &= \prod_{k=0}^{n-1}(1-q^k r), \qquad n \in N, 0<q<1, \; and \; r \in \mathbb{C}. \end{aligned}$$

A complex number r is defined by:

$$\begin{aligned} [r]_q &= 1+q+q^2+...+q^{r-1} \\ &= \frac{1-q^r}{1-q}, \qquad q \in \mathbb{C}-\{1\}; \;\; r \in \mathbb{C} \end{aligned}$$

and the factorial function is

$$\begin{aligned} [r]_q! &= [1]_q[2]_q[3]_q...[r]_q \\ &= \prod_{k=1}^{r}[k]_q, \qquad q \neq 1; \quad k \in \mathbb{N}, \quad 0 \leq q \leq 1. \end{aligned}$$

The q-binomial coefficient $\begin{bmatrix} rk \end{bmatrix}_q$ is defined by:

$$\begin{bmatrix} r \\ k \end{bmatrix}_q = \frac{[r]_q!}{[r]_q![r-k]_q!}, \qquad r = 0, 1, 2, ...k.$$

The function $(f+g)^n$ is defined as:

$$(f+g)_q^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ r \end{array} \right]_q q^{k(k-1)/2} f^{n-k} g^k, n \in \mathbb{N}.$$

The exponential function is defined as:

$$e_q^t = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!}, \qquad 0 < |q| < 1 \;.$$

The functions e_q^t and $e_{q^{-1}}^{-t}$ satisfy:

$$e_q^t \, e_{q^{-1}}^{-t} = 1.$$

The q-derivative $D_q f$ is defined as:

$$D_q f(t) = \frac{f(qt) - f(t)}{qt - t}, \qquad 0 < |q| < 1,$$

$$\begin{aligned} D_q(fg)(t) &= f(qt) D_q g(t) + g(t) D_q f(t) \\ &= g(qt) D_q f(t) + f(t) D_q g(t). \end{aligned}$$

and

$$D_q(\frac{f}{g})(t) = \frac{g(t) D_q f(t) - f(t) D_q g(t)}{g(qt) g(t)}.$$

The partial q-derivative of the function $f(t_1, t_2, ..., t_n)$ to a variable t_i is defined by

$$\begin{cases} D_{q,t_i} f(t) = \dfrac{\partial_q f(\vec{t})}{\partial x_i} = \dfrac{(\epsilon_{q,i} f)(\vec{t}) - f(\vec{t})}{(q-1) t_i}; \; t \neq 0, \; 0 < q < 1, \\ \qquad [D_{q,t_i} f(t)]_{t_i = 0} = \left[\dfrac{\partial_q f(\vec{t})}{\partial t_i} \right]_{t_i = 0} = \lim\limits_{t_i \to 0} \dfrac{\partial_q f(\vec{t})}{\partial t_i}, \end{cases}$$

where

$$(\epsilon_{q,i} f)(\overrightarrow{t}) = f(t_1, t_2, ..., q t_i, ..., t_n).$$

Remark 1.1. I) If $f(t_1, t_2)$ is a function of two variables then

$$\begin{cases} D_{q,t_1} f(t_1, t_2) = \frac{\partial_q}{\partial_q t_1} f(t_1, t_2) = \frac{f(q t_1, t_2) - f(t_1, t_2)}{(q-1)\, t_1}, \quad t_1 \neq 0, \;\; 0 < q < 1, \\ [D_{q,t_1} f(t_1, t_2)]_{t_1 = 0} = \left[\frac{\partial_q}{\partial_q t_1} f(t_1, t_2) \right]_{t_1 = 0} = \lim_{t_1 \to 0} \frac{\partial_q f(t_1, t_2)}{\partial t_1}. \end{cases}$$

and

$$\begin{cases} D_{q,t_2} f(t_1, t_2) = \frac{\partial_q}{\partial_q t_2} f(t_1, t_2) = \frac{f(t_1, q t_2) - f(t_1, t_2)}{(q-1)\, t_2}, \quad t_2 \neq 0, \;\; 0 < q < 1, \\ [D_{q,t_2} f(t_1, t_2)]_{t_2 = 0} = \left[\frac{\partial_q}{\partial_q t_2} f(t_1, t_2) \right]_{t_2 = 0} = \lim_{t_2 \to 0} \frac{\partial_q f(t_1, t_2)}{\partial t_2}. \end{cases}$$

also

$$\begin{aligned} D_{q,t_1,t_2} f(t_1, t_2) &= \frac{\partial_q^2}{\partial_q t_2 \partial_q t_1} f(t_1, t_2) = \frac{\partial_q}{\partial_q t_2} \left[\frac{\partial_q}{\partial_q t_1} f(t_1, t_2) \right] \\ &= \frac{\partial_q}{\partial_q t_2} \left[\frac{f(q t_1, t_2) - f(t_1, t_2)}{(q-1) t_1} \right] \\ &= \frac{\frac{f(q t_1, q t_2) - f(q t_1, t_2)}{(q-1) t_1} - \frac{f(t_1, q t_2) - f(t_1, t_2)}{(q-1) t_1}}{(q-1) y} \\ &= \frac{f(q t_1, q t_2) - f(t_1, q t_2) - f(q t_1, t_2) + f(t_1, t_2)}{(q-1)^2 t_1 t_2} \\ &= \frac{\partial_q^2}{\partial_q t_1 \partial_q t_2} f(t_1, t_2) = D_{q,t_2,t_1} f(t_1, t_2), \;\; t_1 \neq 0, \; t_2 \neq 0, \end{aligned}$$

then

$$\frac{\partial_q^2}{\partial_q t_1 \partial_q t_2} f(t_1,t_2) = \frac{f(qt_1,qt_2) - f(t_1,qt_2) - f(q_1,t_2) + f(t_1,t_2)}{(q-1)^2 t_1 t_2} = \frac{\partial_q^2}{\partial_q t_2 \partial_q t_1} f(t_1,t_2).$$

For example let $f(t_1,t_2) = t_1^2 t_2^3$, then

$$\begin{aligned} D_{q,t_1} f(t_1,t_2) &= \frac{\partial_q}{\partial_q t_1} f(t_1,t_2) = \frac{\partial_q}{\partial_q t_1}(t_1^2 t_2^3) = \frac{(qt_1)^2 t_2^3 - t_1^2 t_2^3}{(q-1)t_1} \\ &= \frac{(q^2-1)t_1^2 t_2^3}{(q-1)t_1} = (1+q)t_1 t_2^3 = [2]_q t_1 t_2^3, \end{aligned}$$

$$\begin{aligned} D_{q,t_2} f(t_1,t_2) &= \frac{\partial_q}{\partial_q t_2} f(t_1,t_2) = \frac{\partial_q}{\partial_q t_2}(t_1^2 t_2^2) = \frac{t_1^2 (qt_2)^3 - t_1^2 t_2^2}{(q-1)t_2} \\ &= \frac{(q^3-1)t_1^2 t_2^3}{(q-1)t_2} = (1+q+q^2)t_1^2 t_2^2 = [3]_q t_1^2 t_2^2, \end{aligned}$$

$$\begin{aligned} \frac{\partial_q^2}{\partial_q t_2 \partial_q t_1} f(t_1,t_2) &= \frac{\partial_q^2}{\partial_q t_2 \partial_q t_1}(t_1^2 t_2^3) = \frac{(qt_1)^2 (qt_2)^3 - t_1^2 (qt_2)^3 - (qt_1)^2 t_2^3 + t_1^2 t_2^3}{(q-1)^2 t_1 t_2} \\ &= \frac{(q^5 - q^3 - q^2 + 1)t_1 t_2^2}{(q-1)^2} = (1+2q+2q^2+q^3)t_1 t_2^2 \\ &= (1+q)(1+q+q^2)t_1 t_2^2 = [2]_q [3]_q t_1 t_2^2 = \frac{\partial_q^2}{\partial_q t_1 \partial_q t_2} f(t_1,t_2). \end{aligned}$$

The formula of q-Leibniz is given as follows.

$$D_q^s (f(z)g(z) = \frac{\partial_q^s}{\partial_q z^s}(f(z)g(z)) = \sum_{r=0}^{s} \left(\begin{array}{c} s \\ r \end{array} \right)_q \frac{\partial_q^r}{\partial_q z^r} f(zq^{s-r}) \frac{\partial_q^{s-r}}{\partial_q t^{s-r}} g(z) \tag{1.1}$$

2 Reducing the q-Differential transform method

In this section, we define transformed function and q-difference inverse transform. Then, we present a new technique to solve partial q-difference equations. Illustrative examples are given to clarify the obtained results.

Definition 2.1. Suppose that all q-differentials of$w(x,y)$ exist in some neighborhood of $y = \lambda$, then

$$W_n(x) = \frac{1}{[n]_q!} \left[\frac{\partial_q^n}{\partial_q t^n} w(x,y) \right]_{y=\lambda}, \tag{2.1}$$

where $W_n(x)$ is the transformed function.

Definition 2.2. The q-difference inverse transform of $W_n(x)$ is defined as follows:

$$w(x,y) = \sum_{n=0}^{\infty} W_n(x)(y-\lambda)^{(n)}, \tag{2.2}$$

The following formula is inferred by substituting (2.1) to (2.2):

$$w(x,y) = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \left[\frac{\partial_q^n}{\partial_q y^n} w(x,y) \right]_{y=\lambda} (y-\lambda)^{(n)}. \tag{2.3}$$

In the following results, we investigate the the properties of some transformations.

Theorem 2.3. **(i)** *If* $u(x, y) = \alpha v(x, y) \pm w(x, y)$*, then* $U_n(x) = \alpha V_n(x) \pm \beta W_n(x)$.

(ii) *If* $u(x, y) = v(x, y)w(x, y)$*, then* $U_n(x) = \sum\limits_{m=0}^{n} V_{n-m}(x)W_m(x)$.

Proof. Let $\lambda = 0$ in (2.3), then

I) By definition (2), we have

$$\begin{aligned} U_n(x) &= \frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q y^n}u(x;y)\right]_{y=0} \\ &= \frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q y^n}\left(\alpha v(x;y) \pm \beta w(x;y)\right)\right]_{y=0} \\ &= \frac{1}{[n]_q!}\left[\alpha\frac{\partial_q^k}{\partial_q y^k}v(x;y)\right]_{y=0} \pm \frac{1}{[n]_q!}\left[\beta\frac{\partial_q^n}{\partial_q y^n}w(x;y)\right]_{y=0} \\ &= \alpha\frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q y^n}v(x)\right]_{y=0} \pm \beta\frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q y^n}w(x;y)\right]_{y=0} \\ &= \alpha V_n(x) \pm \beta W_n(x). \end{aligned}$$

To prove(**ii**), we will use q-Leibniz formula.

$$\begin{aligned} U_n(x) = \frac{1}{[n]_q!}\frac{\partial_q^n}{\partial_q y^n}\left[v(x,y).w(x,y)\right]_{y=0} &= \frac{1}{[n]_q!}D_{q,y}^n\left[v(x,y).w(x,y)\right]_{y=0} \\ &= \frac{1}{[n]_q!}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}_q \left[D_{q,y}^{n-m}v(x,yq^m)D_{q,y}^m w(x,y)\right]_{y=0} \end{aligned}$$

$$= \sum_{m=0}^{n}\frac{1}{[n-m]_q![m]_q!}\left[\sum_{r=0}^{n-m}(-1)^r(1-q)^r\begin{bmatrix} n-m \\ r \end{bmatrix}_q q^{\frac{r}{(}r-1)}y^r\left(D_{q,t}^{n-m+r}v(x,y)\right)D_{q,y}^m w(x,y)\right]_{y=0},$$

at $(y = 0)$ all the limits of the inner series vanish, except for $(r = 0)$ then, we have

$$\begin{aligned} U_n(x) &= \sum_{m=0}^{n}\frac{1}{[n-m]_q![m]_q!}\left[D_{q,y}^{n-m}v(x,y)D_{q,y}^m w(x,y)\right]_{y=0} \\ &= \sum_{m=0}^{n}\left(\frac{1}{[n-m]_q!}\left[D_{q,y}^{n-m}v(x,y)\right]_{y=0}\right)\left(\frac{1}{[m]_q!}\left[D_{q,y}^m w(x,y)\right]_{y=0}\right) \\ &= \sum_{m=0}^{n}V_{n-m}(x)W_m(x). \end{aligned}$$

□

Theorem 2.4. $If\ u(x, y) = x^m y^r\ \ then\ \ U_n(x) = x^m\delta(n - r),$

$$where,\ \delta(n-r) = \begin{cases} 1;\ n = 0, \\ 0;\ n \neq 0. \end{cases}$$

Proof. Let $\lambda = 0$ in (2.3). By definition (2) we have:

$$\begin{aligned} U_n(x) &= \frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q y^n}u(x;y)\right]_{y=0} = \frac{1}{[n]_q!}\left[\frac{\partial_q^n(x^m y^r)}{\partial_q y^n}u(x;y)\right]_{y=0} \\ &= \frac{x^m}{[n]_q!}\left[\frac{\partial_q^n(y^r)}{\partial_q y^n}u(x;y)\right]_{y=0} \\ &= \begin{cases} x^m, \quad n = r\,, \\ x^m\frac{[r]_q[r-1]_q\ldots[r-n+1]_q}{[k]_q}y^{r-n}, \quad n < r, \\ 0; \quad n > r, \end{cases} \\ &= x^m\delta(n-r). \end{aligned}$$

□

Theorem 2.5. (i) $If\ \ u(x,y)=\frac{\partial_q}{\partial_q x}w(x,y)\ \ then\ \ \ U_n(x)=\frac{\partial_q}{\partial_q x}W_n(x).$

(ii) $If\ u(x,y)=\frac{\partial_q^m}{\partial_q x^m}w(x,y)\ \ then\ \ \ U_n(x)=[m+1]_q[m+2]_q...[m+n]_qW_{m+n}(x).$

Proof. Let $\lambda=0$ in (2.3).

(i): By definition (2), we have:

$$\begin{aligned} U_n(x) &= \frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q t^n}u(x,y)\right]_{y=0}=\frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q y^n}(\frac{\partial_q}{\partial_q x}w(x,y))\right]_{y=0}\\ &= \frac{1}{[n]_q!}\left[\frac{\partial_q}{\partial_q x}(\frac{\partial_q^n}{\partial_q y^n}w(x,y))\right]_{y=0}=\frac{\partial_q}{\partial_q x}\left[\frac{1}{[n]_q!}\frac{\partial_q^n}{\partial_q y^n}w(x,y)\right]_{y=0}\\ &= \frac{\partial_q}{\partial_q x}W_k(x). \end{aligned}$$

(ii): By definition (2) we have:

$$\begin{aligned} U_n(x) &= \frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q y^n}u(x,y)\right]_{y=0}=\frac{1}{[n]_q!}\left[\frac{\partial_q^n}{\partial_q y^n}(\frac{\partial_q^m}{\partial_q y^m}w(x,y))\right]_{y=0}\\ &= \frac{1}{[n]_q!}\left[\frac{\partial_q^{n+m}}{\partial_q y^{n+m}}w(x,y)\right]_{y=0}=\frac{[n+m]_q!}{[n]_q!}\left[\frac{1}{[n+m]_q!}\frac{\partial_q^{n+m}}{\partial_q t^{n+m}}w(x,y)\right]_{y=0}\\ &= \frac{[n+m]_q!}{[n]_q!}W_{n+m}(x)=[n+1]_q[n+2]_q...[n+m]_qW_{n+m}(x). \end{aligned}$$

□

In what follow, we construct some examples that illustrate how we can apply the methods investigated in the previous three theorems.

Example 2.6. Consider the equation

$$\frac{\partial_q}{\partial_q y}w(x,y)-\frac{\partial_q^2}{\partial_q y^2}w(x,y)=0, \tag{2.4}$$

With the initial condition

$$w(x;0)=g(x). \tag{2.5}$$

Solution: Let $\lambda=0$. Then

$$\frac{\partial_q}{\partial_q y}w(x,y)-\frac{\partial_q^2}{\partial_q y^2}w(x,y)=0,$$

$$\frac{\partial_q}{\partial_q y}w(x,y)=\frac{\partial_q^2}{\partial_q y^2}w(x,y),$$

$$[n+1]_qW_{n+1}(x)=\frac{\partial_q^2}{\partial_q y^2}W_n(x), \qquad n=0,1,2,... \tag{2.6}$$

using the initial condition (2.5), we get

$$W_0(x)=w(x,0)=g(x). \tag{2.7}$$

Now, substituting (2.7) into (2.6), we have

$$\begin{aligned} W_1(x) &= \frac{1}{[1]_q} g(x) = \frac{1}{[1]_q!} g(x), \\ W_2(x) &= \frac{1}{[2]_q} W_1(x) = \frac{1}{[1]_q[2]_q} g(x) = \frac{1}{[2]_q!} g(x), \\ W_3(x) &= \frac{1}{[3]_q!} g(x) \\ &\vdots \\ W_n(x) &= \frac{1}{[n]_q!} g(x), \end{aligned}$$

then, the analytic solution of (2.4) gives

$$w(x,y) = \sum_{n=0}^{\infty} W_n(x) y^n = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} g(x) y^n = g(x) \sum_{n=0}^{\infty} \frac{y^n}{[n]_q!} = g(x) e_q^y,$$

if $g(x) = e_q^x$ then

$$w(x,y) = e_q^x e_q^y.$$

Example 2.7. Solve

$$\frac{\partial_q}{\partial_q y} k(x,y) = k^2(x,y) + \frac{\partial_q}{\partial_q x} k(x,y), \tag{2.8}$$

with

$$k(x;0) = 1 + 2x. \tag{2.9}$$

Solution: Let $\lambda = 0$. Then

$$[n+1]_q K_{n+1}(x) = \sum_{m=0}^{n} K_{n-m}(x) K_m(x) + \frac{\partial_q}{\partial_q x} K_n(x,y), \quad n = 0, 1, 2, ... \tag{2.10}$$

using the initial condition (2.9), we have

$$K_0(x) = 1 + 2x. \tag{2.11}$$

Now, substituting (2.11) into (2.10), we obtain

$$K_1(x) = 4x^2 + 4x + 3,$$

$$K_2(x) = \frac{1}{1+q}(16x^3 + 24x^2 + 4(6+q)x + 10),$$

$$K_3(x) = \frac{(80+56q)x^4 + (160+16q)x^3 + (16q^2+72q)x^2 + (136+56q)x + (53+13q)}{(1+q)(1+q+q^2)}$$

$$\vdots$$

now, the solution of (2.8) is

$$\begin{aligned} k(x,y) &= \sum_{n=0}^{\infty} K_n(x) y^n \\ &= K_0(x) + K_1(x)y + K_2(x)y^2 + K_3(x)y^3 + ... \\ &= 1 + 2x + (4x^2 + 4x + 3)y + \frac{1}{1+q}(16x^3 + 24x^2 + 4(6+q)x + 10)y^2 \\ &+ \frac{(80+56q)x^4 + (160+16q)x^3 + (16q^2+72q)x^2 + (136+56q)x + (53+13q)}{(1+q)(1+q+q^2)} y^3 + ... \end{aligned}$$

Example 2.8. Consider

$$\frac{\partial_q}{\partial_q x}v(t,x) = \frac{\partial_q^2}{\partial_q x^2}v(t,x) + \frac{\partial_q}{\partial_q t}(tv(t,x)), \tag{2.12}$$

with the initial condition

$$v(t,0) = t^2. \tag{2.13}$$

Solution: Let $\lambda = 0$. Then

$$[n+1]_q V_{n+1}(t) = \frac{\partial_q^2}{\partial_q x^2}V_n(t) + \frac{\partial_q}{\partial_q t}(tV_n(t)), \tag{2.14}$$

using the initial condition (2.13), we have

$$V_0(t) = t^2. \tag{2.15}$$

Now, substituting (2.15) into (2.14), we obtain

$$V_1(t) = \frac{[2]_q + [3]_q t^2}{[1]_q} = \frac{[2]_q + [3]_q t^2}{[1]_q!},$$

$$V_2(t) = \frac{[2]_q(1+[3]_q) + [3]_q^2 t^2}{[1]_q[2]_q} = \frac{[2]_q(1+[3]_q) + [3]_q^2 t^2}{[2]_q!},$$

$$V_3(t) = \frac{[2]_q(1+[3]_q+[3]_q^2) + [3]_q^3 t^2}{[1]_q[2]_q[3]_q} = \frac{[2]_q(1+[3]_q+[3]_q^2) + [3]_q^3 t^2}{[3]_q!},$$

$$V_n(t) = \frac{[2]_q(1+[3]_q+[3]_q^2+...+[3]_q^{k-1}) + [3]_q^k t^2}{[n]_q!}$$
$$= \left(\frac{[2]_q}{[3]_q-1}\left([3]_q^n - 1\right) + [3]_q^n t^2\right)\frac{1}{[n]_q!}.$$

Now, we obtain the series solution of (2.12) as

$$v(t,x) = \sum_{n=0}^{\infty} V_n(t)t^n$$

$$= V_0(t) + V_1(t)x + V_2(t)x^2 + V_3(t)x^3 + ... + V_n(t)x^n + ...$$
$$= t^2 + \frac{[2]_q + [3]_q t^2}{[1]_q!}x + \frac{[2]_q(1+[3]_q) + [3]_q^2 t^2}{[2]_q!}x^2 + \frac{[2]_q(1+[3]_q+[3]_q^2) + [3]_q^3 t^2}{[3]_q!}x^3$$
$$+ ... + \left(\frac{[2]_q}{[3]_q-1}\left([3]_q^n - 1\right) + [3]_q^n t^2\right)\frac{x^n}{[n]_q!} + ...$$
$$= \frac{[2]_q}{[3]_q-1}\left(\sum_{n=0}^{\infty}\frac{[3]_q^n x^n}{[n]_q!} - \sum_{k=0}^{\infty}\frac{x^n}{[n]_q!}\right) + t^2\sum_{n=0}^{\infty}\frac{[3]_q^n x^n}{[n]_q!}$$
$$= \frac{[2]_q}{[3]_q-1}\left(\sum_{n=0}^{\infty}\frac{([3]_q x)^n}{[n]_q!} - \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}\right) + t^2\sum_{n=0}^{\infty}\frac{([3]_q x)^n}{[n]_q!}$$
$$= \frac{[2]_q}{[3]_q-1}\left(e_q^{[3]_q x} - e_q^{x}\right) + t^2 e_q^{[3]_q x}.$$

Example 2.9. q-Laplace equation

$$\frac{\partial_q^2}{\partial_q x^2}v(x,t) + \frac{\partial_q^2}{\partial_q t^2}v(x,t) = 0, \tag{2.16}$$

with the boundary conditions

$$\begin{aligned} v(x,0) &= 0, \\ \frac{\partial_q}{\partial_q t} v(x,0) &= \frac{\sin ax}{a}, \; a \neq 0. \end{aligned} \tag{2.17}$$

where $a \neq 0$ is an integer.
Solution: Let $\lambda = 0$. Then

$$\frac{\partial_q^2}{\partial_q x^2} v(x,t) = -\frac{\partial_q^2}{\partial_q t^2} v(x,t),$$

$$[n+1]_q[n+2]_q V_{n+2}(x) = -\frac{\partial_q^2}{\partial_q x^2} V_n(x), \tag{2.18}$$

using the boundary conditions (2.17), we have

$$\begin{aligned} V_0(x) &= 0, \\ V_1(x) &= \frac{\sin ax}{a}. \end{aligned} \tag{2.19}$$

Now, substituting (2.19) into (2.18), we obtain

$$V_2(x) = 0, \tag{2.20}$$

$$V_3(x) = -\frac{a \sin ax}{[2]_q[3]_q} = -\frac{a \sin ax}{[3]_q!},$$

$$V_4(x) = 0,$$

$$V_5(x) = -\frac{a^3 \sin ax}{[5]_q!},$$

$$\vdots$$

$$V_{2n}(x) = 0,$$

$$V_{2n+1}(x) = \frac{a^{2n-1} \sin ax}{[2n+1]_q!}.$$

Now, the solution of (2.16) is

$$v(x;t) = \sum_{n=0}^{\infty} V_n(x) t^n$$

$$= V_1(x)t + V_3(x)t^3 + V_5(x)t^5 + ... + V_{2n+1}(x)t^{2n-1} + ...$$

$$= \frac{\sin_q ax}{a} t + \frac{a \sin_q ax}{[3]_q!} t^3 + \frac{a^3 \sin_q ax}{[5]_q!} t^5 + ... + \frac{a^{2n-1} \sin_q ax}{a^2[2n+1]_q!} t^{2n+1} - ...$$

$$= \frac{\sin_q ax}{a^2} \sum_{n=0}^{\infty} \frac{a^{2n+1}}{[2a+1]_q!} t^{2n+1} = \frac{\sin_q ax \sinh_q at}{a^2}, \; a \neq 0.$$

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